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On asymptotically $\lambda-$ statistical equivalent sequences of interval numbers

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ABSTRACT. In this paper we introduce and study the concepts of asymptotically $\overline{s}_{\lambda}-$statistical equivalent and strongly asymptotically $\lambda-$ equivalent sequences for interval numbers and prove some inclusion relations. In the last section we introduce the concept of Cesaro asymptotically $\lambda-$ equivalent sequences defined by Orlicz function of multiple $\overline{L}$ and we have established some relation between those classes.

Keywords: $\lambda-$sequence, asymptotic equivalence, interval numbers.

Sequências estatísticas equivalentes assintoticamente a $\lambda-$ de números intervalares

RESUMO. Este artigo apresenta e analisa os conceitos de equivalente estatístico assintoticamente a $\overline{s}_{\lambda}-$ e sequências equivalentes fortemente assintóticas a $\overline{s}_{\lambda}-$ para números intervalares e comprova algumas relações de inclusão. Na última seção, apresentamos o conceito de sequência equivalente assintótica a $\lambda-$ de Cesaro definido pela função de Orlicz de múltiplo $\overline{L}$ e estabelecemos algumas relações entre essas classes.

Palavras chave: sequência $\lambda-$, equivalência assintótica, números intervalares.

Introduction

Currently the sequence of interval numbers and usual convergence of sequences of interval numbers are studied by Chiao (2002). Later, is introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space by Sengonul and Eryilmaz (2010). Recently Esi (2011, 2012) introduced and studied strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers and lacunary sequence spaces of interval numbers, respectively. For more information about interval numbers one may refer to Sengonul and Eryilmaz (2010), Dwyer (1951, 1953), Fischer (1958), Moore (1959), Moore and Yang (1958, 1962), Markov (2005) and may refer to Esi (2011), Tripathy and Borgogain (2011), Tripathy and Dutta (2010a and b), Tripathy and Sarma (2011), Tripathy and Das (2011), Fast (1952), Krasnoselskii and Rutitsky (1961) and Chiao (2002).

The idea of statistical convergence for single sequences was introduced by Fast (1952). Schoenberg (1959) studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable.

A set consisting of a closed interval of real numbers x such that $a \le x \le b$ is called an interval number. A real interval can also be considered as a set. We denote the set of all real valued closed intervals by $I\mathbb{R}$ Any elements of $I\mathbb{R}$ is a closed interval and denoted by $\overline{x}$. That is $\overline{x}=\{x\in\mathbb{R} : a\le x\le b\}$. Hence an interval number $\overline{x}$ is a closed subset of real numbers (CHIAO, 2002). Let x_l and x_r be first and last points of $\overline{x}$ interval number, respectively. For $\overline{x}_1,\overline{x}_2\in I\mathbb{R}$, we have,

$$\overline{x}_1=\overline{x}_2 \Leftrightarrow x_{1_l}=x_{2_l},\ x_{1_r}=x_{2_r}.$$

$$\overline{x}_1+\overline{x}_2=\left\{x\in\mathbb{R} : x_{1_l}+x_{2_l}\le x\le x_{1_r}+x_{2_r}\right\},$$

and if $\alpha\ge 0$, then $\alpha\overline{x}=\left\{x\in\mathbb{R} :\ \alpha x_{1_l}\le x\le \alpha x_{1_r}\right\}$ and if $\alpha<0$, then $\alpha\overline{x}=\left\{x\in\mathbb{R} :\ \alpha x_{1_r}\le x\le \alpha x_{1_l}\right\}$,

$$\overline{x_1 x_2}=\left\{\begin{matrix} x\in\mathbb{R} : \min\left\{x_{1_l}x_{2_l}, x_{1_l}x_{2_r}, x_{1_r}x_{2_l}, x_{1_r}x_{2_r}\right\}\le x \\ \le \max\left\{x_{1_l}x_{2_l}, x_{1_l}x_{2_r}, x_{1_r}x_{2_l}, x_{1_r}x_{2_r}\right\}\end{matrix}\right\},$$

$$\frac{\overline{x_1}}{\overline{x_2}} = \left\{ \begin{array}{l} x \in \mathbb{R} : \min\left\{ x_{1_l}/x_{2_l}, x_{1_l}/x_{2_r}, x_{1_r}/x_{2_l}, x_{1_r}/x_{2_r} \right\} \le x \\ \le \max\left\{ x_{1_l}/x_{2_l}, x_{1_l}/x_{2_r}, x_{1_r}/x_{2_l}, x_{1_r}/x_{2_r} \right\} \end{array} \right\},$$

if $0 \notin \overline{x_2}$.

The set of all interval numbers $I\mathbb{R}$ is a complete metric space defined by $d(\overline{x}_1, \overline{x}_2) = \max\left\{ \left| x_{1_l} - x_{2_l} \right|, \left| x_{1_r} - x_{2_r} \right| \right\}$ (CHIAO, 2002)

In the special case $\overline{x}_1 = [a,a]$ and $\overline{x}_2 = [b,b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f : \mathbb{N} \to \mathbb{R}$ by $k \to f(k) = \overline{x}$, $\overline{x} = (\overline{x}_k)$. Then $\overline{x} = (\overline{x}_k)$ is called sequence of interval numbers. The $\overline{x}_k$ is called k^{th} term of sequence $\overline{x} = (\overline{x}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in Chiao (2002).

Now we give the definition of convergence of interval numbers:

Definition 1.1. Chiao (2002) a sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be convergent to the interval number $\overline{x}_o$ if for each $\varepsilon > 0$ there exists a positive integer k_o such that $d(\overline{x}_k, \overline{x}_o) < \varepsilon$ for all $k \ge k_o$ and we denote it by $\lim_k \overline{x}_k = \overline{x}_o$.

Thus, $\lim_k \overline{x}_k = \overline{x}_o \Leftrightarrow \lim_k x_{k_l} = x_{o_l}$ and $\lim_k x_{k_r} = x_{o_r}$.

Main results

In this paper, we introduce and study the concepts of asymptotically $\overline{s}_\lambda -$ statistical equivalent and strongly $\lambda -$ asymptotically equivalent sequences for interval numbers.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers such that $\lambda_{n+1} \le \lambda_n + 1$, for all $n \in \mathbb{N}$. $\lambda_1 = 1, \lambda_n \to \infty$ as $n \to \infty$ and $I_n = [n - \lambda_n + 1, n]$ and let Λ denote the set of all non-decreasing sequences $\lambda = (\lambda_n)$.

Definition 2.1. Two sequences $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ of interval numbers with $0 \notin \overline{y}_k$ for all $k \in \mathbb{N}$ are said to be asymptotically equivalent if

$$\lim_k d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{1} \right) = 0, \ \left(\text{denoted by } \overline{x}_k \sim \overline{y}_k \right).$$

In Esi (2011), introduced the concept of statistical $\lambda -$convergence of interval numbers as follows:

Definition 2.2. A sequence $\overline{x} = (\overline{x}_k)$ of interval numbers is said to be $\lambda -$ statistically convergent to interval number $\overline{x}_o$ if for every $\varepsilon > 0$

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \ d(\overline{x}_k, \overline{x}_o) \ge \varepsilon \right\} \right| = 0.$$

In this case we write $\overline{s}_\lambda - \lim \overline{x}_k = \overline{x}_o$. If $\lambda_n = n$, then $\lambda -$ statistically convergence reduces to statistically convergence as follows:

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \ d(\overline{x}_k, \overline{x}_o) \ge \varepsilon \right\} \right| = 0.$$

In this case we write $\overline{s} - \lim \overline{x}_k = \overline{x}_o$.

Following this result we introduce two new notions, namely asymptotically $\lambda -$statistical equivalent of multiple $\overline{L}$ and strong asymptotically $\lambda -$ equivalent of multiple $\overline{L}$.

The next definition is natural combination of Definition 2.1 and 2.2.

Definition 2.3. Two sequences $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ of interval numbers with $0 \notin \overline{y}_k$ for all $k \in \mathbb{N}$ are said to be asymptotically $\overline{s}_\lambda -$statistical equivalent of multiple $\overline{L}$ provided that for every $\varepsilon > 0$.

$$\lim_n \frac{1}{\lambda_n} \left| \left\{ k \in I_n : \ d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \ge \varepsilon \right\} \right| = 0, \left(\text{denoted by } \overline{x}_k \overset{\overline{s}_\lambda^{\overline{L}}}{\sim} \overline{y}_k \right)$$

and simply asymptotically $\overline{s}_\lambda -$statistical equivalent if $\overline{L} = \overline{1}$.

If we take $\lambda_n = n$, the above definition reduces to following definition:

Definition 2.4. Two sequences $\overline{x} = (\overline{x}_k)$ and $\overline{y} = (\overline{y}_k)$ of interval numbers with $0 \notin \overline{y}_k$ for all $k \in \mathbb{N}$ are said to be asymptotically statistical equivalent of multiple $\overline{L}$ provided that for every $\varepsilon > 0$

$$\lim_n \frac{1}{n} \left| \left\{ k \le n : \ d\left(\frac{\overline{x}_k}{\overline{y}_k}, \overline{L} \right) \ge \varepsilon \right\} \right| = 0, \left(\text{denoted by } \overline{x}_k \overset{\overline{s}^{\overline{L}}}{\sim} \overline{y}_k \right)$$

and simply asymptotically $\overline{s} -$statistical equivalent if $\overline{L} = \overline{1}$.

Definition 2.5. Two sequences $\overline{x}=(\overline{x}_k)$ and $\overline{y}=(\overline{y}_k)$ of interval numbers with $0 \notin \overline{y}_k$ for all $k \in \mathbb{N}$ are said to be strongly asymptotically $\lambda-$ equivalent of multiple $\overline{L}$ provided that

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)=0, \quad \left(\text{denoted by } \overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k\right)$$

and simply strongly $\lambda-$asymptotically equivalent if $\overline{L}=\overline{1}$.

If we take $\lambda_n = n,$ the above definition reduces to following definition:

Definition 2.6. Two sequences $\overline{x}=(\overline{x}_k)$ and $\overline{y}=(\overline{y}_k)$ of interval numbers with $0 \notin \overline{y}_k$ for all $k \in \mathbb{N}$ are said to be strongly Cesaro asymptotically equivalent of multiple $\overline{L}$ provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)=0, \quad \left(\text{denoted by } \overline{x}_k \overset{C_\lambda^{\overline{L}}}{\sim} \overline{y}_k\right)$$

and simply strongly Cesaro asymptotically equivalent if $\overline{L}=\overline{1}$.

Theorem 2.1.
Let $\lambda \in \Lambda$.

(i) If $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$, then $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$.

(ii) If $\overline{x}=(\overline{x}_k)$ and $\overline{y}=(\overline{y}_k)$ are in $\overline{\ell}_\infty$ and $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$, then $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$ and hence $\overline{x}_k \overset{C_\lambda^{\overline{L}}}{\sim} \overline{y}_k$,

(iii) If $\overline{x},\overline{y} \in \overline{\ell}_\infty$, then $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$, if and only if $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$

where

$$\overline{\ell}_\infty=\left\{\overline{x}=(\overline{x}_k):\ \sup_k\left\{\left|x_{k_l}\right|,\left|x_{k_r}\right|\right\}<\infty\right\}.$$

Proof. (i) If $\varepsilon>0$ and $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$ then

$$\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right) \ge \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon}} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)$$

$$\ge \frac{\varepsilon}{\lambda_n}\left|\left\{k\in I_n : \mathrm{d}\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\right|.$$

Therefore $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$.

(ii) Suppose that $\overline{x}=(\overline{x}_k)$ and $\overline{y}=(\overline{y}_k)$ are in $\overline{\ell}_\infty$ and $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$. Then we can assume that $d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\le A$ for all $k\in\mathbb{N}$.

Given $\varepsilon>0$

$$\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right) = \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon}} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right) + \frac{1}{\lambda_n}\sum_{\substack{k\in I_n\\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)<\varepsilon}} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)$$

$$\le \frac{A}{\lambda_n}\left|\left\{k\in I_n :\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\right|+\varepsilon.$$

Therefore $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$. Further, we have

$$\frac{1}{n}\sum_{k=1}^{n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)=\frac{1}{n}\sum_{k=1}^{n-\lambda_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)+\frac{1}{n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)$$

$$\le \frac{1}{\lambda_n}\sum_{k=1}^{n-\lambda_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)+\frac{1}{\lambda_n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)$$

$$\le \frac{2}{\lambda_n}\sum_{k\in I_n} d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right).$$

Hence $\overline{x}_k \overset{C_\lambda^{\overline{L}}}{\sim} \overline{y}_k$, since $\overline{x}_k \overset{V_\lambda^{\overline{L}}}{\sim} \overline{y}_k$.

(iii) Follows from (ii) and (iii).

Theorem 2.2.

$\overline{x}_k \overset{S^{\overline{L}}}{\sim} \overline{y}_k$ implies $\overline{x}_k \overset{S_\lambda^{\overline{L}}}{\sim} \overline{y}_k$ if $\liminf_n \frac{\lambda_n}{n}>0$.

Proof. For given $\varepsilon>0$, we have

$$\left\{k\le n:\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\supset\left\{k\in I_n:\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}.$$

Therefore

$$\frac{1}{n}\left|\left\{k\le n:\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\right|\ge\frac{1}{n}\left|\left\{k\in I_n : d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\right|$$

$$\ge\frac{\lambda_n}{n}.\frac{1}{\lambda_n}\left|\left\{k\in I_n :\ d\left(\frac{\overline{x}_k}{\overline{y}_k},\overline{L}\right)\ge\varepsilon\right\}\right|.$$

Taking limit as $n\to\infty$ and using $\liminf_n \frac{\lambda_n}{n}>0$, we get desired result.

Theorem 2.3. $\bar{x}_k \overset{\bar{s}_\lambda^{\bar{L}}}{\sim} \bar{y}_k$ implies $\bar{x}_k \overset{\bar{s}^{\bar{L}}}{\sim} \bar{y}_k$ if $\limsup_n \frac{\lambda_n}{\lambda_{n-1}} < \infty.$

Proof. Let $\limsup_n \frac{\lambda_n}{\lambda_{n-1}} < \infty,$ then there exists a $n_0 > 0$ such that $\frac{\lambda_n}{\lambda_{n-1}} < n_0$, for every $n \in \mathbb{N}$. Let $\delta > 0$. From $\bar{x}_k \overset{\bar{s}_\lambda^{\bar{L}}}{\sim} \bar{y}_k$, we can find constants M and K such that

$$A_n = \frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(\frac{\bar{x}_k}{\bar{y}_k}, \bar{L}\right) \geq \epsilon\right\}\right| < \delta$$

and $$\frac{1}{\lambda_n}\left|\left\{k \in I_n : d\left(\frac{\bar{x}_k}{\bar{y}_k}, \bar{L}\right) \geq \epsilon\right\}\right| < K$$

for all $n \in \mathbb{N}$. Now let t be any integer number such that $\lambda_{r-1} < t \leq \lambda_r$, where r>M. Then we have:

$$\frac{1}{t}\left|\left\{t \leq n : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| \leq \frac{1}{\lambda_{r-1}}\left|\left\{t \leq \lambda_r : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| =$$

$$\frac{1}{\lambda_{r-1}}\left|\left\{t \in I_1 : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| + \frac{1}{\lambda_{r-1}}\left|\left\{t \in I_2 : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| +$$

$$\cdots + \frac{1}{\lambda_{r-1}}\left|\left\{t \in I_M : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| + \cdots + \frac{1}{\lambda_{r-1}}\left|\left\{t \in I_r : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| =$$

$$\frac{\lambda_1}{\lambda_{r-1}\cdot\lambda_1}\left|\left\{t \in I_1 : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| + \frac{\lambda_2}{\lambda_{r-1}\cdot\lambda_2}\left|\left\{t \in I_2 : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| +$$

$$\cdots + \frac{\lambda_M}{\lambda_{r-1}\cdot\lambda_M}\left|\left\{t \in I_M : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| + \cdots + \frac{\lambda_r}{\lambda_{r-1}\cdot\lambda_r}\left|\left\{t \in I_r : d\left(\frac{\bar{x}_t}{\bar{y}_t}, \bar{L}\right) \geq \epsilon\right\}\right| =$$

$$\frac{\lambda_1}{\lambda_{r-1}}A_1 + \frac{\lambda_2}{\lambda_{r-1}}A_2 + \cdots + \frac{\lambda_M}{\lambda_{r-1}}A_M + \cdots + \frac{\lambda_r}{\lambda_{r-1}}A_r \leq \delta.$$

Finally we conclude this section by stating a definition which generalizes Definition 2.5. of this section and Theorem 2.2 related to this definition.

Definition 2.7. Let $p \in (0, \infty)$. Two sequences $\bar{x} = (\bar{x}_k)$ and $\bar{y} = (\bar{y}_k)$ of interval numbers with $0 \notin \bar{y}_k$ for all $k \in \mathbb{N}$ are said to be strong asymptotically λ_p- equivalent of multiple $\bar{L}$ provided that

$$\lim_n \frac{1}{\lambda_n}\sum_{k\in I_n}\left[d\left(\frac{\bar{x}_k}{\bar{y}_k}, \bar{L}\right)\right]^p = 0, \left(\text{denoted by } \bar{x}_k \overset{\bar{V}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k\right)$$

and simply strong asymptotically λ_p- equivalent if $\bar{L} = \bar{1}$. If $\lambda_n = n$, then strong asymptotically λ_p- equivalence reduces to strong Cesaro asymptotically λ_p- equivalence as follows:

$$\lim_n \frac{1}{n}\sum_{k=1}^{n}\left[d\left(\frac{\bar{x}_k}{\bar{y}_k}, \bar{L}\right)\right]^p = 0, \left(\text{denoted by } \bar{x}_k \overset{\bar{C}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k\right).$$

The following theorem is similar to that of Theorem 2.1., so the proof omitted.

Theorem 2.4. Let $\lambda \in \Lambda$.

(i) If $\bar{x}_k \overset{\bar{V}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k$, then $\bar{x}_k \overset{\bar{s}_\lambda^{\bar{L}}}{\sim} \bar{y}_k$.

(ii) If $\bar{x} = (\bar{x}_k)$ and $\bar{y} = (\bar{y}_k)$ are in $\overline{\ell_\infty}$ and $\bar{x}_k \overset{\bar{s}_\lambda^{\bar{L}}}{\sim} \bar{y}_k$, then $\bar{x}_k \overset{\bar{V}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k$ and hence $\bar{x}_k \overset{\bar{C}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k$,

(iii) If $\bar{x}, \bar{y} \in \overline{\ell_\infty}$, then $\bar{x}_k \overset{\bar{s}_\lambda^{\bar{L}}}{\sim} \bar{y}_k$ if and only if $\bar{x}_k \overset{\bar{V}_{\lambda_p}^{\bar{L}}}{\sim} \bar{y}_k$.

Asympotically $\lambda-$ statically equivalent with respect to Orlicz function

In this section we will introduce Cesaro asymptotically $\lambda-$statistical equivalence with respect to an Orlicz function. An Orlicz function is a function $M : (0, \infty] \to (0, \infty]$ which is continuous, nondecreasing and convex with $M(0) = 0$, $M(x) > 0$ for x > 0 and $M(x) \to \infty$ as $x \to \infty$. It is well known that if M is a convex function and $M(0) = 0$ then $M(\lambda x) \leq \lambda \cdot M(x)$ for all λ with $0 < \lambda \leq 1$.

An Orlicz function M is said to satisfy the Δ_2-condition for all values u, if there exists a constant L > 0 such that $M(2u) \leq LM(u), u \geq 0$ (KRASNOSELSKII; RUTITSKY, 1961). We will define the following asymptotically $\lambda-$ statistical equivalences:

Definition 3.1. Two sequences $(\bar{x}_k)$ and $(\bar{y}_k)$ with $0 \notin \bar{y}_k$ for all $k \in \mathbb{N}$ are Cesaro asymptotically

$\lambda-$ equivalent of multiple $\bar{L}$ with respect to an Orlicz function provided that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} M\left(d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\right)=0,$$

(denoted by $\bar{x}_k \overset{\bar{\sigma}_1^{\bar{L}}(M)}{\sim} \bar{y}_k$), and simply Cesaro asymptotically λ equivalent with respect to an Orlicz function if $\bar{L}=\bar{1}$.

Definition 3.2. Two sequences $(\bar{x}_k)$ and $(\bar{y}_k)$ with $0\notin\bar{y}_k$ for all $k\in\mathbb{N}$ are Cesaro strong asymptotically $\lambda-$ equivalent of multiple $\bar{L}$ with respect to an Orlicz function provided that

$$\lim_r \frac{1}{\lambda_r}\sum_{k\in I_r} M\left(d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\right)=0,$$

(denoted by $\bar{x}_k \overset{\bar{\sigma}_1 S^{\bar{L}}(M)}{\sim} \bar{y}_k$), and simply Cesaro strong asymptotically $\lambda-$ equivalent with respect to an Orlicz function if $\bar{L}=\bar{1}$.

Theorem 3.1. Let M be an Orlicz function which satisfies the Δ_2-conditions. Two sequences $(\bar{x}_k)$ and $(\bar{y}_k)$ are said to be $\bar{x}_k \overset{\bar{S}^{\bar{L}}(M)}{\sim} \bar{y}_k$ of multiple $\bar{L}$ with respect to Orlicz function, provided that for every $\epsilon>0$,

$$\lim_r \frac{1}{\lambda_r}\left\{\text{the number of } \quad i\in I_r : M\left(d\left(\frac{\bar{x}_i}{\bar{y}_i},\bar{L}\right)\right)\geq\epsilon\right\}=0.$$

Then $\bar{x}_k \overset{\bar{s}^{\bar{L}}}{\sim} \bar{y}_k$ implies $\bar{x}_k \overset{\bar{S}^{\bar{L}}(M)}{\sim} \bar{y}_k$.

Proof. From fact that M satisfies Δ_2-conditions it follows that:

$$M\left(d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\right)\leq K\cdot d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right),$$

for some constant $K>0$, in both cases where $d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\leq 1$ and $d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\geq 1$. Really in first case it follows directly from definition of the Orlicz function. In second case we have:

2.1 $$d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)=2\cdot R^{(1)}=2^2\cdot R^{(2)}=\cdots=2^s\cdot R^{(s)},$$

such that $R^{(s)}\leq 1$. Now taking into consideration Δ_2 conditions of Orlicz functions, we get the following estimation:

2.2 $$M\left(d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right)\right)\leq T\cdot R^{(s)}\cdot M(1)=K\cdot d\left(\frac{\bar{x}_k}{\bar{y}_k},\bar{L}\right),$$

where T and K are constants. Now proof of Theorem follows from relations (2.1) and (2.2).

Proposition 3.2. Let M be an Orlicz function and $\sup_n \frac{\lambda_n}{\lambda_{n-1}}<\infty$. Then for any two sequences $(\bar{x}_k)$ and $(\bar{y}_k)$ we have: $\bar{x}_k \overset{\bar{S}^{\bar{L}}(M)}{\sim} \bar{y}_k$ implies $\bar{x}_k \overset{\bar{s}^{\bar{L}}}{\sim} \bar{y}_k$.

Proof. Proof of the proposition is similar to that given in Theorem 2.3, using into consideration that M is non-decreasing function, so it is omitted.

Conclusion

In this paper are given and studied the concepts of asymptotically $\bar{s}_\lambda$–statistical equivalent and strongly asymptotically $\lambda-$ equivalent sequences for interval numbers and are proved some inclusion relations. This results are extension of the known results from asymptotically equivalent real numerical sequences. Also are given the concept of Cesaro asymptotically $\lambda-$ equivalent sequences defined by Orlicz function of multiple $\bar{L}$ and we have established some relation between those classes.

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