Application of Chybeshev Polynomials in Factorizations of Balancing and Lucas-Balancing Numbers

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ABSTRACT: In this paper, with the help of orthogonal polynomials especially Chybeshev polynomials of first and second kind, number theory and linear algebra intertwined to yield factorization of balancing and Lucas-balancing numbers.

Key Words: Balancing numbers, balancers, Lucas-balancing numbers, triangular numbers.

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1. Introduction

As usual, see [1], the balancing number \( n \) is defined by the solution of the Diophantine equation

\[
1 + 2 + \cdots + (n - 1) = (n + 1) + (n + 2) + \cdots + (n + r),
\]

where \( r \) is the balancer corresponding to the balancing number \( n \). The first few balancing numbers are 1, 6, 35 with corresponding balancers 0, 2, 14. If \( B_n \) is the \( n^{th} \) balancing number, the recurrence relation for balancing numbers is given by

\[
B_{n+1} = 6B_n - B_{n-1}, \quad n \geq 2,
\]

(1.1)

with \( B_1 = 1, B_2 = 6 \).

In [1] it is shown that, if \( n \) is a balancing number, \( n^2 \) is a triangular number, that is, \( 8n^2 + 1 \) is a perfect square and for all \( n \), \( \sqrt{8n^2 + 1} \) generates a sequence called as the sequence of Lucas-balancing numbers [3], whose first few terms are given by 1, 3 and 17 and if \( C_n \) is the \( n^{th} \) Lucas-balancing number, its recurrence relation is given by

\[
C_{n+1} = 6C_n - C_{n-1}, \quad n \geq 2,
\]

(1.2)

with \( C_1 = 3, C_2 = 17 \).

In the recent years many number theorists from all over the world are taking interest in this beautiful number system. Liptai [2] proved that the only Fibonacci number

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in the sequence of balancing numbers is 1. In [3], he also proved that there is no Lucas number in the sequence of balancing numbers. Balancing numbers and its related sequences are available in the literature. Interested reader may follow [4], [6], [7].

In this paper, we observe that, with the help of orthogonal polynomials, number theory and linear algebra intertwined to yield factorization of balancing and Lucas-balancing numbers. In section 2 and 3 we derive the following factorization of these numbers:

\[ B_n = \prod_{1 \leq k \leq n-1} (6 - 2 \cos \frac{k\pi}{n}) \quad (1.3) \]

\[ C_n = \frac{1}{2} \prod_{1 \leq k \leq n} (6 - 2 \cos \frac{(2k - 1)\pi}{2n}) \quad (1.4) \]

In order to derive (1.3) and (1.4) we present the following theorem whose proof is included for completeness.

**Theorem 1.1** If the sequence of tridiagonal matrices \( \{A_n, n = 1, 2, \cdots \} \) is of the form

\[
A_n = \begin{pmatrix}
A_{11} & A_{12} & \cdots & A_{1(n-1)} \\
A_{21} & A_{22} & \cdots & A_{2(n-1)} \\
\vdots & \vdots & \ddots & \vdots \\
A_{n-1,1} & A_{n-1,2} & \cdots & A_{n-1,n} \\
A_{n,1} & A_{n,2} & \cdots & A_{nn}
\end{pmatrix},
\]

then the successive determinant of \( A_n \) are given by the recursive formulas:

\[
det(A_1) = A_{11} \\
det(A_2) = A_{11}A_{22} - A_{12}A_{21} \\
det(A_n) = A_{nn} \det(A_{n-1}) - A_{(n-1)n} A_{n(n-1)} \det(A_{n-2}).
\]

**Proof.** Using Induction one can easily check that the theorem is true for \( n = 1, 2 \) and 3 and assume that it is true for all \( k, 3 \leq k \leq n \), that is

\[
det(A_k) = A_{kk} \det(A_{k-1}) - A_{(k-1)k} A_{k(k-1)} \det(A_{k-2}).
\]
Now,

\[
det(A_{k+1}) = det\left( \begin{array}{cccc}
A_{11} & A_{12} & & \\
A_{21} & A_{22} & A_{23} & \\
& & \ddots & \\
& & & A_{k(k+1)} \\
& & & \end{array} \right)
\]

\[
= A_{k(k+1)} det(A_k) - A_{k(k+1)} det\left( \begin{array}{cccc}
A_{11} & A_{12} & & \\
A_{21} & A_{22} & A_{23} & \\
& & \ddots & \\
& & & A_{k(k-1)} \\
& & & \end{array} \right)
\]

Thus the theorem is true for all natural number \( n \).

2. Factorization of Balancing Numbers

In order to derive the factorization of balancing numbers (1.3), let us introduce the sequence of matrices \( \{D_n, n = 1, 2, \cdots \} \) where \( D_n \) is an \( n \times n \) tridiagonal matrix with entries \( d_{kk} = 6, \ 1 \leq k \leq n \) and \( d_{(k-1)k} = -i, d_{k(k-1)} = i, \ 2 \leq k \leq n \), where \( i = \sqrt{-1} \). That is

\[
D_n = \begin{pmatrix}
6 & -i & & \\
-i & 6 & -i & \\
& i & 6 & \ddots \\
& & \ddots & \ddots \\
& & & i & 6
\end{pmatrix},
\]

By virtue of Theorem 1.1, we find

\[
det(D_1) = 6
\]

\[
det(D_2) = 36 + i^2 = 35
\]

\[
det(D_n) = 6 \ det(D_{n-1}) - det(D_{n-2}),
\]

which is nothing but the sequence of balancing numbers starting with \( B_2 \). Thus,

\[
B_n = det(D_{n-1}), \quad n \geq 2.
\] (2.1)

Since the determinant of a matrix can be found by taking the product of its eigenvalues, we will now find the spectrum of \( D_n \) in order to find an alternate formulation for \( det(D_n) \).

Let us introduce another sequence of matrices \( \{S_n, n = 1, 2, \cdots \} \) where \( S_n \) is
an \( n \times n \) tridiagonal matrix with entries \( s_{kk} = 0, \ 1 \leq k \leq n \) and \( s_{(k-1)k} = -i, s_{k(k-1)} = i, \ 2 \leq k \leq n \). That is,

\[
S_n = \begin{pmatrix}
0 & -i & & & \\
& 0 & -i & & \\
& & i & 0 & \\
& & & \iddots
& \iddots \iddots \iddots \iddots \\
& & & & i & 0
\end{pmatrix}.
\]

Clearly \( D_n = 6I + S_n \), where \( I \) be the identity matrix same order as \( S_n \). Let \( \lambda_k, k = 1, 2, 3 \cdots, n \), be the eigenvalues of \( S_n \) with corresponding eigenvectors \( X_k \). Then for each \( j \),

\[
D_nX_j = [6I + S_n]X_j \\
= 6IX_j + S_nX_j \\
= 6X_j + \lambda_jX_j \\
= (6 + \lambda_j)X_j.
\]

Thus \( \delta_k = 6 + \lambda_k, \ k = 1, 2, \cdots, n \), be the eigenvalues of \( D_n \). Therefore,

\[
det(D_n) = \prod_{1 \leq k \leq n} (6 + \lambda_k), \quad n \geq 1. \tag{2.2}
\]

In order to find \( \lambda_k, k = 1, 2, \cdots, n \), we recall that each \( \lambda_k \) is zero of the characteristic polynomial \( p_n(\lambda) = det(S_n - \lambda I) \).

Since \( S_n - \lambda I \) is a tridiagonal matrix we have,

\[
S_n - \lambda I = \begin{pmatrix}
-\lambda & -i & & & \\
& -\lambda & -i & & \\
& & i & -\lambda & \\
& & & \iddots
& \iddots \iddots \iddots \\
& & & & i & -\lambda
\end{pmatrix}.
\]

Using Theorem 1.1, we get the following recursive formula for the characteristic polynomials:

\[
p_1(\lambda) = -\lambda \\
p_2(\lambda) = \lambda^2 - 1 \\
p_n(\lambda) = -\lambda p_{n-1}(\lambda) - p_{n-2}(\lambda).
\]

This family of polynomials can be transformed into another family \( \{M_n, n \geq 1\} \) by the transformation \( \lambda = -2x \) to get,

\[
M_1(x) = 2x \\
M_2(x) = 4x^2 - 1 \\
M_n(x) = 2xM_{n-1}(x) - M_{n-2}(x).
\]
We observe that the family \( \{M_n, n \geq 1\} \) is the set of Chebyshev polynomials of second kind. It is well known that for \( x = \cos \theta \), the Chebyshev polynomials of the second kind can be written as

\[
M_n(x) = \frac{\sin((n + 1)\theta)}{\sin \theta}
\]

which when equal to zero gives

\[
\theta_k = \frac{\pi k}{n + 1}, \quad k = 1, 2, \ldots, n.
\]

Thus,

\[
x_k = \cos \theta_k = \cos \frac{\pi k}{n + 1}, \quad k = 1, 2, \ldots, n.
\]

Now applying the transformation \( \lambda = -2x \), the eigenvalues of \( S_n \) are given by

\[
\lambda_k = -2 \cos \frac{\pi k}{n + 1}, \quad k = 1, 2, \ldots, n.
\]

Combining (2.1), (2.2) and (2.3), we get

\[
B_{n+1} = \det(D_n) = \prod_{1 \leq k \leq n} (6 - 2 \cos \frac{k\pi}{n}), \quad n \geq 1,
\]

which is identical to the factorization (1.3).

3. Factorization of Lucas-Balancing Numbers

In a similar manner we can derive (1.4) by considering the sequence of matrices \( \{E_n, n = 1, 2, \ldots\} \) where \( E_n \) is an \( n \times n \) tridiagonal matrix with entries \( e_{11} = 3, e_{kk} = 6, \quad 2 \leq k \leq n \) and \( e_{(k-1)k} = -i, e_{k(k-1)} = i, \quad 2 \leq k \leq n \). That is,

\[
E_n = \begin{pmatrix}
3 & -i &  & \\
 & i & 6 & -i \\
 & & i & 6 & \\
 & & & & \ddots & i & -i \\
 & & & & & i & 6
\end{pmatrix}
\]

Again using Theorem 1.1, we obtain

\[
det(E_1) = 3 \\
det(E_2) = 18 + i^2 = 17 \\
det(E_n) = 6 \det(E_{n-1}) - \det(E_{n-2}).
\]
We observe that each member in this sequence is a Lucas-balancing number. Thus, we get
\[ C_n = det(E_n), \quad n \geq 1. \quad (3.1) \]
If \( e_j \) is the \( j^{th} \) column of the identity matrix \( I \), we see that \( det(I + e_1 e_1^T) = 2 \). Therefore, we may write
\[ det(E_n) = \frac{1}{2} det[(I + e_1 e_1^T)E_n]. \quad (3.2) \]
Also we observe that the right hand side of (3.2) can be expressed as
\[ \frac{1}{2} det[(I + e_1 e_1^T)E_n] = \frac{1}{2} det[6I + S_n - ie_1 e_2^T] \]
where \( S_n \) is the matrix defined earlier.
If \( \alpha_k, \quad k = 1, 2, 3, \ldots, n \), be the eigenvalues of \( S_n - ie_1 e_2^T \) with corresponding eigenvectors \( Y_k \), then for each \( j \),
\[ [6I + S_n - ie_1 e_2^T]Y_j = 6iY_j + (S_n - ie_1 e_2^T)Y_j = 6Y_j + \alpha_j Y_j = (6 + \alpha_j)Y_j. \]
Therefore,
\[ \frac{1}{2} det[6I + S_n - ie_1 e_2^T] = \frac{1}{2} \prod_{1 \leq k \leq n} (6 + \alpha_k), \quad n \geq 1. \quad (3.3) \]
In order to find \( \alpha_k \)'s, we recall that each \( \alpha_k \) is a zero of the characteristic polynomial \( q_n(\alpha) = det(S_n - ie_1 e_2^T - \alpha I) \). Since \( det(I + \frac{1}{2} e_1 e_1^T) = \frac{1}{2} \), we can express the characteristic polynomial as
\[ q_n(\alpha) = 2det[(I - \frac{1}{2} e_1 e_1^T)(S_n - ie_1 e_2^T - \alpha I)] \]
\[ = 2det \left( \begin{array}{ccccc} -\alpha & -i & & & \\
\frac{1}{2} i & -\alpha & -i & & \\
& i & -\alpha & \ddots & \\
& & \ddots & \ddots & -i \\
& & & i & -\alpha \end{array} \right). \]
Since \( q_n(\alpha) \) is the twice of a tridiagonal matrix, we can use Theorem 1.1 to get the following recursive formulas:
\[ q_1(\alpha) = -\frac{\alpha}{2} \]
\[ = \frac{\alpha^2}{2} - 1 \\
= -\alpha q_{n-1}(\alpha) - q_{n-2}(\alpha). \]
Using the transformation $\alpha = -2x$, the family of the above polynomial can be transformed to a new family $\{T_n(x), \ n \geq 1\}$ where,

\[
\begin{align*}
T_1(x) &= x \\
T_2(x) &= 2x^2 - 1 \\
T_n(x) &= 2xT_{n-1}(x) - T_{n-2}(x).
\end{align*}
\]

Once again we observe that the family $\{T_n(x), \ n \geq 1\}$ is the set of Chebyshev polynomials of first kind. It is well known that for $x = \cos \theta$ the Chebyshev polynomials of the first kind can be written as

\[T_n(x) = \cos n\theta\]

which when equal to zero gives,

\[\theta_k = \frac{\pi(2k - 1)}{2n}, \ k = 1, 2, \ldots, n.\]

Therefore,

\[
x_k = \cos \theta_k = \cos \frac{\pi(2k - 1)}{2n}, \ k = 1, 2, \ldots, n.
\]

Applying the transformation $\alpha = -2x$, the eigenvalues of $S_n - i\epsilon_1 \epsilon_2^T$ is given by

\[\alpha_k = -2 \cos \frac{\pi(2k - 1)}{2n}, \ k = 1, 2, \ldots, n. \tag{3.4}\]

Thus, from (3.1), (3.3) and (3.4), we have

\[C_n = \frac{1}{2} \left[ \prod_{1 \leq k \leq n} \left( 6 - 2 \cos \frac{(2k - 1)\pi}{2n} \right) \right]\]

which is identical to the factorization (1.4).

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**References**


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