Warfield $p$-Invariants in Abelian Group Rings of Characteristic $p$

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ABSTRACT: We calculate Warfield $p$-invariants $W_{\alpha,p}(V(RG))$ of the group of normalized units $V(RG)$ in a commutative group ring $RG$ of prime $\text{char}(RG) = p$ in each of the following cases:

1. $G_0/G_p$ is finite and $R$ is an arbitrary direct product of indecomposable rings;
2. $G_0/G_p$ is bounded and $R$ is a finite direct product of fields;
3. $\text{id}(R)$ is finite (in particular, $R$ is finitely generated).

Moreover, we give a general strategy for the computation of the above Warfield $p$-invariants under some restrictions on $R$ and $G$. We also point out an essential incorrectness in a recent paper due to Mollov and Nachev in Commun. Algebra (2011).

Key Words: Abelian groups, commutative rings, indecomposable rings, units, Warfield $p$-invariants.

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1. Introduction

Everywhere in the text, let $R$ be a commutative unital ring of prime characteristic $p$ and $G$ an Abelian group written multiplicatively as is customary when discussing group rings. For such $R$ and $G$, suppose $RG$ is the group ring of $G$ over $R$ with unit group $U(RG)$ and its normalized component $V(RG)$; note that the decomposition $U(RG) = V(RG) \times U(R)$ holds, where $U(R)$ is the unit group (that is, the multiplicative group of units of $R$). As usual, $\text{id}(R) = \{ e \in R \mid e^2 = e \}$ is the set of all idempotents of $R$.

Imitating [11], for any multiplicative group $A$ we define the following ordinal-to-cardinal functions, called in the existing literature Warfield $p$-invariants

$$W_{\alpha,p}(A) = r(A^p\alpha/(A^{\alpha+1}p)),$$

where $\alpha \geq 0$ is an ordinal.

These invariants were the object of a series of explorations [1]-[6]. They were calculated for both $U(RG)$ and $V(RG)$ under some limitations on $R$ and $G$ only in their terms and divisions. The most important achievements are these:

(i) $G_0 = G_p$ (i.e., $G$ is $p$-mixed) and $R$ is arbitrary;
(ii) $G_0/G_p$ is bounded and $R$ is perfect;
(iii) $G_0/G_p$ is bounded and $R$ is a field;

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(iv) $G_0/G_p$ is finite and $R$ is indecomposable;
(v) $G$ is arbitrary and $R$ is perfect indecomposable.

Actually, the last result is proved in [1] for a perfect integral domain and in [2] for a perfect field, but according to the main theorem of [7] the same idea also works for an indecomposable ring.

Some other useful estimations of $W_{\alpha,p}(U(RG))$ and $W_{\alpha,p}(V(RG))$ are also obtained there.

Mollov and Nachev [10] have duplicated the results of ours from [1], [2], [3] and [4]. Even more, they have partly plagiarized results (i) and (v) as well as the ideas for their proofs without any concrete correct citation of the articles [2], [3] and [4].

Moreover, they wrongly cited in ([10], p.2300, the last sentence before Section 2) that [1] is the unique article of the current author which treated the problem for calculation of $W_{\alpha,p}(V(RG))$, but seeing the cited bibliography listed below this is apparently false.

The main purpose here is to add two more points to the list (i)-(v) given above, that are:

(vi) $G_0/G_p$ is finite and $R$ is an arbitrary direct product of indecomposable rings - thus extending (iv).

(vii) $G_0/G_p$ is bounded and $R$ is a finite direct product of fields - thus extending (iii).

We also give a general strategy for the computation of $W_{\alpha,p}(U(RG))$ over some special rings $R$.

2. Main Results

We first begin with a crucial technicality (see also [2]).

Lemma 2.1. Let $A = \prod_{i \in I} A_i$ be an abelian group. Then, for any ordinal $\alpha$,

$$W_{\alpha,p}(A) = \sum_{i \in I} W_{\alpha,p}(A_i).$$

Proof: Observe that for any ordinal $\beta$ we have $A^p^\beta = \prod_{i \in I} A_i^p^\beta$, and hence $A^p^\beta = \prod_{i \in I} (A_i^p^\beta)_p = \prod_{i \in I} (A_i^p)^\beta$. Therefore, $A^p^\alpha / (A^p^{\alpha+1} A_p^\alpha) = \prod_{i \in I} [A_i^p^\alpha / (A_i^p)^\alpha]$, whence by a simple appeal to the additive property of the rank of an abelian group we derive that $r(A^p^\alpha / (A^p^{\alpha+1} A_p^\alpha)) = \sum_{i \in I} r(A_i^p^\alpha / (A_i^p)^\alpha)$. The last is just equivalent to the desired equality. \qed

If $\{R_i\}_{i \in I}$ is a system of commutative unital rings for some finite or infinite index set $I$, then by $\prod_{i \in I} R_i$ we will denote the arbitrary direct product of rings in the following sense: Any element $r \in \prod_{i \in I} R_i$ is of the form of a vector (finite or infinite) $r = (\cdots, r_i, \cdots)$ equipped with the operations for an other element $f = (\cdots, f_i, \cdots)$ given by $r + f = (\cdots, r_i + f_i, \cdots)$ and $rf = (\cdots, r_if_i, \cdots)$. Clearly the zero element is $0 = (\cdots, 0_i, \cdots)$ where $0_i$ is the corresponding zero element.
in $R_i$, and the identity element is $1 = (\cdots, 1_i, \cdots)$ where $1_i$ is the corresponding identity element in $R_i$.

Under these circumstances, it is not difficult to check that $U(\prod_{i \in I} R_i) = \prod_{i \in I} U(R_i)$ which fact will be used in the sequel without a concrete referring.

Note that in some existing literature such a product is also called a coproduct of these rings $R_i$.

The next statement is well known but we will prove it for completeness and for the reader’s convenience.

**Proposition 2.2.** Let $A$ be a finite group and let $K = \times_{j \in J} K_j$ be a finite direct product of rings. Then the following isomorphisms hold:

$$(a) \ (\prod_{i \in I} R_i)A \cong \prod_{i \in I} (R_iA)$$

where $I$ is an arbitrary index set.

$$(b) \ KG \cong \times_{j \in J} (K_jG).$$

**Proof:** (a) For any $v = \sum_{a \in A_v} r_\alpha a$ where $r_\alpha = (\cdots, r_{\alpha a}, \cdots) \in \prod_{i \in I} R_i$ and $A_v$ is a finite subset of $A$ depending on the element $v$, define the map $\phi : (\prod_{i \in I} R_i)A \rightarrow \prod_{i \in I} (R_iA)$ via the equality $\phi(v) = (\cdots, \sum_{a \in A_v} r_{\alpha a} a, \cdots)$. Furthermore, it is only a routine technical exercise to verify that $\phi$ is an isomorphism of $R$-algebras, as required.

(b) Follows in the same manner. \hfill \square

**Remark 2.1.** We will further identify with no loss of generality $(\prod_{i \in I} R_i)A$ with $\prod_{i \in I} (R_iA)$, and $(\times_{j \in J} K_j)G$ with $\times_{j \in J} (K_jG)$, so that the two isomorphisms in points (a) and (b) will be formal equalities, indeed.

We are now ready to state and prove the following first main result.

**Theorem 2.3.** Suppose $G$ is a group whose factor $G_0/G_p$ is finite and $R = \prod_{i \in I} R_i$ where each $R_i$ is indecomposable for $i \in I$. Then the following formula is valid:

$$W_{\alpha,p}(U(RG)) = \mu \cdot W_{\alpha,p}(G) + \sum_{i \in I} \sum_{d \mid \exp (G_0/G_p)} (l_d/(R_i(\zeta_d) : R_i)) \cdot W_{\alpha,p}(U(R_i(\zeta_d))),$$

where $l_d = |\{a \in G_0/G_p : o(a) = d\}|$.

**Proof:** Since $\prod_{q \neq p} G_q$ is finite and pure in $G$, one may write $G = (\prod_{q \neq p} G_q) \times M$ for some $p$-mixed group $M$. Consequently, Proposition 2.2 (a) leads to $RG = [R(\prod_{q \neq p} G_q)]M = [(\prod_{i \in I} R_i) (\prod_{q \neq p} G_q)]M = (\prod_{i \in I} R_i (\prod_{q \neq p} G_q)) M$. Furthermore, as in [6], $W_{\alpha,p}(U(RG)) = \mu \cdot W_{\alpha,p}(G) + W_{\alpha,p}(U(\prod_{i \in I} R_i (\prod_{q \neq p} G_q)))$ where
Let $G$ be an abelian group for which $G_0/G_p$ is infinite bounded and $R = F_1 \times \cdots \times F_n$ where every $F_i$ is a field; $i \in [1, n]$, where $n$ is natural. Then

$$W_{\alpha, p}(U(RG)) = |\text{id}(R)| \cdot |\prod_{q \notin p} G_q| \cdot W_{\alpha, p}(G) + \sum_{i=1}^{n} \prod_{m=0}^{\infty} W_{\alpha, p}(F_i(\zeta_m))$$

with $a_i(m) = |\{g \in \prod_{q \notin p} G_q : \alpha(g) = d\}|/(F_i(\zeta_m) : F_i)$.

**Proof:** Since $RG = F_1 G \times \cdots \times F_n G$, we derive $U(RG) = U(F_1 G) \times \cdots \times U(F_n G)$. Therefore, using Lemma 2.1, we deduce that $W_{\alpha, p}(U(RG)) = \sum_{i=1}^{n} W_{\alpha, p}(U(F_i G))$. Utilizing ([6], Theorem 2.2 (1)), $W_{\alpha, p}(U(F_i G))$ are completely computed, so that the wanted equality follows. \[\square\]

**Remark 2.2.** When $G_0/G_p$ is finite bounded, things are settled in Theorem 2.3 listed above.

The next statement somewhat supersedes Theorem 3.9 from [10].

**Theorem 2.5.** Suppose $R$ is a perfect ring with a finite number of idempotents (in particular, $R$ is perfect finitely generated). Then the following formula holds:

$$W_{\alpha, p}(U(RG)) = \left( \sum_{k=1}^{n} \sum_{d/k} l(d)/\lambda(d) \right) \cdot W_{\alpha, p}(G),$$

provided $W_{\alpha, p}(G) \neq 0$ and $\prod_{q \notin p} G_q$ is finite of exponent $k$ where $l_d = |\{g \in \prod_{q \notin p} G_q : \alpha(g) = d\}|$, $\lambda(d)$ is the boundary defined as in ([10], (3.8)) and

$$W_{\alpha, p}(U(RG)) = \max(|\prod_{q \notin p} G_q|, W_{\alpha, p}(G)),$$

provided $W_{\alpha, p}(G) \neq 0$ and $\prod_{q \notin p} G_q$ is infinite,

or

$$W_{\alpha, p}(U(RG)) = 0,$$

provided $W_{\alpha, p}(G) = 0$.
Proof: Since \(id(R)\) is finite, \(R\) possesses \(2^n\) idempotents where \(n\) is the number of primitive idempotents of \(R\), say \(\{e_1, \cdots, e_n\}\) is such a system. Furthermore, owing to a folklore ring-theoretic fact, one may decompose \(R\) like this:

\[
R = (Re_1) \oplus \cdots \oplus (Re_n) = (Re_1) \times \cdots \times (Re_n)
\]

where each \(Re_i\) is an indecomposable subring of \(R\); \(i \in [1, n]\). Thus, in view of Proposition 2.2 (b), one can write that \(RG = (Re_1)G \times \cdots \times (Re_n)G\), whence \(U(RG) = U((Re_1)G) \times \cdots \times U((Re_n)G)\). Applying Lemma 2.1, \(W_{\alpha,p}(U(RG)) = W_{\alpha,p}(U((Re_1)G)) + \cdots + W_{\alpha,p}(U((Re_n)G))\). It is readily seen that every \(Re_i\) is a perfect ring of characteristic \(p\) as well; \(1 \leq i \leq n\). Moreover, ([2], Theorem 6 - see also [10], Theorem 3.9) applies to calculate all functions \(W_{\alpha,p}(U((Re_i)G))\) where \(i \in [1, n]\). Thus we obtain the explicit form of \(W_{\alpha,p}(U(RG))\) stated above. 

Remark 2.3. Unfortunately, there is no result of that type for infinite decompositions of \(R\). For example, take \(R = \prod_{n=1}^{\infty} F_n / \oplus_{n=1}^{\infty} F_n\) where all \(F_n\) are fields. Therefore, the set of idempotents in \(R\) is a quotient of boolean algebras: \(id(R) = B/J\) where \(B\) is the boolean algebra of subsets of the set \(\mathbb{N}\) of natural numbers and \(J\) is the ideal of finite subsets. Since \(|B| = 2^{\aleph_0}\) and \(|J| = \aleph_0\), we get that \(|id(R)| = 2^{\aleph_0}\). However, \(id(R)\) has no atoms (= primitive idempotents), so no ring direct summand of \(R\) is indecomposable.

One source of the problem is that cardinality information is much stronger in the finite case: in fact, any finite boolean algebra is generated by its atoms, so if \(|id(R)| = 2^n\), then \(id(R)\) is set-theoretically isomorphic to the boolean algebra of subsets of \(\{1, \cdots, n\}\) and thus \(id(R)\) always possesses primitive idempotents. Consequently, a more promising hypothesis would be to assume that \(id(R)\) is isomorphic to the boolean algebra \(2^n\) of subsets of an infinite set \(I\). Nevertheless, it looks like even this is not completely sufficient. For instance, start with \(S = \prod_{n=1}^{\infty} F_n\) where each \(F_n\) is a copy of some large field \(F\) (larger than its prime subfield), choose a nontrivial maximal ideal \(M\) in \(S\) (meaning one that contains \(\oplus_{n=1}^{\infty} F_n\)), and take \(R = K \cdot 1 + M\), where \(K\) is a proper subfield of \(F\). Then \(R\) contains all the idempotents of \(S\), so that \(|id(R)| \cong 2^{\aleph_0}\), but \(R\) is not an infinite direct product of indecomposable rings. E.g., since \(R\) is a commutative von Neumann regular ring, it could only be a direct product of indecomposable rings if it were a direct product of fields. That fact would imply \(R\) is self-injective, but it is not - in fact, its injective hull, equal to its maximal quotient ring, is \(S\).

We now start the procedure for giving up of a useful algorithm calculating successfully \(W_{\alpha,p}(U(RG))\) in a rather general situation for an arbitrary \(p\)-divisible group \(G\) and with a restriction only on the coefficient ring \(R\). To this aim, suppose \(R\) is a ring in which every finitely generated (in particular, each indecomposable) subring is pure – we may also take \(R\) to be perfect finitely generated.

And so, let \(x \in U(RG)/U^p(RG) = U(RG)/(U(RpG)^p) = U(RG)/U(RpG)\), where the last equality follows by taking into account that \(G = C^p\). Thus \(x \in U(LG)U(RpG)/U(RpG) \cong U(LG)/(U(LG) \cap U(RpG)) = U(LG)/U((L \cap R^p)G) = U(LG)/U((L \cap R^p)G) = U(LG)/U((L \cap R^p)G) = U((L \cap R^p)G)\).
$U(LG)/U(L^pG)$ for some finitely generated subring $L$ of $R$ containing the same identity as that of $R$. Furthermore, $L \cong R_1 \times \cdots \times R_n$ where each $R_i$ is indecomposable $(1 \leq i \leq n)$, and hence $LG \cong R_1G \times \cdots \times R_nG$ with $U(LG) \cong U(R_1G) \times \cdots \times U(R_nG)$ and $U(L^pG) \cong U(R_1^pG) \times \cdots \times U(R_n^pG)$ under the same isomorphism. We consequently will have $U(LG)/U(L^pG) \cong [U(R_1G)/U(R_1^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)]$, whence we may formally write $x \in [U(R_1G)/U(R_1^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)]$. Finally, $U(RG)/U(R^pG) = \bigcup [U(R_iG)/U(R_i^pG)] \times \cdots \times [U(R_nG)/U(R_n^pG)]$, where the union is taken over each finite family $\{R_i\}_{1 \leq i \leq n}$ of indecomposable subrings $R_i$ of $R$.

On the other hand, if we calculate separately $W_{\alpha,p}(U(R_iG))$ for each index $i$, then utilizing some set-theoretical gymnastics, there is a way to compute $W_{\alpha,p}(U(RG))$ as well. However, this will be the theme of some other research exploration.

**Remark 2.4.** Note also that if $G_0/G_p$ is finite, then $G = M \times K$ where $M$ is finite $p$-divisible and $K$ is $p$-mixed. Therefore, $U(RG) \cong U(RM) \times V((RM)K)$ and $V(RG) \cong V(RM) \times V((RM)K)$. Thus, in accordance with Lemma 2.1, the Warfield $p$-invariants of $U(RG)$ and $V(RG)$ are respectively sums of the Warfield $p$-invariants of $U(RM)$ plus these of $V((RM)K)$, and of the Warfield $p$-invariants of $V(RM)$ plus these of $V((RM)K)$. But the Warfield $p$-invariants of $V((RM)K)$ are completely calculated in [4] because $\text{char}(RM) = p$. So, what remains to compute are $W_{\alpha,p}(U(RM))$ or $W_{\alpha,p}(V(RM))$. In this aspect does it follow that $|V(RM)/V(R^pM)| = |R/R^p|^?$

Finally, we assert that if $K$ is a commutative indecomposable unital ring and $G$ is a finite abelian group of exponent which inverts in $K$, then $KG \cong KH$ for some group $H$ if, and only if, $H$ is finite of the same exponent as that of $G$ and $KG_p \cong KH_p$ for every prime $p$. The complete proof will be the theme of some other research exploration.

**Correction:** In [6], pp.7-8 there is a series of identical typos. In fact, in ([6], p. 8, Claim) the equality $|\bigcup_{i \in I} A_i| = \sum_{i \in I} |A_i|$ should be read and written as $|\bigcup_{i \in I} A_i| \leq \sum_{i \in I} |A_i|$. In general, an equality cannot be happen. The next two examples manifestly demonstrate this.

If $A_i = A_j$ for all indexes $i$ and $j$, or $A_i \supset A_{i+1}$ for all indices $i \in I$, the equality is trivially false.

A less trivial construction is the following: There exist continuum ($= c$) countable subsets $A_i$ ($i \in c$) of $\mathbb{Z} \oplus \mathbb{Z}$ such that $A_i \cap A_j$ is finite for all $i \neq j$, and $\bigcup_{i \in c} A_i = \mathbb{Z} \oplus \mathbb{Z}$. Therefore, $|\bigcup_{i \in c} A_i| = c$ while $\sum_{i \in c} |A_i| = c$. The examples are shown.

However, if all sets $A_i$ are disjoint (i.e., $A_i \cap A_j = \emptyset$ for all indices $i$ and $j$), the desired equality holds, that is, $|\bigcup_{i \in I} A_i| = \sum_{i \in I} |A_i|$ - see, e.g., Dugundji, Topology, Allyn and Bacon, Boston, 1966, p.30.

So, the statement of Proposition 2.8 on p.7, the equality for $W_{\alpha,p}(U(RG))$ should be written as the inequality "$\leq\$". The same correction appears two more times on lines 4 and 8 after the Claim.
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References

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