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Statistical and lacunary statistical convergence of interval numbers in topological groups

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ABSTRACT. In this paper we introduce the concepts of statistical and lacunary statistical convergence of interval numbers in topological groups. We prove some inclusion relations and study some of their properties.

Keywords: lacunary sequence, statistical convergence, interval numbers, topological groups.

Convergência estatística e estatístico-lacunar de números de intervalos em grupos topológicos

RESUMO. Nesse artigo apresentamos os conceitos de convergência estatística e estatístico-lacunar de números de intervalos em grupos topológicos. Provamos relações de inclusão a estudamos algumas de suas propriedades.

Palavras-chave: sequência lacunar, convergência estatística, números de intervalos, grupos topológicos.

Introduction

Interval arithmetic was first suggested by Dwyer (1951). Development of interval arithmetic as a formal system and evidence of its value as a computational device was provided by Moore (1959) and Moore and Yang (1962). Furthermore, Moore and others Dwyer (1951), Dwyer (1953), and Moore and Yang (1958) have developed applications to differential equations.

Chiao (2002) introduced sequence of interval numbers and defined usual convergence of sequences of interval number. Şengönül and Eryilmaz (2010) introduced and studied bounded and convergent sequence spaces of interval numbers and showed that these spaces are complete metric space. Recently Esi (2011) introduced and studied strongly almost $\lambda-$convergence and statistically almost $\lambda-$convergence of interval numbers.

The idea of statistical convergence for single sequences was introduced by Fast (1951). Schoenberg (1959) studied statistical convergence as a summability method and listed some of elementary properties of statistical convergence. Both of these authors noted that if bounded sequence is statistically convergent, then it is Cesaro summable. Existing work on statistical convergence appears to have been restricted to real or complex sequence, but several authors extended the idea to apply to sequences of fuzzy numbers and also introduced and discussed the concept of statistically sequences of fuzzy numbers.

Preliminaries

Let $p=(p_k)$ be a positive sequence of real numbers. If $0<h=\inf_k p_k \le p_k \le H=\sup_k p_k<\infty$ and $D=max(1,2^{H-1})$, then for $a_k,b_k\in\mathbb{C}$ for all $k\in\mathbb{N}$ we have $|a_k+b_k|^{p_k}\le D(|a_k|^{p_k}+|b_k|^{p_k})$.

By a lacunary sequence $\theta=(k_r)$, $r=0,1,2,\ldots$, where:

$k_o=0$, we shall mean increasing sequence of non-negative integers $h_r=k_r-k_{r-1}\to\infty$ as $r\to\infty$.

The intervals determined by θ are denoted by $I_r=(k_{r-1},k_r]$ and the ratio $\frac{k_r}{k_{r-1}}$ will be denoted by q_r. The space of lacunary strongly convergent sequence N_θ was defined by Freedman et al. (1978) as follows:

$$N_\theta=\left\{x=(x_i):\lim_{r\to\infty}\frac{1}{h_r}\sum_{i\in I_r}|x_i-L|=0 \text{ for some } L\right\}.$$

A set consisting of a closed interval of real numbers such that $a\le x\le b$ is called an interval number. A real interval can also be considered as a set. Thus we can investigate some properties of interval numbers, for instance arithmetic properties or analysis properties.

We denote the set of all real valued closed intervals by $I\mathbb{R}$. Any elements of $I\mathbb{R}$. is called closed interval and denoted by $\overline{A}$. That is $\overline{A}=\{x\in R: a\leq x\leq b\}$. An interval number $\overline{A}$ is a closed subset of real numbers (CHIAO, 2002). Let a_l and a_r be first and last points of $\overline{A}$ interval number, respectively. For $\overline{A},\overline{B}\in I\mathbb{R}$, we have $\overline{A}=\overline{B}\Leftrightarrow a_{1_l}=a_{2_l}, a_{1_r}=a_{2_r}$. $\overline{A}+\overline{B}=\{x\in\mathbb{R}: a_{1_l}+a_{2_l}\leq x\leq a_{1_r}+a_{2_r}\}$, and if $\alpha\geq 0$, then $\alpha\overline{A}=\{x\in R:\alpha a_{1_l}\leq x\leq \alpha a_{1_r}\}$ and if $\alpha<0$, then $\alpha\overline{A}=\{x\in R:\alpha a_{1_r}\leq x\leq \alpha a_{1_l}\}$, $\overline{A}.\overline{B}=\{x\in\mathbb{R}:\min\{a_{1_l}.a_{2_l}, a_{1_l}.a_{2_r}, a_{1_r}.a_{2_l}, a_{1_r}.a_{2_r}\}\leq x\leq \max\{a_{1_l}.a_{2_l}, a_{1_l}.a_{2_r}, a_{1_r}.a_{2_l}, a_{1_r}.a_{2_r}\}\}$. The set of all interval numbers $I\mathbb{R}$. is a complete metric space defined by $d(\overline{A},\overline{B})=\max\{|a_{1_l}-a_{2_l}|, |a_{1_r}-a_{2_r}|\}$ (CHIAO, 2002).

In the special case $\overline{A}=[a,a]$ and $\overline{B}=[b,b]$, we obtain usual metric of $\mathbb{R}$.

Let us define transformation $f:N\to R$ by $k\to f(k)=\overline{A}$, $\overline{A}=(\overline{A}_k)$. Then $\overline{A}=(\overline{A}_k)$ is called sequence of interval numbers. The $\overline{A}_k$ is called k^{th} term of sequence $\overline{A}=(\overline{A}_k)$. w^i denotes the set of all interval numbers with real terms and the algebraic properties of w^i can be found in (ŞENGÖNÜL; ERYILMAZ, 2010).

Now we give the definition of convergence of interval numbers:

A sequence $\overline{A}=(\overline{A}_k)$ of interval numbers is said to be convergent to the interval number $\overline{A}_o$ if for each $\varepsilon>0$ there exists a positive integer k_o such that $d(\overline{A}_k,\overline{A}_o)<\varepsilon$ for all $k\geq k_o$ and we denote it by $\lim_k\overline{A}_k=\overline{A}_o$, (CHIAO, 2002).

Thus, $\lim_k\overline{A}_k=\overline{A}_o\Leftrightarrow \lim_k a_{k_l}=a_{o_l}$ and $\lim_k a_{k_r}=a_{o_r}$.

By X, we will denote an abelian topological Hausdorff group, written additively which satisfies the first axiom of countability. In Cakalli (1995), a single sequence $x=(x_k)$ in X is said to be statistically convergent to an element $L\in X$ if for each neighborhood U of 0,

$$\lim_n\frac{1}{n}\left|\{k\leq n: x_k-L\notin U\}\right|=0,$$

where the vertical bars indicate the number of elements in the enclosed set. In this case we write $st(X)-\lim_k x_k=L$ or $x_k\to L(st(X))$.

In Cakalli (1996), for single sequence $x=(x_k)$ in X the concept of lacunary statistical convergence was defined by Cakallı (1996) as follows: Let $\theta=(k_r)$ be a lacunary sequence; the single sequence $x=(x_k)$ in X is said to be $st^\theta-convergent$ to L (or lacunary statistically convergent to L in X) if for each neighborhood U of 0,

$$\lim_r\frac{1}{h_r}\left|\{k\in I_r: x_k-L\notin U\}\right|=0.$$

In this case we write $st^\theta(X)-\lim_k x_k=L$ or $x_k\to L(st^\theta(X))$.

In this paper, we introduce and study the concepts of statistical convergence, strongly convergence, lacunary statistical convergence and lacunary strongly convergence for interval numbers in topological groups as follows.

Main results

In this section we give some definition and prove the results of this paper.

Definition 3.1 - An interval numbers sequence $\overline{A}=(\overline{A}_k)$ is said to be statistically convergent to an element of $\overline{A}_o$ of X if for each neighborhood system U of $\overline{0}=[0,0]$

$$\lim_n\frac{1}{n}\left|\{k\leq n:\overline{x}_k-\overline{x}_o\notin U\}\right|=0,$$

in this case we write $\overline{A}_k\to\overline{A}_o(\overline{s}(X))$ or $\overline{s}(X)-\lim\overline{A}_k=\overline{A}_o$. The set of all statistically convergent sequences of interval number sequences is denoted by $\overline{s}(X)$.

Definition 3.2 - An interval numbers sequence $\overline{A}=(\overline{A}_k)$ is called strongly convergent in X if for each neighborhood system U of $\bar{0}=[0,0]$ such that

$$\lim_n \frac{1}{n}\sum_{k=1}^{n} d\left(\bar{A}_k,\bar{A}_o\right)=0$$

In this case we write $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$ or $\bar{N}(X)-lim\bar{A}_k=\bar{A}_o$.

Definition 3.3 - Let $\theta=(k_r)$ be a lacunary sequence. A sequence $\bar{A}=\left(\bar{A}_k\right)$ of interval numbers is said to be lacunary statistically convergent to interval number $\bar{A}_o$ in X, if for each neighborhood system of $\bar{0}=[0,0]$

$$\lim_r \frac{1}{h_r}\left|\left\{k\in I_r:\bar{A}_k-\bar{A}_o\notin U\right\}\right|=0.$$

In this case we write $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$ or $\bar{s}_\theta(X)-\lim\bar{A}_k=\bar{x}_o$. The set of all lacunary statistically convergent sequences of interval number sequences is denoted by $\bar{s}_\theta(X)$. In the special case $\theta=(2^r)$, we shall write $\bar{s}(X)$ instead of $\bar{s}_\theta(X)$.

Definition 3.4. - An interval numbers sequence $\bar{x}=\left(\bar{x}_k\right)$ is called strongly lacunary convergent in X if for each neighborhood system U of $\bar{0}=[0,0]$ such that

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r,\bar{x}_k-\bar{x}_o\notin U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}=0$$

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r,\bar{x}_k-\bar{x}_o\notin U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}=0$$

in this case we write $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$ or $\bar{N}_\theta^p(X)-lim\bar{x}_k=\bar{x}_o$. In the special case $\theta=(2^r)$, we shall write $\bar{N}^p(X)$ instead of $\bar{N}_\theta^p(X)$.

Theorem 3.1 - Let $\theta=(k_r)$ be a lacunary sequence and $\bar{A}=\left(\bar{A}_k\right)$ be a sequence of interval numbers. Then

(i) $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$ implies $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$,

(ii) $\bar{A}=\left(\bar{A}_k\right)\in\bar{m}(X)$ and $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$ imply $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$,

(iii) If $\bar{A}=\left(\bar{A}_k\right)\in\bar{m}(X)$, then $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$ and $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$,

where:

$$\bar{m}(X)=\left\{\bar{A}=\left(\bar{A}_k\right):\sup_k d\left(\bar{A}_k,\bar{A}_o\right)<\infty\right\}.$$

Proof. (i) Let $\varepsilon>0$ and $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$. Then we write

$$\left|\left\{k\in I_r:\bar{A}_k-\bar{A}_o\notin U\right\}\right|\le\sum_{k\in I_r,\bar{x}_k-\bar{x}_o\notin U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}$$

and

$$\lim_r \frac{1}{h_r}\sum_{k\in I_r,\bar{A}_k-\bar{A}_o\notin U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}=0.$$

This implies that

$$\lim_r \frac{1}{h_r}\left|k\in I_r:\bar{A}_k-\bar{A}_o\notin U\right|=0.$$

Hence $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$.

(ii) Suppose that $\bar{A}=\left(\bar{A}_k\right)\in\bar{m}(X)$ and $\bar{A}_k \to \bar{A}_o\left(\bar{s}_\theta(X)\right)$. Since $\bar{A}=\left(\bar{A}_k\right)\in\bar{m}(X)$, there is a constant $C>0$ such that $d\left(\bar{A}_k,\bar{A}_o\right)\le C$. Given $\varepsilon>0$, we have

$$\frac{1}{h_r}\sum_{k\in I_r}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}$$

$$\frac{1}{h_r}\sum_{k\in I_r,\bar{A}_k-\bar{A}_o\notin U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}+\frac{1}{h_r}\sum_{k\in I_r,\bar{A}_k-\bar{A}_o\in U}\left[d\left(\bar{A}_k,\bar{A}_o\right)\right]^{p_k}$$

$$\le\frac{1}{h_r}\sum_{k\in I_r,\bar{A}_k-\bar{A}_o\notin U}\max\left(C^h,C^H\right)+\frac{1}{h_r}\sum_{k\in I_r,\bar{A}_k-\bar{A}_o\in U}\varepsilon^{p_k}$$

$$\le\max\left(C^h,C^H\right)\frac{1}{h_r}\left|\left\{k\in I_r:\bar{A}_k-\bar{A}_o\notin U\right\}\right|+\max\left(\varepsilon^h,\varepsilon^H\right).$$

Thus we obtain $\bar{A}_k \to \bar{A}_o\left(\bar{N}_\theta^p(X)\right)$.

(iii) It follows from (i) and (ii).

Theorem 3.2. - Let $\theta=(k_r)$ be a lacunary sequence and $\overline{A}=(\overline{A_k})$ be a sequence of interval numbers. Then

(i) For $\liminf_r q_r>1$, then $\overline{A_k}\to\overline{A_o}(\overline{s}(X))$ implies $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$,

(ii) For $\limsup_r q_r<\infty$, then $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$ implies $\overline{A_k}\to\overline{A_o}(\overline{s}(X))$,

(iii) If $1<\liminf_r q_r\le\limsup_r q_r<\infty$, then $\overline{A_k}\to\overline{A_o}(\overline{s}(X))$ if and only if $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$.

Proof. (i) Suppose that $\liminf_r q_r>1$, then there exists a $\delta>0$ such that $q_r\ge 1+\delta$ for sufficiently large r, which implies

$$\frac{h_r}{k_r}\ge\frac{\delta}{1+\delta}$$

if $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$, then for sufficiently large r, we have

$$\frac{1}{k_r}\left|\left\{k\le k_r:\overline{A_k}-\overline{A_o}\notin U\right\}\right|\ge\frac{1}{k_r}\left|\left\{k\in I_r:\overline{A_k}-\overline{A_o}\notin U\right\}\right|$$

$$\ge\frac{\delta}{1+\delta}.\frac{1}{h_r}\left|\left\{k\in I_r:\overline{A_k}-\overline{A_o}\notin U\right\}\right|.$$

Hence $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$

(ii) If $\limsup_r q_r<\infty$, then there exists $C>0$ such that $q_r<C$ for all $r\ge 1$. Let $\overline{A_k}\to\overline{A_o}(\overline{s_\theta}(X))$ and set $A_r=\left|\left\{k\in I_r:d(\overline{A_k},\overline{A_o})\ge\varepsilon\right\}\right|$. Then there exists an $r_o\in\mathbb{N}$ such that

$$\frac{A_r}{h_r}<\varepsilon \text{ for all } r>r_o. \tag{3.1}$$

Now let $N=max\{A_r:1\le r\le r_o\}$ and choose n such that $k_{r-1}<n\le k_r$. Then we have

$$\frac{1}{k_r}\left|\left\{k\le n:\overline{A_k}-\overline{A_o}\notin U\right\}\right|\le\frac{1}{k_{r-1}}\left|\left\{k\le k_r:\overline{A_k}-\overline{A_o}\notin U\right\}\right|$$

$$=\frac{1}{k_{r-1}}\left\{A_1+A_2+\ldots+A_{r_o}+A_{(r_o+1)}+\ldots+A_r\right\}$$

$$\le\frac{N}{k_{r-1}}r_o+\frac{1}{k_{r-1}}\left\{h_{r_o+1}\frac{A_{(r_o+1)}}{h_{r_o+1}}+\ldots+h_r\frac{A_r}{h_r}\right\}$$

$$\le\frac{N}{k_{r-1}}r_o+\frac{1}{k_{r-1}}\left(\sup_{r>r_o}\frac{A_r}{h_r}\right)\left\{h_{r_o+1}+\ldots+h_r\right\}$$

$$\le\frac{N}{k_{r-1}}r_o+\varepsilon\frac{k_r-k_{r_o}}{k_{r-1}},by\,(3.1)$$

$$\le\frac{N}{k_{r-1}}r_o+\varepsilon q_r\le\frac{N}{k_{r-1}}r_o+\varepsilon C.$$

Thus we obtain $\overline{A_k}\to\overline{A_o}(\overline{s}(X))$.

(iii) It follows from (i) and (ii).

Conclusion

The concept of interval arithmetic was first suggested by Dwyer (1951). After then Chiao (2002) introduced usual convergence of sequences of interval numbers. Recently, interval numbers sequences studied by several authors. The results obtained in this paper are much more general than those obtained earlier.

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