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The homotopy exact sequence of a pair of graphs

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ABSTRACT. Higher homotopy of graphs has been defined in several articles. However, the existence of a long exact sequence associated to a pair (G, A) has not been touched at. We treat it here. Applied to the discrete spheres, this lead to interesting open questions.

Keywords: graph, homotopy, exact sequence.

A sequência exata de homotopia de um par de gráficos

RESUMO. A homotopia superior de gráficos foi definida em vários artigos. Todavia, a existência de uma longa sequência exacta associada a um par (G, A) de gráfico não foi abordada. Vamos tratar disto aqui. Aplicada à as esferas discretas, isto levavanta interessantes questões abertas.

Palavras-chave: gráficos, homotopia, sequência exata.

Introduction

Homotopy groups of graphs have been defined in Benayat and Kadri (1997) and Babson et al. (2006). One of the main property of any consistent homotopy theory is the existence of a long exact sequence associated to any based pair (G, H, v_0) of graphs. In this paper, we define a relative homotopy theory and prove such a sequence exists. We give some immediate applications and formulate two important open questions related to the homotopy of the discrete spheres.

Definitions and notations

As usual, an undirected graph is a pair $G = (V, E)$ where $V = V(G)$ and $E = E(G)$ are, respectively, the sets of vertices and edges of G. We consider only simple (i.e. without multiple edges) and connected graphs. The neighborhood (or the neighbors) of $a \in V$ is $N(a) = \{b \in V : (a,b) \in E\}$. We use the same letter $\boldsymbol{N}$ for all graphs; this will cause no confusion. We will adopt the convention that any vertex is a neighbor of itself: $\forall a \in G$ then $a \in N(a)$. Recall that a graph can be defined by giving the neighbors of its vertices. We write $N^\star(G)$ (or just $N^\star$) for all the neighborhoods of the graph G A pair (G,A) of graphs is just a graph G and a subgraph A. A morphism $f : G \rightarrow G'$ is an application $f : V(G) \rightarrow V(G')$ such that $f(N(a)) \subset N(f(a))$ for any $a \in G$. Graphs and morphisms define a category G. A morphism of pairs of graphs $f : (G, A) \rightarrow (G', A')$ is a morphism $f : G \rightarrow G'$ such that $f_{|A} : A \rightarrow A'$ is also a morphism. A based pair (of graphs) is a pair (G, A) and a distinguished vertex $x_0 \in A$. A morphism of based pairs sends base vertices to base vertices.

The set $\boldsymbol{G}(G,G')$ of morphisms can be endowed with a natural graph structure. If f, g: $G \rightarrow G'$ are morphisms, they are contiguous if, $\forall a \in V(G)$, $\exists \sigma \in N^*(G')$, depending upon a, such that $f(N(a)) \cup g(N(a)) \subset \sigma$. The graph structure on $\boldsymbol{G}(G,G')$ is defined by the neighborhoods $N(f) = \{g \in$ G $(G, G') : f$ and are contiguous$\}$.

The infinite path (graph) is □ with the neighborhoods $N(m) = \{m-1, m, m+1\}$, $\forall m \in$ □. We will also consider the subgraphs □ and, if $p < q$, $[p,q] = \{p, p+1, ..., q\}$ of □. The particular graph $[0,m]$ will be written I_m.

The graphs □, □ and I_m are based by 0. A sequence in G is a morphism $s : \mathrm{N} \rightarrow G$; It is convergent if $\exists m \in \mathrm{N}$ such that $s(n) \in N(s(m))$, $\forall n \geq m$. Notice that, if we put $x_n = s(n)$, then $x_n \in N(x_{n-1})$ for $n \geq 1$. As in topology, we have the following result.

Proposition 1. The application $G \xrightarrow{f} G'$ is a morphism $\Leftrightarrow$ for any convergent sequence s in G, then $f \circ s$ is convergent in G'.

Proof. The necessary part is obvious. We prove the converse by contradiction. Assuming f is not a morphism, then there exists $x_0 \in G$ such that $f(N(x_0)) \not\subset N(y_0)$ where $y_0 = f(x_0)$. Let $x_1 \in N(x_0)$ and $y_1 = f(x_1) \notin N(y_0)$; Consider the sequence $s(2n) = x_0$ and

$s(2n+1)=x_1$. This sequence is convergent since $\mathrm{Im}(s)=\{x_0,x_1\}\subset N(x_0)$ but $f\circ s(\mathbb{N})=\{y_0,y_1\}$ is neither included in $N(y_0)$ nor in $N(y_1)$. Recall that $y_0\notin N(y_1)$ because $y_1\notin N(y_0)$.

Without restricting the generality, we can and will assume that a convergent sequence $s:\mathbb{N}\to G$ satisfies $s(n)=s(m),\ \forall n\geq m$ for some m; we call it a path of length m in G and write it $s: I_m\to G$. Moreover, by trivial continuation, we may assume that any finite number of paths $\gamma_i,\ i=1,2,\dots,r$ are defined on the same I_m.

The Cartesian product of graphs G and G' is written G x G' and defined by the neighborhoods:

$$N(x,y)=\left\{\begin{array}{l}(x',y')\in G\times G' :\\ \left(x'\in N(x) \text{ and } y'=y\right)\\ \text{or}\\ \left(x'=x \text{ and } y'\in N(y)\right)\end{array}\right\}$$

Higher homotopy of graphs

If $I_m^n=\underbrace{I_m\times\cdots\times I_m}_{n \text{ times}}$, its boundary is:

$$\partial I_m^n=\left\{(p_1,\dots,p_m)\in I^n \ :\ \text{at least one of the } p_j^{'}\text{s is 0 or } m\right\}$$

Let (G,x_0) be a based graph. An n-spheroid of (G,x_0) is a morphism $\gamma: I_m^n\to G$ such that $\gamma(\partial I_m^n)=\{x_0\}$ and we put $\Omega^n(G,x_0)$ for the set of all of them. We can assume that any finite number of n-spheroids are defined on the same I_m^n, by extending them outside their domain by the constant value x_0. If $\gamma_i: I_m^n\to G,\ i=1,2,$ are two n-spheroids, they are homotopic, written $\gamma_1\sim\gamma_2$, if there is a morphism $H: I_m^n\times I_p\to G$ such that $H(s,0)=\gamma_1(s)$, $H(s,p)=\gamma_2(s), \forall s\in I_p$ and $H(\partial I_m^n,t)=\{x_0\}, \forall t\in I_p$. Homotopy is an equivalence relation on $\Omega^n(G,x_0)$. The n^{th} homotopy group of the based graph (G,x_0) is defined as $\Pi_n(G,x_0)=\Omega^n(G,x_0)/\sim$. It is known that $\Pi_n(G,x_0)$ is a group for $n\geq 1$ which is abelian for $n\geq 2$. The following results have been proved in Benayat and Kadri (1997) and Babson et al. (2006).

Proposition 2. 1) We have the isomorphism: $\Pi_n(G,x_0)\approx\Pi_1(\Omega^{n-1}(G,x_0),c_{x_0})$, $n\geq 2$, where c_{x_0} is the constant spheroid at x_0; 2) $\Pi_n(G,x_0)$ is abelian for $n\geq 2$.

Definition 1. A graph G is simply connected if $\Pi_1(G,x_0)=\{0\}$ and n-connected ($n\ \geq\ 2$) if $\Pi_j(G,x_0)=\{0\}$, $1\leq j\leq n-1$ and $\Pi_n(G,x_0)\neq\{0\}$, for some base vertex x_0.

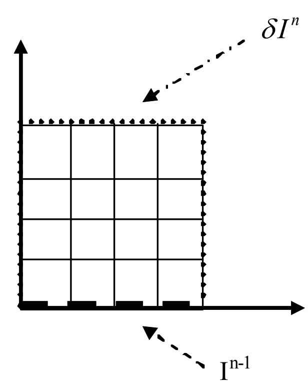
Relative homotopy

Let (G,A,x_0) be a based pair of graphs. For $n\geq 1$, we put:

$$I_m^{n-1}=\underbrace{I_m\times\cdots\times I_m}_{(n-1)\text{ factors}}\times\{0\} \qquad \text{and}$$

$$\delta I_m^n=\{(q_1,\dots,q_n)\in I^n : (\text{one of the } q_j^{'}s=0 \text{ or } m \text{ for } 1\leq j\leq n-1) \text{ or } (q_n=m)\}$$

$$=\left(\bigcup_{1\leq j\leq n-1}\underbrace{I_m\times\cdots\times I_m\times\{0,m\}}_{j}\times I_m\times\cdots\times I_m\right)\cup\left(I_m\times\cdots\times I_m\times\{m\}\right)$$



We have $\partial I_m^n=I^{n-1}\cup\delta I^n$.

Definition 2. An n-spheroid of (G,A,x_0) is a morphism of triples $\gamma:(I^n,I^{n-1},\delta I^n)\to(G,A,x_0)$.

Remark 1. We have $\gamma(I^{n-1})\subset A$, $\gamma_{|I^{n-1}}\in\Omega^{n-1}(A,x_0)$ and $\gamma(\delta I^n)=\{x_0\}$. When $A=\{x_0\}$, we get back $\Omega^n(G,x_0)$.

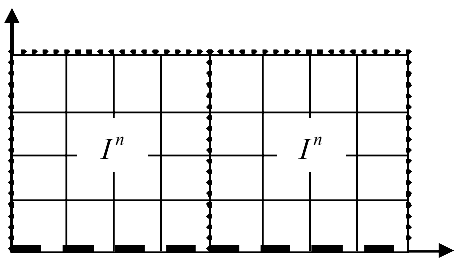
We write $\Omega^n(G,A,x_0)$ for the set of n-spheroids of (G,A,x_0). Let us recall that any finite number of n-spheroids can and will be defined on the same I_m^n which will be written I^n.

Definition 3. Let $\gamma_i:(I^n,I^{n-1},\delta I^n)\to(G,A,x_0)$, $i=1,2$, be two n-spheroids. They are homotopic if there is a morphism $H: I^n\times I_p\to G$ such that $H(\cdot,0)=\gamma_1$, $H(\cdot,p)=\gamma_2$ and $H(\cdot,t)\in\Omega^n(G,A,x_0), \forall t\in I_p$.

Homotopy of relative spheroids is an equivalence relation on $\Omega^n(G,A,x_0)$; the set of equivalence classes is written $\Pi_n(G,A,x_0)$. The n-spheroid $\gamma\in\Omega^n(G,A,x_0)$ is nullhomotopic if it is homotopic to a $\gamma'\in\Omega^n(G,A,x_0)$ such that $\gamma'(I^n)\subset A$.

Law of composition in $\Pi_n(G,A,x_0)$, $n\geq 2$

Let $\gamma_i:(I_{m_i}^n,I_{m_i}^{n-1},\delta I_{m_i}^n)\to(G,A,x_0)$, $i=1,2$, be representants of two homotopy classes $\overline{\gamma_i}$, $i=1,2$ in $\Pi_n(G,A,x_0)$. Let J^n be the domain shown underneath where the second square is actually a translated copy of I^n. The base J^{n-1} is just the juxtaposition of I^{n-1} and a translation of itself.



We define $\gamma_1 * \gamma_2$ as the morphism $\gamma : J^n \to (G, A, x_0)$ which is γ_1 on I^n and γ_2 on the translated of I^n. It is easy to show that the homotopy class $\bar{\gamma}$ of γ depends only upon the homotopy classes $\bar{\gamma}_i$.

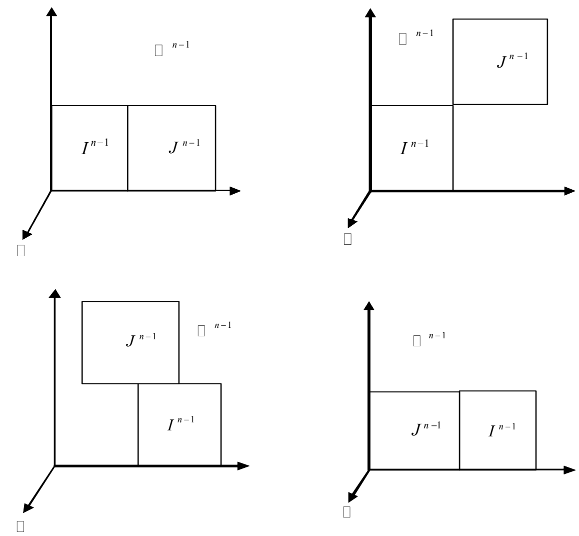
Proposition 3. The $\Pi_n(G, A, x_0)$ are sets for n = 1, groups for $n \geq 2$ which are abelian for $n \geq 3$.

Proof. The associativity of the law is easy but lengthy; we omit it. The class of the constant spheroid at x_0 is clearly the identity element. Let us show that every class has an inverse. We assume $n \geq 2$. Let $\gamma : (I_m^n, I_m^{n-1}, \delta I_m^n) \to (G, A, x_0)$ and define $\tilde{\gamma}: I^n \to G$ by $\tilde{\gamma}(q_1, \dots, q_n) = \gamma(m - q_1, q_2, \dots, q_n)$ for $(q_1, \dots, q_n) \in I^n$. The composed morphism $\gamma * \tilde{\gamma}$ is defined on $K = I_{2m} \times I_m \times \dots \times I_m$. We have $(\gamma * \tilde{\gamma})(K^{n-1}) \subset A$ and $(\gamma * \tilde{\gamma})(\delta K) = \{x_0\}$. So $(\gamma * \tilde{\gamma})$ is an n-spheroid of (G, A, x_0). Let us show that it is nullhomotopic. So let us consider $H : K \times I_m \to G$ defined by:

$$H(q_1, s, t) = \begin{cases} \gamma(q_1, s) & \text{if} \quad q_1 \leq t \\ \gamma(t, s) & \text{if} \quad t \leq q_1 \leq m - t \\ \gamma(m - q_1, s) & \text{if} \quad 2m - t \leq q_1 \leq 2m \end{cases}$$

where $s = (q_2, \dots, q_n)$. The application H is a morphism since all parts of its definition glue together. Moreover, for t = 0, we have $H(q_1, s, 0) = \gamma(0, q_2, \dots, q_n) = \{x_0\}$ and, for t = m, we have where $q = (q_1, q_2, \dots, q_n)$. For all $t \in I_m$, the values of $H(q_1, s, t)$ are given in term of values of γ and, consequently, define an n-spheroid of (G, A, x_0). So is homotopic to the constant n-spheroid.

We show that $\Pi_n(G, A, x_0)$ is abelian for $n \geq 3$. Let $\gamma_1 : I^n \to G$ and $\gamma_2 : J^n \to G$ two elements of $\Omega^n(X, A, x_0)$. The following displacements of the domains I^n and J^n in the hyperplane $q_n = 0$ and around the axis $(0, \dots, 0) \times \mathbb{R}$ define a homotopy between $\gamma_1 * \gamma_2$ and $\gamma_2 * \gamma_1$. Points of the empty space are sent to x_0. This has been possible only because of the extra degree of freedom allowing rotation around the axis $0 \times \mathbb{R}$:



Meaning of $\Pi_0[G,x_0]$

Elements of $\Pi_0[G, x_0]$ are homotopy classes of morphisms y : ({0++ $\gamma : (\{0\}, 0) \to (G, A)$; two such morphisms γ and ρ, i.e. two vertices $\gamma(0)$ and $\rho(0)$ of G, are homotopic if there is a path in G joining them. So $\Pi_0[G, x_0]$ is in bijection with the (path) connected components of G.

The homotopy exact sequence

Let (G, A, x_0) be a based pair of graphs, $i : (A, x_0) \subset (G, x_0)$ and $j : (G, \{x_0\}, x_0) \subset (G, A, x_0)$ be the obvious inclusions which are morphisms. By functoriality of the Π_n we get homomorphisms $i_* : \Pi_n[A, x_0] \to \Pi_n[G, x_0]$ and $j_* : \Pi_n[G, x_0] \to \Pi_n[G, A, x_0]$, for $n \geq 2$; i_* is also a homomorphism when $n = 1$. We define a boundary operator $\partial : \Pi_{n+1}[G, A, x_0] \to \Pi_n[A, x_0]$ as follows: $\partial[\gamma] = [\gamma_{|I^n}]$ where $\gamma : (I^{n+1}, I^n, 0) \to (G, A, x_0)$ is an element of $\Omega^{n+1}(G, A, x_0)$. We get a long homotopy sequence:

$$\cdots \to \Pi_n[A, x_0] \xrightarrow{i_*} \Pi_n[G, x_0] \xrightarrow{j_*} \Pi_n[G, A, x_0] \xrightarrow{\partial} \Pi_{n-1}[A, x_0] \xrightarrow{i_*} \cdots$$

$$\cdots \to \Pi_1[A, x_0] \to \Pi_1[G, x_0] \to \Pi_1[G, A, x_0] \to \Pi_0[A, x_0] \to \Pi_0[G, x_0]$$

Theorem 1. The long homotopy sequence of a based triple (G, A, x_0) is exact in degrees $n \geq 1$.

Proof. 1) $j_* \circ i_* = 0$: let $\gamma : I^n \to A$ represent a class in $\Pi_n[A, x_0]$. Then $j \circ i \circ \gamma : (I^n, I^{n-1}, 0) \to (G, A, x_0)$ represents the null class in $\Pi_n[G, A, x_0]$ since its image is already in A.

2) $\partial \circ j_* = 0$: let $\gamma : I^n \to G$ defining a class in $\Pi_n[G, x_0]$. Then $\gamma_{|I^{n-1}} : I^{n-1} \to \{x_0\}$ is a constant map whose class is 0.

3) $i_* \circ \partial = 0$: let $\gamma : I^n \to G$ represents a class in $\Pi_n[G, A, x_0]$. Then $\gamma_1 = i \circ \gamma_{|I^{n-1}} : I^{n-1} \to A?G$ represents $i_* \circ \partial[\gamma]$. The map $H = \gamma : I^{n-1} \times I_m = I^n \to G$ is a homotopy between $H(\alpha, 0) = \gamma_1(\alpha)$ and $H(\alpha, m) = x_0$.

4) $Ker(j_*) \subset \mathrm{Im}(i_*)$: let $\gamma : I^n \to G$ represents a class in $\Pi_n[G, x_0]$ homotopic, as an element of $\Omega^n(G, A, x_0)$, to a morphism $\delta : I^n \to A$; thus we have $[\gamma] = [i \circ \delta] = i_*[\delta]$.

5) $Ker(i_*) \subset \mathrm{Im}(\delta)$: we consider an element $\gamma : I^n \to A$ of $\Omega^n(A, x_0)$ whose image $\gamma : I^n \to G$ of $\Omega^n(G, x_0)$ is nullhomotopic and let $K : I^n \times I_p \to G$ be such a homotopy. So $K(\alpha, p) = x_0, \forall \alpha \in I^n$. The restriction of K to I^{n-1} is precisely γ and the rest of the border of $I^n \times I_p$ is sent to x_0. We have gotten an element in $\Omega^{n+1}(G, A, x_0)$ whose image is γ

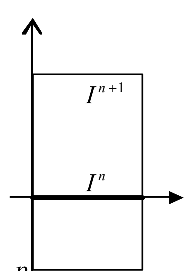
6) We prove, now, exactness at $\Pi_{n+1}[G, A, x_0]$. Let $\gamma : (I^{n+1}, I^n, \partial I^{n+1}) \to (G, A, x_0)$ an element of $\Omega^{n+1}(A, x_0)$ such that $\gamma_0 = \gamma_{|I^n} : (I^n, 0) \to (A, x_0)$ is homotopic, in A to the constant loop x_0 and let $H : I^n \times I_p \to A$ such a homotopy where, for convenience, we take $I_p = \{-p, -p+1, \ldots, 0\}$. So we have:

$$H(q_1, \ldots, q_n, 0) = \gamma_0(q_1, \ldots, q_n);$$

$$H(q_1, \ldots, q_n, s) \in A, \forall s \in I_p;$$

$$H(q_1, \ldots, q_n, -p) = x_0, \forall (q_1, \ldots, q_n) \in I^n.$$

We have to show that γ is homotopic in G relatively to A to an $(n + 1)$ - spheroid $\omega \in \Omega^n(G, x_0)$. Let us consider the following extension $\tilde{\gamma}$ of γ to $J = I^{n+1} \cup I^n \times I_p$:



$\tilde{\gamma}(q_1, \ldots, q_{n+1}) = \gamma(q_1, \ldots, q_{n+1})$ if $(q_1, \ldots, q_{n+1}) \in I^{n+1}$;
$\tilde{\gamma}(q_1, \ldots, q_n, t) = \gamma(q_1, \ldots, q_n, 0)$ if $(q_1, \ldots, q_n, t) \in I^n \times I_p$.

It is clear that γ and $\tilde{\gamma}$ are homotopic in $\Omega^{n+1}(G, A, x_0)$. Let us construct a relative homotopy, that is in (G, A), between $\tilde{\gamma}$ and an (n+1)-spheroid $\omega \in \Omega^n(G, x_0)$.

For $t \in [0, p]$, we define:

$$K(q_1, \ldots, q_{n+1}, t) = \gamma(q_1, \ldots, q_{n+1}), \forall (q_1, \ldots, q_{n+1}) \in I^{n+1}$$

$$K(q_1, \ldots, q_n, s, t) = \begin{cases} H(q_1, \ldots, q_n, -t + s + p) & -p \le s \le t - p \\ H(q_1, \ldots, q_n, 0) & t - p \le s \le 0 \end{cases}$$

$$K(q_1, \ldots, q_n, s, t) = \begin{cases} H(q_1, \ldots, q_n, -t + s + p) & -p \le s \le t - p \\ H(q_1, \ldots, q_n, 0) & t - p \le s \le 0 \end{cases}$$

For each value of t, the application $K : J \times \{t\} \to G$ is an element of $\Omega^{n+1}(G, A, x_0)$. For $t = 0$, we start with γ_0 and we end up, for $t = p$, with $K(*, p)$ which is in $\Omega^{n+1}(G, x_0)$ since all the boundary of J is sent to $\{x_0\}$.

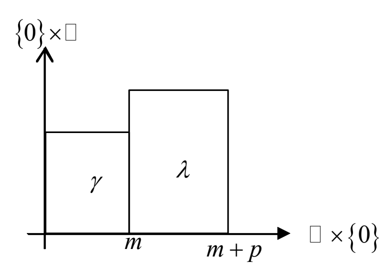
Corollary 1. a) If A is contractible, then $\Pi_n[G, x_0] \xrightarrow{j_*} \Pi_n[G, A, x_0]$ is an isomorphism for $n \ge 2$.

b) If G is contractible, then $\Pi_n[G, A, x_0] \xrightarrow{\partial} \Pi_{n-1}[A, x_0]$ is an isomorphism for $n \ge 2$.

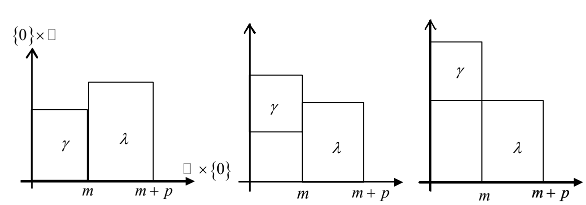
Proof. These are direct consequences of the exact sequence.

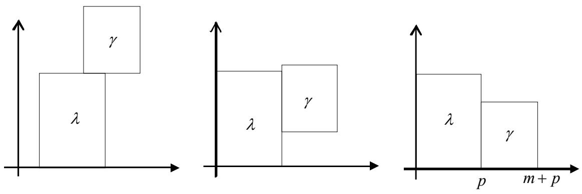
Proposition 4. The image of the homomorphism $\Pi_2[G, x_0] \xrightarrow{j_*} \Pi_2[G, A, x_0]$ is included in the center of $\Pi_2[G, A, x_0]$.

Proof. Let $\gamma : I_m \times I_p \to G$ be a 2-spheroid of (G, x_0) representing a class in $\Pi_2[G, x_0]$ and $\lambda : I_{m'} \times I_{p'} \to G$ representing a class in $\Pi_2[G, A, x_0]$. We have the following representation for $j_*(\gamma) * \lambda$:



Using the same trick we used before, we can move the domain of γ everywhere in $□^2$ since $\partial(I_m \times I_p)$ is sent to $\{x_0\}$ by γ.





This is not the case for δ which is constrained by $\delta(I_{m'} \times \{0\}) \subset A$. So we move the domain of γ round the domain of δ. All these moves are globally a homotopy between $j_*(\gamma) * \lambda$ and $\lambda * j_*(\gamma)$.

Example 1. The path graph $I_m = [0, m]$ is contractible; a homotopy of the identity map with a constant is $H : I_m \times I_m \to I_m$ given by:

$$H(n,t) = \begin{cases} 0 & \textbf{if } n < t \\ n - t & \textbf{if } n \geq t \end{cases}$$

Proposition 5. Let $G = \bigcup_{k \in \mathrm{N}} A_k$ where, for any k, A_k is contractible and $A_k \subset A_{k+1}$. Then $\Pi_n[G, x_0] = 0, \forall n \geq 1$.

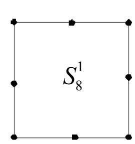
Proof. Let $\gamma : I^n \to G$ be an n-spheroid at x_0. Then, there is $k \in \mathrm{N}$, such that $\gamma(I^n) \subset A_k$. The latter being contractible, γ is homotopic to a constant.

Since $\mathrm{Z}^m = \underline{\lim}[-k, k]^m$, we deduce:

Corollary 2. We have $\Pi_n[\mathrm{Z}^m, 0] = 0, \forall n \geq 1$ and $m \geq 1$.

The discrete n-sphere

The circle with m vertices is the quotient graph $\mathbf{S}^1_m = [0, m]/\{0, m\}$. A visual representation of $\mathbf{S}^1_8$ is:

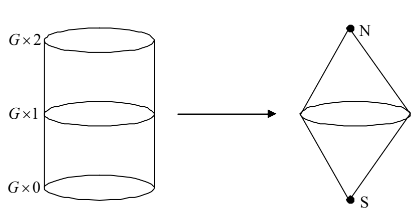


A graph $G = (V, E)$ is trivial if $\exists v \in V$ such that $N(v) = V$. In particular, they are contractible and have trivial homotopy. All the complete graphs are trivial. So the minimal non-trivial circle is $\mathbf{S}^1_4$.

Suspension of a graph

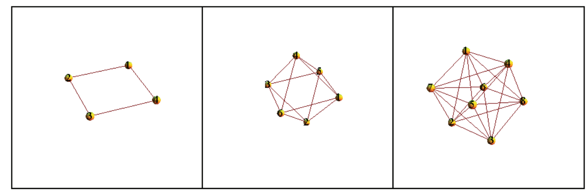
Let $G = (V, E)$ be a graph; the suspension SG of G is the quotient of G x [0,2] by the relation G x {0} = S and G x {2} = N:

$\mathsf{S}G = G \times [0,2]/\{G \times \{0\} = S; G \times \{2\} = N\}$



A useful remark is that a suspension SG is a union $\boldsymbol{S}^+G \cup \boldsymbol{S}^-G$ of two contractible subgraphs which are the neighborhoods of the 'opposite' poles N and S, and whose intersection is G.

We define the discrete n-sphere as $\mathbf{S}^n = \boldsymbol{S}(\mathbf{S}^{n-1})$ starting with the circle $\mathbf{S}^1 = \mathbf{S}^1_4$.



Proposition 6. Let $f : G \to H$ be a morphism of graphs. Then we have a natural morphism $\mathsf{S}f : \mathsf{S}G \to \mathsf{S}H$.

Proof. We define $\mathsf{S}f$ as follows: $\mathsf{S}f(x,1) = (f(x),1), \forall x \in G$, and North and South poles goes to the same named vertices respectively.

It is clearly a morphism of graphs

Proposition 7. We have a canonical homorphism of suspension:

$$\mathbf{S} : \Pi_n[\mathrm{G}, \mathrm{x}_0] \to \Pi_{n+1}[\mathsf{S}\mathrm{G}, \mathrm{x}_0], \mathrm{n} \geq 1.$$

Proof. The suspension homomorphism is the result of the following compositions:

$$\Pi_n[G] \xrightarrow{\approx} \Pi_{n+1}[\mathsf{S}^+G, G] \longrightarrow \Pi_{n+1}[\mathsf{S}G, \mathsf{S}^-G] \xrightarrow{\approx} \Pi_{n+1}[\mathsf{S}G]$$

The first isomorphism comes from the HES (homotopy exact sequence) applied to the pair $(\boldsymbol{S}^+G, G)$ and the contractibility of $\boldsymbol{S}^+G$; a similar argument gives the second isomorphism. The middle homomorphism comes from functoriality applied to the inclusion of pairs $(\boldsymbol{S}^+G, G) \subset (\boldsymbol{S}G, \boldsymbol{S}^-G)$.

In particular, we have homomorphisms of suspension $\boldsymbol{S}_n : \Pi_n[\mathbf{S}^n, 1] \to \Pi_{n+1}[\mathbf{S}^{n+1}, 1]$, $n \geq 1$. As mentioned before, we have, in several ways, $\mathbf{S}^n = \mathbf{S}^+ \cup \mathbf{S}^-$ with $\mathbf{S}^+ \cap \mathbf{S}^- = \mathbf{S}^{n-1}$. The number of vertices and edges of S^n is 2 (n + 1) and 2n (n + 1) respectively. Applying the HES (homotopy exact sequence) to the pair $(\mathbf{S}^2, \mathbf{S}^1)$ and using results from Benayat and Kadri (1997), we get the exact sequence:

$\prod_2 [\mathrm{S}^1, 1] = \{0\} \xrightarrow{i_*} \prod_2 [\mathrm{S}^2, 1] \xrightarrow{j_*} \prod_2 [\mathrm{S}^2, \mathrm{S}^1, 1] \xrightarrow{\partial} \prod_1 [\mathrm{S}^1, 1] \approx$ □ $\to 0$

Conclusion

By analogy with topology, there is a strong indication that $\Pi_2[\mathbf{S}^2,1] \approx \mathbb{Z}$. This leads to the following future work:

1) Use technology to prove the latter isomorphism.
2) We have defined (not yet published) a notion of discrete fibration of graphs and proved the existence of a long exact sequence. Use the topological Hopf fibration $S^1 \to S^3 \to S^2$ to construct a combinatorial version of it. This would imply, using the mentioned exact sequence that $\Pi_2[\mathbf{S}^3,1] = \{0\}$ and $\Pi_3[\mathbf{S}^3,1] \approx \Pi_3[\mathbf{S}^2,1]$.
3) Do we have $\Pi_n[\mathbf{S}^n,1] \approx \mathbb{Z}$ for $n > 1$.

This would show that the discrete spheres can play the role of n-dimensional holes in a graph and that the homotopy of graphs is able to detect them.

4) Ultimately, compute the $\Pi_n[\mathbf{S}^m,1]$ for small values of m and n, and compare them with the topological homotopy groups of spheres.

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