Some separation axioms in generalized topological spaces

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Abstract: We give different definitions for g-closed sets, $R_0$ and $R_1$ spaces in generalized topological spaces, characterize such spaces and compare with the existing definitions and results.

Key Words: generalized topology, $\mu$-closed and $\mu$-open sets; $\delta$-open and $\delta$-closed sets, connected, irreducible, $\mu$-regular, $R_0$ and $R_1$ generalized topological spaces.

Contents

1 Introduction and preliminaries

2 Strong generalized spaces

3 $g^\lambda$-closed sets

4 $R_0$ and $R_1$-spaces

5 $G_{\mu}$-regular generalized spaces

1. Introduction and preliminaries

A generalized topology or simply GT $\mu$ [3] on a nonempty set $X$ is a collection of subsets of $X$ such that $\emptyset \in \mu$ and $\mu$ is closed under arbitrary union. Elements of $\mu$ are called $\mu$-open sets. A subset $A$ of $X$ is said to be $\mu$-closed if $X - A$ is $\mu$-open. The pair $(X, \mu)$ is called a generalized topological space (GTS). If $A$ is a subset of a space $(X, \mu)$, then $c_{\mu}(A)$ is the smallest $\mu$-closed set containing $A$ and $i_{\mu}(A)$ is the largest $\mu$-open set contained in $A$. If $\gamma : \wp(X) \rightarrow \wp(X)$ is a monotonic function defined on a nonempty set $X$ and $\mu = \{A \mid A \subset \gamma(A)\}$, the family of all $\gamma$-open sets is also a GT [2], $i_{\gamma} = i_{\mu}$, $c_{\gamma} = c_{\mu}$, and $\mu = \{A \mid A = i_{\mu}(A)\}$ [4, Corollary 1.3]. The family of all monotonic functions defined on $X$ is denoted by $\Gamma$. By a space $(X, \mu)$, we will always mean a GTS $(X, \mu)$. A subset $A$ of a space $(X, \mu)$ is said to be $\alpha$-open [4] (resp., semiopen [4], preopen [4], $b$-open [14], $\beta$-open [4]) if $A \subset i_{\mu}c_{\mu}i_{\mu}(A)$ (resp., $A \subset c_{\mu}i_{\mu}(A)$, $A \subset i_{\mu}c_{\mu}i_{\mu}(A)$, $A \subset i_{\mu}c_{\mu}i_{\mu}(A)$). We will denote the family of all $\alpha$-open sets by $\alpha$, the family of all semiopen sets by $\sigma$, the family of all preopen sets by $\pi$, the family of all $b$-open sets by $b$ and the family of all $\beta$-open sets by $\beta$. If

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(X, μ) is a GTS, then we say that a subset \( A \in \delta \subset \wp(X) \) \[6\] if for every \( x \in A \), there exists a \( \mu \)-closed set \( Q \) such that \( x \in i_\mu(Q) \subset A \). Then \((X, \delta)\) is a GTS \[6, \text{Proposition 2.1}\] such that \( \delta \subset \mu \[6, \text{Theorem 1}\]. Elements of \( \delta \) are called the \( \delta \)-open sets of \((X, \mu)\). For \( A \subset X \), \( i_\delta(A) \) and \( c_\delta(A) \) are the interior and closure of \( A \) in \((X, \delta)\). We will denote by \( \nu \) (resp. \( \xi, \eta, \varepsilon, \psi \)), the family of all \( \alpha \)-open (resp. semiopen, preopen, \( b \)-open, \( \beta \)-open) sets of the generalized space \((X, \delta)\). If \( \kappa \in \{ \mu, \alpha, \sigma, \pi, \beta, \delta, \nu, \xi, \eta, \varepsilon, \psi \} \) and \( A \) is a subset of a space \((X, \kappa)\), then \( c_\kappa(A) \) is the smallest \( \kappa \)-closed set containing \( A \) and \( i_\kappa(A) \) is the largest \( \kappa \)-open set contained in \( A \). Note that the operator \( c_\kappa \) is monotonic, increasing and idempotent and the operator \( i_\kappa \) is monotonic, decreasing and idempotent. Clearly, \( A \) is \( \kappa \)-open if and only if \( A = i_\kappa(A) \) and \( A \) is \( \kappa \)-closed if and only if \( A = c_\kappa(A) \). Also, for every subset \( B \) of a space \((X, \kappa)\), \( X - i_\kappa(A) = c_\kappa(X - A) \). If \( \lambda \subset \wp(X) \) is a GT, then \( \gamma \in \Gamma \) is said to be \( \lambda \)-friendly \[5\] if \( \gamma(A) \cap L \subset \gamma(A \cap L) \) for \( A \subset X \) and \( L \in \lambda \). In \[14\], it is denoted that \( \Gamma_4 = \{ \gamma \mid \gamma \text{ is } \mu \text{-friendly where } \mu \text{ is the GT of all } \gamma \text{-open sets} \} \) and if \( \gamma \in \Gamma_4 \), the space \((X, \gamma)\) (resp. \((X, \mu)\)) is called a \( \gamma \)-space. By \[14, \text{Theorem 2.1}\], the intersection of two \( \mu \)-open sets is again a \( \mu \)-open set and so every \( \gamma \)-space is a quasi-topological space \[5\]. By \[14, \text{Theorem 2.3}\], it is established that in a \( \gamma \)-space, \( i_\gamma \) and \( c_\mu \) preserve finite intersection and finite union, respectively. Later, in \[5\], it is established that the above result is also true for quasi-topological spaces. A space \((X, \mu)\) is said to be strong if \( X \in \mu \). The following lemma is essential to proceed further where the easy proof is omitted.

**Lemma 1.1.** Let \((X, \mu)\) be a space where \( \mu \) is the family of all \( \gamma \)-open sets of a \( \gamma \in \Gamma_4 \). Then the following hold.

(a) The intersection of two \( \delta \)-open sets is a \( \delta \)-open set.

(b) \( i_\delta(A) \cap i_\delta(B) = i_\delta(A \cap B) \) for every subsets \( A \) and \( B \) of \( X \).

(c) \( c_\delta(A) \cup c_\delta(B) = c_\delta(A \cup B) \) for every subsets \( A \) and \( B \) of \( X \).

(d) \( i_\delta \in \Gamma_4 \).

2. Strong generalized spaces

If \((X, \mu)\) is any generalized space which is not strong, then in \[7, \text{Proposition 1.2}\], it is established that \( X \in \sigma \) and so it follows that always \( X \in b \) and \( X \in \beta \). The following Example 2.1 shows that in general, if \( X \not\in \lambda \mu \), then \( X \not\in \lambda \) for \( \lambda \in \{ \mu, \delta, \alpha, \pi, \nu, \xi, \eta \} \) and Theorem 2.1 below shows that \( X \in \xi \) and hence \( X \in \varepsilon \) and \( X \in \psi \).

**Example 2.1.** Let \( X \) be the set of all real numbers and \( \mu = \{ \emptyset, \{0\} \} \). Then \( X \not\in \lambda \) where \( \lambda \in \{ \mu, \delta, \alpha, \pi, \nu, \eta \} \).

**Theorem 2.1.** If \((X, \mu)\) is a generalized space which is not strong, then the following hold.

(a) \( X \not\in \pi \) and hence \( X \not\in \alpha \).

(b) \( X \not\in \delta \) and hence \( X \not\in \eta \) and \( X \not\in \nu \).

(c) \( X \in \xi \) and hence \( X \in \varepsilon \) and \( X \in \psi \).

**Proof:** (a) Suppose \( X \in \pi \). But always, \( X \in \sigma \) and so \( X \in \sigma \cap \pi = \alpha \). Therefore, \( X \subset i_\mu c_\mu i_\mu(X) \subset i_\mu c_\mu(X) = i_\mu(X) \). Hence \( X \in \mu \), a contradiction and so \( X \not\in \pi \).
and hence $X \notin \alpha$.

(b) Since $X \notin \mu$, $X \notin \delta$, since $\delta \subset \mu$. Since $\eta = \pi(\delta)$, by (a), $X \notin \eta$ and hence $X \notin \nu$, since $\nu = \alpha(\delta)$.

(c) Since $\xi = \sigma(\delta)$, $X \in \xi$ and so $X \in \varepsilon$ and $X \in \psi$. \hfill \Box

3. \(g^*_\lambda\)-closed sets

Let \((X, \mu)\) be a generalized space. A subset \(A\) of \(X\) is said to be \(g_\mu\)-closed \cite{9} if \(c_\mu(A) \subset M\) whenever \(A \subset M\) and \(M \in \mu\). Various properties of \(g_\mu\)-closed are discussed and characterizations are given in \cite{9} and these properties are valid for the generalized topologies induced by \(\mu\) and \(\delta\). Given a topological space \((X, \tau)\) and a generalized topology \(\mu\) on \(X\), a subset \(A\) of \(X\) is said to be \(g_\mu\)-closed \cite{11} if \(c_\mu(A) \subset M\) whenever \(A \subset M\) and \(M \in \tau\). If \(\mu = \tau\), then the \(g_\mu\)-closed sets coincide with the \(g\)-closed sets of Levine \cite{8}. If \(\tau\) is fixed and \(\mu\) is any one of the generalized topologies, namely \(\alpha, \sigma, \pi, b\) and \(\beta\) of the topological space \((X, \tau)\), where all these family contains \(X\), then we have \(g_\alpha\)-closed, \(g_\sigma\)-closed, \(g_\pi\)-closed, \(g_b\)-closed and \(g_\beta\)-closed sets in \((X, \tau)\) and all the results established in \cite{11} are valid for these sets. If \(\mu\) is a fixed generalized topology, and instead of \(\tau\), if we consider \(\sigma, b\) and \(\beta\), the generalized topologies induced by \(\mu\), which contains \(X\), then we can define \(g\sigma(\mu)\)-closed, \(gb(\mu)\)-closed and \(g_\beta(\mu)\)-closed sets in the space \((X, \mu)\) and for these family of sets also, all the results established in \cite{11} are valid.

The difference between the two definitions is that the definition of \(g_\mu\)-closed sets uses elements of the topology \(\tau\) on \(X\) where \(X \in \tau\) where as the definition of \(g_\mu\)-closed sets uses elements of the generalized topology \(\mu\) where \(X\) may or may not be in \(\mu\). Therefore, the definition of \(g_\mu\)-closed sets is more general, since the definition uses a large class of generalized topologies which also contains the class of all topological spaces. Moreover, similar results established for \(g_\mu\)-closed sets in \cite{11} are already established for \(g_\mu\)-closed sets in \cite{9}. We give below a new definition for generalized closed sets in a generalized space, which is common for both strong spaces and non-strong spaces, and discuss the relation between these three kinds of sets in the following Examples 3.1 to 3.3. A subset \(A\) of \(\mathcal{M}_\mu = \cup\{B \mid B \in \mu\}\) of a generalized space \((X, \mu)\) is said to be \(g^*_\mu\)-closed if \(c_\mu(A) \cap \mathcal{M}_\mu \subset M\) whenever \(A \subset M\) and \(M \in \mu\). Note that, if the space is strong, then this definition coincides with the definition of \(g_\mu\)-closed sets.

Example 3.1. Let \(X\) be a nonempty set and \(\mu\) be a generalized topology on \(X\). Suppose \(\mathcal{M}_\mu = \cup\{A \mid A \in \mu\} \neq X\) and \(\tau = \wp(\mathcal{M}_\mu)\cup \{X\}\). Then every \(\mu\)-closed subset of \(X\) contains \(X - \mathcal{M}_\mu\). Therefore, every subset \(A\) of \(\mathcal{M}_\mu\) is neither a \(g_\mu\)-closed set nor a \(g_\mu\)-closed set. \(g^*_\mu\)-closed sets depend on the generalized topology \(\mu\). Every nonempty subset \(B\) of \(X\) such that \(B \cap (X - \mathcal{M}_\mu) \neq \emptyset\) or \(B \subset (X - \mathcal{M}_\mu)\) is not contained in any \(\mu\)-open set which implies that such sets are trivially \(g_\mu\)-closed. Clearly, such sets are \(g_\mu\)-closed, since \(X\) is the only open set containing such sets.

Example 3.2. \cite[Example 2.1]{1} Let \(X = \mathbb{J}_n = \{1, 2, 3, \ldots, n\}\). Define \(\kappa : \wp(\mathbb{J}_n) \rightarrow \wp(\mathbb{J}_n)\) by \(\kappa(A) = A\) if \(\mathbb{J}_n \setminus \{i\} \subseteq A\) for some \(i \in \{1, 2, 3, \ldots, n\}\) and \(\kappa(A) = \emptyset\).
otherwise. Then \( \mu = \{\emptyset, X\} \cup \{A \subseteq \mathcal{I}_n \mid A = \mathcal{I}_n - \{i\}, i = 1, 2, 3, \ldots, n\} \), the co-singleton generalized topology defined on a finite set. The only \( \mu \)-closed sets are \( \emptyset, X \) and singleton subsets of \( \mathcal{I}_n \). In this space, the family of all \( g^*_\mu \)-closed sets, the family of all \( g^*_\mu \)-closed sets and family of all \( \mu \)-closed sets coincide. For the topology \( \tau = \{\emptyset\} \cup \{G \subseteq X \mid \{1, 2\} \subseteq G\} \) on \( X \), the \( \mu \)-closed sets are precisely the \( g^*_\mu \)-closed sets.

**Example 3.3.** Consider the space \((X, \tau)\) and generalized topology \( \mu \) of the Example 2.3 of [11]. In this space, \( \{a, c\} \) is \( g\mu \)-closed but it is not \( g^*_\mu \)-closed and also not \( g^*\mu \)-closed.

Throughout the paper, if \( \mu \) is a generalized topology on \( X \), let \( \mathcal{M}_\mu = \cup\{A \mid A \in \mu\} \), \( X \notin \mu \) and \( \lambda \in \{\mu, \alpha, \pi, \sigma, b, \beta, \delta, \nu, \xi, \eta, \varepsilon, \psi\} \). Then, by Theorem 2.1, we have \( \mathcal{M}_\lambda \neq X \) if \( \lambda \in \{\mu, \alpha, \pi, \sigma, b, \beta, \delta, \nu, \xi, \eta, \varepsilon, \psi\} \) and \( \mathcal{M}_\lambda = X \) if \( \lambda \in \{\sigma, b, \beta, \xi, \varepsilon, \psi\} \). Moreover, \( \mathcal{M}_\lambda = \mathcal{M}_\mu \), if \( \mathcal{M}_\lambda \neq X \). The following Lemma 3.1 is essential to proceed further.

**Lemma 3.1.** Let \( X \) be a nonempty set, \( \mu \) be a generalized topology on \( X \) and \( A \subseteq X \). Then the following hold.

(a) \((X - \mathcal{M}_\lambda)\) is a \( \lambda \)-closed set contained in every \( \lambda \)-closed set.
(b) \(c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = c_\lambda(A) \cap \mathcal{M}_\lambda\).
(c) If \( A \) is \( \lambda \)-closed, then \( c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda = A \cap \mathcal{M}_\lambda\).
(d) \(c_\lambda(A) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup \{X - \mathcal{M}_\lambda\}\).
(e) If \( A \) is \( \lambda \)-closed, then \( A = (A \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda)\).
(f) \( (\mathcal{M}_\lambda, \lambda^*) \) is a strong generalized topology where \( \lambda^* = \lambda \mid \mathcal{M}_\lambda \) is the subspace generalized topology.
(g) If \( A \subseteq \mathcal{M}_\lambda \), then \( c_{\lambda^*}(A) = c_\lambda(A) \cap \mathcal{M}_\lambda \) and \( i_{\lambda^*}(A) = i_\lambda(A) \) where \( c_\lambda(A) / i_\lambda(A) \) is the closure (resp. interior) of \( A \) in \( \mathcal{M}_\lambda\).
(h) \( A \subseteq \mathcal{M}_\lambda \) is \( \lambda^* \)-closed in \( \mathcal{M}_\lambda \) if and only if \( A = c_\lambda(A) \cap \mathcal{M}_\lambda\).
(i) \( A \subseteq \mathcal{M}_\lambda \) is \( \lambda^* \)-closed in \( \mathcal{M}_\lambda \) if and only if \( c_\lambda(A) \cap \mathcal{M}_\lambda = A \cup (X - \mathcal{M}_\lambda)\).

**Proof:** (a) follows from the fact that if \( G \) is \( \lambda \)-open, then \( G \subseteq \mathcal{M}_\lambda\).
(b) Clearly, \( c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda \subseteq c_\lambda(A) \cap \mathcal{M}_\lambda\). Let \( x \in c_\lambda(A) \cap \mathcal{M}_\lambda\). Then \( x \in c_\lambda(A) \) and \( x \in \mathcal{M}_\lambda\). Now \( x \in c_\lambda(A) \) implies that \( G \cap A \neq \emptyset \) for every \( \lambda \)-open set \( G \) containing \( x \) and so \( G \cap (A \cap \mathcal{M}_\lambda) \neq \emptyset \) for every \( \lambda \)-open set \( G \) containing \( x \). Therefore, \( x \in c_\lambda(A \cap \mathcal{M}_\lambda) \) and so \( x \in c_\lambda(A) \cap \mathcal{M}_\lambda\). Hence \( c_\lambda(A) \cap \mathcal{M}_\lambda \subseteq c_\lambda(A \cap \mathcal{M}_\lambda) \cap \mathcal{M}_\lambda\).
This completes the proof.
(c) The proof follows from (b).
(d) \( c_\lambda(A) = c_\lambda(A) \cap X = c_\lambda(A) \cap (\mathcal{M}_\lambda \cup (X - \mathcal{M}_\lambda)) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup (c_\lambda(A) \cap (X - \mathcal{M}_\lambda)) = (c_\lambda(A) \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda\), by (a).
(e) If \( A \) is \( \lambda \)-closed, by (d), we have \( A = (A \cap \mathcal{M}_\lambda) \cup (X - \mathcal{M}_\lambda\).
The proofs of (f), (g), (h) and (i) are clear.

As per the present definition, the \( g^*_\lambda \)-closed sets must be subsets of \( \mathcal{M}_\lambda\). Moreover, \( g^*_\lambda \)-closed subsets coincide with \( g_\lambda \)-closed subsets if \( X \) is \( \mu \)-open. In Example 3.2, the space is strong and the \( g^*_\lambda \)-closed sets are exactly the \( g\lambda \)-closed sets.
It is easy to note that \(g_\lambda^*\)-closed subsets are \(g_\lambda^*\)-closed subsets of the subspace \((M_\lambda, \lambda^*)\). In Example 3.1, there is no \(g_\lambda^*\)-closed subset and here also, the two concepts coincide. The following Theorem 3.4 gives some properties of \(g_\lambda^*\)-closed sets. Example 3.4 shows that the converse of Theorem 3.1(a) is not true.

**Theorem 3.1.** Let \((X, \mu)\) be a generalized space and \(A \subset X\). Then the following hold,

(a) If \(A\) is a \(\lambda\)-closed subset of \(X\), then \(A \cap M_\lambda\) is a \(g_\lambda^*\)-closed set.

(b) \(c_\lambda(A) \cap M_\lambda\) is a \(g_\lambda^*\)-closed set for every subset \(A\) of \(X\).

**Proof:**
(a) Let \(A \cap M_\lambda \subset M\) and \(M\) be \(\lambda\)-open. Since \(c_\lambda(A \cap M_\lambda) \cap M_\lambda = c_\lambda(A) \cap M_\lambda\), by Lemma 3.1(b), we have \(c_\lambda(A \cap M_\lambda) \cap M_\lambda = c_\lambda(A) \cap M_\lambda = A \cap M_\lambda \subset M\). Therefore, we have \(c_\lambda(A \cap M_\lambda) \cap M_\lambda \subset M\) and so \(A \cap M_\lambda\) is \(g_\lambda^*\)-closed.

(b) The proof follows from (a). □

**Example 3.4.** Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, \{a\}, \{b, c\}\}\). Then \(\mu\)-closed sets are \(X\), \{\(a\), \(c\)\}, \{\(b, c\)\} and \(\emptyset\). If \(A = \{a, b\}\), then \(A \cap M_\mu = \{a, b\}\) and \(A \cap M_\mu\) is a \(g_\mu^*\)-closed set but \(A\) is not \(\mu\)-closed.

The following Theorem 3.2 gives a characterization of \(g_\lambda^*\)-closed sets.

**Theorem 3.2.** Let \((X, \mu)\) be a space. Then a subset \(A\) of \(M_\lambda\) is \(g_\lambda^*\)-closed if and only if \(F \subset c_\lambda(A) - A\) and \(F\) is \(\lambda\)-closed imply that \(F = X - M_\lambda\).

**Proof:** Let \(F\) be a \(\lambda\)-closed subset of \(c_\lambda(A) - A\). Since \(A \subset X - F\) and \(A\) is \(g_\lambda^*\)-closed, \(c_\lambda(A) \cap M_\lambda \subset X - F\) and so \(F \subset X - (c_\lambda(A) \cap M_\lambda) = (X - c_\lambda(A)) \cup (X - M_\lambda)\). Since \(F \subset c_\lambda(A)\), we have \(F \subset (X - M_\lambda)\). Therefore, by Lemma 3.1(a), \(F = X - M_\lambda\). Conversely, suppose the condition holds and \(A \subset M\) and \(M \in \lambda\). Suppose \((c_\lambda(A) \cap M_\lambda) \cap (X - M)\) is a nonempty subset. Then \((c_\lambda(A) \cap M_\lambda) \cap (X - M) \subset c_\lambda(A) \cap (X - M) \subset c_\lambda(A) \cap (X - A) \subset c_\lambda(A) - A\). Thus \(c_\lambda(A) \cap (X - M)\) is a \(\lambda\)-closed subset contained in \(c_\lambda(A) - A\). Therefore, \(c_\lambda(A) \cap (X - M) = X - M_\lambda\) which implies that \((c_\lambda(A) \cap M_\lambda) \cap (X - M) = \emptyset\), a contradiction to the assumption. Therefore, \(c_\lambda(A) \cap M_\lambda \subset M\) which implies that \(A\) is a \(g_\lambda^*\)-closed set. □

**Theorem 3.3.** Let \((X, \mu)\) be a generalized space. Then a \(g_\lambda^*\)-closed subset \(A\) of \(M_\lambda\) is a \(\lambda\)-closed set, if \(c_\lambda(A) - A\) is a \(\lambda\)-closed set.

**Proof:** By Theorem 3.2, \(c_\lambda(A) - A = X - M_\lambda\). Then \(c_\lambda(A) = A \cup (X - M_\lambda)\). By Lemma 3.1(i), \(A\) is \(\lambda\)-closed. □

The following Theorem 3.4 shows that in a \(\gamma\)-space \((X, \mu)\), the union of two \(g_\lambda^*\)-closed sets (resp. \(g_\mu^*\)-closed sets) is again a \(g_\lambda^*\)-closed set (resp. \(g_\mu^*\)-closed sets). Example 3.5 shows that the condition \(\gamma\)-space on the space cannot be replaced by generalized topology. Example 3.6 below shows that the intersection of two \(g_\lambda^*\)-closed sets need not be a \(g_\lambda^*\)-closed set in a strong generalized space. Theorem 3.5 shows that, the intersection of a \(g_\lambda^*\)-closed set with a \(\lambda\)-closed is a \(g_\lambda^*\)-closed set.
Theorem 3.4. Let \((X, \mu)\) be a \(\gamma\)-space. Then the following hold.
(a) If \(A\) and \(B\) are \(g^*_\delta\)-closed subsets of \(M_\delta\), then \(A \cup B\) is also a \(g^*_\delta\)-closed set.
(b) If \(A\) and \(B\) are \(g^*_\mu\)-closed subsets of \(M_\mu\), then \(A \cup B\) is also a \(g^*_\mu\)-closed set.

Proof: (a) Suppose \(A\) and \(B\) are \(g^*_\delta\)-closed sets. Let \(M \in \delta\) such that \(A \cup B \subseteq M\). Since \(A\) and \(B\) are \(g^*_\delta\)-closed sets, \(c_\delta(A) \cap \mathcal{M}_\delta \subseteq M\) and \(c_\delta(B) \cap \mathcal{M}_\delta \subseteq M\) and so \((c_\delta(A) \cap \mathcal{M}_\delta) \cup (c_\delta(B) \cap \mathcal{M}_\delta) \subseteq M\) and so \((c_\delta(A) \cup c_\delta(B)) \cap \mathcal{M}_\delta \subseteq M\). By Lemma 1.1(c), it follows that \(c_\delta(A \cup B) \cap \mathcal{M}_\delta \subseteq M\) and so the proof follows.
(b) The proof follows from (a) and Lemma 1.1(c).

Example 3.5. Let \(X = \{a, b, c\}\) and \(\mu = \{\emptyset, \{b\}, \{c\}, \{a, b\}, \{a, c\}, \{b, c\}, X\}\). Then \(\mu\) is a GT but not a quasi-topology. If \(A = \{b\}\) and \(B = \{c\}\), then \(A\) and \(B\) are \(g^*_\delta\)-closed sets but their union is not a \(g^*_\delta\)-closed set.

Example 3.6. Consider the space \((X, \mu)\) where \(X = \{a, b, c, d, e\}\) with \(\mu = \{\emptyset, \{a, b\}, \{a, c\}, \{a, b, c\}, X\}\). If \(A = \{a, c, d\}\) and \(B = \{b, c, e\}\), then \(A\) and \(B\) are \(g^*_\delta\)-closed sets. But \(A \cap B = \{c\}\), is not a \(g^*_\delta\)-closed set, since \(c\) is not a \(g^*_\delta\)-closed set.

Theorem 3.5. Let \((X, \mu)\) be a generalized space. If \(A\) is \(g^*_\lambda\)-closed subset of \(M_\lambda\) and \(B\) is \(\lambda\)-closed, then \(A \cap B\) is a \(g^*_\lambda\)-closed set.

Proof: Suppose \(A \cap B \subseteq M\) where \(M\) is \(\lambda\)-open. Then \(A \subseteq (M \cup (X - B))\). Since \(A\) is \(g^*_\lambda\)-closed, \(c_\lambda(A) \cap \mathcal{M}_\lambda \subseteq M \cup (X - B)\) and so \((c_\lambda(A) \cap \mathcal{M}_\lambda) \subseteq M\) which implies that \(c_\lambda(A \cap B) \cap 

\mathcal{M}_\lambda \subseteq M\) and so \(A \cap B\) is a \(g^*_\lambda\)-closed set.

A subset \(A\) of \(M_\lambda\) in a space \((X, \mu)\) is said to be \(g^*_\lambda\)-open if \(M_\lambda - A\) is \(g^*_\lambda\)-closed. The following Theorem 3.6 gives a characterization of \(g^*_\lambda\)-open sets. Since the intersection of two \(g^*_\lambda\)-closed sets need not be a \(g^*_\lambda\)-closed set, the union of two \(g^*_\lambda\)-open sets need not be a \(g^*_\lambda\)-open set. Theorem 3.7 below gives a characterization of \(g^*_\lambda\)-open sets and Theorem 3.8 below gives a property of \(g^*_\lambda\)-closed sets. Theorem 3.9 below gives a characterization of \(g^*_\lambda\)-closed sets in terms of \(g^*_\lambda\)-open sets.

Theorem 3.6. A subset \(A\) of \(M_\lambda\) in a space \((X, \mu)\) is \(g^*_\lambda\)-open if and only if \(F \cap \mathcal{M}_\lambda \subseteq i_\lambda(A)\) whenever \(F\) is \(\lambda\)-closed and \(F \cap \mathcal{M}_\lambda \subseteq A\).

Proof: Let \(A\) be a \(g^*_\lambda\)-open subset of \(M_\lambda\) and \(F\) be a \(\lambda\)-closed subset of \(X\) such that \(F \cap M_\lambda \subseteq A\). Then \(M_\lambda - A \subseteq (M_\lambda \cap M_\lambda) = M_\lambda - F\). Since \(M_\lambda - F\) is \(\lambda\)-open and \(M_\lambda - A\) is \(g^*_\lambda\)-closed, \(c_\lambda(M_\lambda - A) \cap \mathcal{M}_\lambda \subseteq M_\lambda - F\) and so \(F \cap (M_\lambda - (c_\lambda(M_\lambda - A) \cap \mathcal{M}_\lambda)) = M_\lambda \cap (M_\lambda - c_\lambda(M_\lambda - A)) = i_\lambda(A) \cap \mathcal{M}_\lambda = i_\lambda(A)\). Conversely, suppose the condition holds. Let \(A\) be a subset of \(M_\lambda\) and \(F\) be a \(\lambda\)-closed such that \(F \cap M_\lambda \subseteq A\). By hypothesis, \(F \cap M_\lambda \subseteq i_\lambda(A)\) which implies that \(M_\lambda - i_\lambda(A) \subseteq M_\lambda - (F \cap M_\lambda)\) and \(c_\lambda(M_\lambda - A) \subseteq M_\lambda - F\). Then \(c_\lambda(M_\lambda - A) \cap \mathcal{M}_\lambda \subseteq (M_\lambda - F) \cap \mathcal{M}_\lambda = M_\lambda - F\) which implies that \(M_\lambda - A\) is \(g^*_\lambda\)-closed and so \(A\) is \(g^*_\lambda\)-open.
Theorem 3.7. Let \((X, \mu)\) be a space. A subset \(A\) of \(M_\lambda\) is \(g^*_\lambda\)-open if and only if \(M = M_\lambda\) whenever \(M\) is \(\lambda\)-open and \(i_\lambda(A) \cup (M_\lambda - A) \subset M\).

Proof: Suppose \(A\) is \(g^*_\lambda\)-open subset of \(M_\lambda\) and \(M\) is \(\lambda\)-open such that \(i_\lambda(A) \cup (M_\lambda - A) \subset M\). Then \(M_\lambda - M \subset (M_\lambda - i_\lambda(A)) \cap A = c_\lambda(M_\lambda - A) \cap A = c_\lambda(M_\lambda - A) - (M_\lambda - A)\) and so \((M_\lambda - M) \cup (X - M_\lambda) \subset c_\lambda(M_\lambda - A) - (M_\lambda - A)\). By Theorem 3.2, \((M_\lambda - M) \cup (X - M_\lambda) = X - M_\lambda\) and so \(M_\lambda - M = \emptyset\) which implies that \(M_\lambda = M\). Conversely, suppose the condition holds. Let \(F\) be a \(\lambda\)-closed set such that \(F \cap M_\lambda \subset A\). Since \(i_\lambda(A) \cup (M_\lambda - A) \subset i_\lambda(A) \cup (M_\lambda - F) \cup (M_\lambda - M_\lambda) = i_\lambda(A) \cup (M_\lambda - F)\) and \(i_\lambda(A) \cup (M_\lambda - F)\) is \(\lambda\)-open, by hypothesis, \(M_\lambda = i_\lambda(A) \cup (M_\lambda - F)\) and so \(F \cap M_\lambda \subset (i_\lambda(A) \cup (M_\lambda - F)) \cap F = (i_\lambda(A) \cap F) \cup ((M_\lambda - F) \cap F) = i_\lambda(A) \cap F \subset i_\lambda(A)\).

By Theorem 3.6, \(A\) is \(g^*_\lambda\)-open.

\(\square\)

Theorem 3.8. Let \((X, \mu)\) be a space and \(A\) and \(B\) be subsets of \(M_\lambda\). If \(i_\lambda(A) \subset B \subset A\) and \(A\) is \(g^*_\lambda\)-open, then \(B\) is \(g^*_\lambda\)-open.

Proof: The proof follows from Theorem 3.7.

\(\square\)

Theorem 3.9. Let \((X, \lambda)\) be a space. Then a subset \(A\) of \(M_\lambda\) is \(g^*_\lambda\)-closed if and only if \((c_\lambda(A) - A) \cap M_\lambda\) is \(g^*_\lambda\)-open.

Proof: Suppose \((c_\lambda(A) - A) \cap M_\lambda\) is \(g^*_\lambda\)-open. Let \(A \subset M\) and \(M\) is \(\lambda\)-open. Since \(c_\lambda(A) \cap (M_\lambda - M) \subset c_\lambda(A) \cap (M_\lambda - A) = (c_\lambda(A) - A) \cap M_\lambda\), \((c_\lambda(A) - A) \cap M_\lambda\) is \(g^*_\lambda\)-open and \(c_\lambda(A) \cap (M_\lambda - M)\) is \(\lambda\)-closed, by Theorem 3.6, \(c_\lambda(A) \cap (M_\lambda - M) \subset i_\lambda((c_\lambda(A) - A) \cap M_\lambda) \subset i_\lambda(c_\lambda(A)) \cap i_\lambda(M_\lambda - A) \subset i_\lambda(c_\lambda(A)) \cap i_\lambda(X - A) = i_\lambda(c_\lambda(A)) \cap (X - c_\lambda(A)) = \emptyset\). Therefore, \(c_\lambda(A) \cap M_\lambda \subset M\) which implies that \(A\) is \(g^*_\lambda\)-closed. Conversely, suppose \(A\) is \(g^*_\lambda\)-closed and \(F \cap M_\lambda \subset (c_\lambda(A) - A) \cap M_\lambda\), where \(F\) is \(\lambda\)-closed. Then \(F \subset (c_\lambda(A) - A)\) and so by Theorem 3.2, \(F = X - M_\lambda\) and so \(\emptyset = (X - M_\lambda) \cap M_\lambda = F \cap M_\lambda \subset (c_\lambda(A) - A) \cap M_\lambda\) which implies that \(F \cap M_\lambda \subset i_\lambda((c_\lambda(A) - A) \cap M_\lambda)\). By Theorem 3.6, \((c_\lambda(A) - A)\) is \(g^*_\lambda\)-open.

\(\square\)

4. \(R_0\) and \(R_1\)-spaces

In this section, we define and discuss generalized \(R_0\) and \(R_1\) spaces which are not strong and establish that all the results established already will follow as a corollary. Generalized \(R_0\) and \(R_1\) spaces are independently defined by Sivagami and Sivaraj [15], Roy [12] and Sarasak [13]. Unless otherwise stated, in this section, \((X, \mu)\) is a generalized space which is not strong and \(\lambda \in \{\mu, \delta, \alpha, \sigma, \pi, b, \beta, \nu, \xi, \eta, \varepsilon, \psi\}\). The following definitions and Lemma 4.1 are essential to proceed further. For \(A \subset M_\lambda\), we define \(\wedge_\lambda(A) = \cap\{U \subset X \mid A \subset U\ and \ U \in \lambda\}\) [15]. The following Lemma 4.1 gives the properties of the operator \(\wedge_\lambda\), the proof is similar to the corresponding result in [15].

Lemma 4.1. [15, Theorem 3.1] Let \((X, \mu)\) be a generalized space and \(A, B\) and \(C_i\) for \(i \in \Delta\) be subsets of \(M_\lambda\). Then the following hold.
(a) If $A \subset B$, then $\Lambda_\lambda(A) \subset \Lambda_\lambda(B)$.
(b) $A \subset \Lambda_\lambda(A)$.
(c) $\Lambda_\lambda(\Lambda_\lambda(A)) = \Lambda_\lambda(A)$.
(d) $\Lambda_\lambda(\cup \{C_i \mid i \in \Delta\}) = \cup \{\Lambda_\lambda(C_i) \mid i \in \Delta\}$.
(e) $\Lambda_\lambda(\cap \{C_i \mid i \in \Delta\}) \subset \cap \{\Lambda_\lambda(C_i) \mid i \in \Delta\}$.
(f) If $A \in \lambda$, then $\Lambda_\lambda(A) = A$.
(g) $\Lambda_\lambda(A) = \{x \in M_\lambda \mid c_\lambda(\{x\}) \cap A \neq \emptyset\}$.
(h) For every $x, y \in M_\lambda$, $y \in \Lambda_\lambda(\{x\})$ if and only if $x \in c_\lambda(\{y\}) \cap M_\lambda$.
(i) $\Lambda_\lambda(\{x\}) \neq \Lambda_\lambda(\{y\})$ if and only if $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$ for every $x, y \in M_\lambda$.

A space $(X, \lambda)$ is said to be a $\lambda-R_0$ space \cite{15,12,13} if every $\lambda-$open subset of $X$ contains the $\lambda-$closure of its singletons. $(X, \lambda)$ is said to be a $\lambda-R_1$ space \cite{15,12,13} if for $x, y \in X$ with $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$, there exist disjoint $\lambda-$open sets $G$ and $H$ such that $c_\lambda(\{x\}) \subset G$ and $c_\lambda(\{y\}) \subset H$. The results on generalized $R_0$ and $R_1$ spaces are independently established in \cite{15,12,13}. The space in Example 3.1 is neither $\lambda-R_0$ nor $\lambda-R_1$. Example 3.2 is $\lambda-R_0$, since each point is $\lambda-$closed but is not $\lambda-R_1$, since no disjoint $\lambda-$open sets exist. In particular, if a space is not strong, then it is neither $\lambda-R_0$ nor $\lambda-R_1$ (Refer Example 3.1). To rectify it, we redefine $R_0$ and $R_1$ spaces as follows.

A generalized space $(X, \lambda)$ is said to be a $\lambda^*-R_0$ space if for every $\lambda-$open subset $G$ of $M_\lambda$ and $x \in G$, $c_\lambda(\{x\}) \cap M_\lambda \subset G$. $(X, \lambda)$ is said to be a $\lambda^*-R_1$ space if for $x, y \in M_\lambda$ with $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$, there exist disjoint $\lambda-$open sets $G$ and $H$ such that $c_\lambda(\{x\}) \cap M_\lambda \subset G$ and $c_\lambda(\{y\}) \cap M_\lambda \subset H$. Clearly, for strong spaces, $\lambda^*-R_1$ spaces coincide with $\lambda-R_1$ spaces and every $\lambda^*-R_1$ space is a $\lambda^*-R_0$ space but the converse is not true (Refer to Example 3.2). Also, for $i = 1, 2, (X, \lambda)$ is $\lambda-R_i$ implies that $(X, \lambda)$ is $\lambda^*-R_i$. The following Example 4.1 shows that the converses are not true and it shows that non strong generalized spaces may be $\lambda^*-R_0$ and $\lambda^*-R_1$ spaces. Theorems in this section give characterizations of $\lambda^*-R_i, i = 1, 2$ generalized spaces which are true for both strong and non strong generalized spaces.

Example 4.1. Let $X = \{a, b, c\}$ and $\mu = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$. Since $c_\mu(\{a\}) = \{a, c\}$ and $c_\mu(\{b\}) = \{b, c\}$, it is easy to show that $(X, \mu)$ is neither $\mu-R_1$ nor $\mu-R_0$ but $(X, \mu)$ is both $\mu^*-R_1$ and $\mu^*-R_0$.

Theorem 4.1. For a generalized space $(X, \mu)$, the following are equivalent.
(a) $(X, \lambda)$ is $\lambda^*-R_0$.
(b) For each $\lambda-$closed set $F$ and $x \notin F$, there exists $U \in \lambda$ such that $F \cap M_\lambda \subset U$ and $x \notin U$.
(c) For every $\lambda-$closed set $F$ and $x \notin F$, $F \cap c_\lambda(\{x\}) = X - M_\lambda$.
(d) For any two distinct points $x, y \in M_\lambda$, either $c_\lambda(\{x\}) = c_\lambda(\{y\})$ or $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$.

Proof: (a)⇒(b). Let $F$ be a $\lambda-$closed set and $x \notin F$. Then by hypothesis, $c_\lambda(\{x\}) \cap M_\lambda \subset X - F$ and so $F \subset (X - c_\lambda(\{x\})) \cup (X - M_\lambda)$. Therefore, $F \cap M_\lambda \subset (X - c_\lambda(\{x\})) \cap M_\lambda \subset X - c_\lambda(\{x\})$. If $U = X - c_\lambda(\{x\})$, then $x \notin U$
and $U \in \lambda$ such that $F \cap M_\lambda \subset U$.

(b)$\Rightarrow$(c). Let $F$ be a $\lambda$–closed set and $x \notin F$. Then by hypothesis, there exists $U \in \lambda$ such that $x \notin U$ and $F \cap M_\lambda \subset U$, $x \notin U$ implies that $U \cap c_\lambda(\{x\}) = 0$ and so $(F \cap M_\lambda) \cap c_\lambda(\{x\}) = \emptyset$ which implies that $F \cap c_\lambda(\{x\}) = X - M_\lambda$.

(c)$\Rightarrow$(d). Let $x$, $y \in M_\lambda$ such that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. Then there exists $z \in c_\lambda(\{x\})$ such that $z \notin c_\lambda(\{y\})$. Then there exists $z \in V \in \lambda$ such that $y \notin V$ and $x \in V$. Hence $x \notin c_\lambda(\{y\})$. By hypothesis, $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$.

(d)$\Rightarrow$(a). Let $G$ be a $\lambda$–open set such that $x \in G$. If $y \notin G$, then $x \neq y$ and $x \notin c_\lambda(\{y\})$ which implies that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By hypothesis, $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$ and so $y \notin c_\lambda(\{x\}) \cap M_\lambda$. Hence $c_\lambda(\{x\}) \cap M_\lambda \subset G$ which implies that $(X, \lambda)$ is a $\lambda^* - R_0$ space.

**Theorem 4.2.** Let $(X, \mu)$ be generalized space. Then, $(X, \lambda)$ is a $\lambda^* - R_0$ space if and only if for $x$, $y \in M_\lambda$, $\lambda_\lambda(\{x\}) \neq \lambda_\lambda(\{y\})$ implies that $\lambda_\lambda(\{x\}) \cap \lambda_\lambda(\{y\}) = 0$.

**Proof:** Suppose $(X, \lambda)$ is a $\lambda^* - R_0$ space. Let $x$, $y \in M_\lambda$ such that $\lambda_\lambda(\{x\}) \neq \lambda_\lambda(\{y\})$. By Lemma 4.1(i), $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By Theorem 4.1, it follows that $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$. Let $z \in \lambda_\lambda(\{x\}) \cap \lambda_\lambda(\{y\})$. Then $z \in \lambda_\lambda(\{x\})$ and $z \in \lambda_\lambda(\{y\})$ and so by Lemma 4.1(h), $x \in c_\lambda(\{z\}) \cap M_\lambda$ and $y \in c_\lambda(\{z\}) \cap \lambda_\lambda(\{z\})$ implies that $x$, $y \in c_\lambda(\{z\})$. Therefore, $c_\lambda(\{x\}) \cup c_\lambda(\{y\}) \subset c_\lambda(\{z\})$. Now $x \in c_\lambda(\{z\}) \cap M_\lambda$ implies that $x \in c_\lambda(\{x\}) \cap M_\lambda$ and so $c_\lambda(\{x\}) \cap c_\lambda(\{z\}) \cap \lambda_\lambda(\{z\}) = \emptyset$. By Theorem 4.1(d), $c_\lambda(\{x\}) = c_\lambda(\{z\})$. Similarly, $c_\lambda(\{y\}) = c_\lambda(\{z\})$ and so $c_\lambda(\{x\}) = c_\lambda(\{y\})$, a contradiction. Therefore, $\lambda_\lambda(\{x\}) \cap \lambda_\lambda(\{y\}) = 0$. Conversely, suppose the condition holds. Let $x$, $y \in X$ such that $c_\lambda(\{x\}) \neq c_\lambda(\{y\})$. By Lemma 4.1(i), $\lambda_\lambda(\{x\}) \neq \lambda_\lambda(\{y\})$. By hypothesis, $\lambda_\lambda(\{x\}) \cap \lambda_\lambda(\{y\}) = 0$. We prove that $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$. Suppose $z \in M_\lambda$ such that $z \in c_\lambda(\{x\}) \cap c_\lambda(\{y\})$. Then $z \in c_\lambda(\{x\})$ and $z \in c_\lambda(\{y\})$. Now $z \in c_\lambda(\{x\})$ implies that $x \in \lambda_\lambda(\{z\})$ and $z \in \lambda_\lambda(\{x\}) \cap \lambda_\lambda(\{z\}) \neq 0$. Similarly, we can prove that $\lambda_\lambda(\{y\}) \cap \lambda_\lambda(\{z\}) \neq 0$. So by hypothesis, $c_\lambda(\{x\}) = c_\lambda(\{y\}) = c_\lambda(\{z\})$, a contradiction. Thus $c_\lambda(\{x\}) \cap c_\lambda(\{y\}) = X - M_\lambda$. By Theorem 4.1, $X$ is a $\lambda^* - R_0$ space.

**Theorem 4.3.** For a generalized space $(X, \mu)$, the following are equivalent.

(a) $(X, \lambda)$ is a $\lambda^* - R_0$ space.

(b) For any nonempty subset $A$ of $M_\lambda$ and a $\lambda$–open set $G$ such that $A \cap G \neq \emptyset$, there exists a $\lambda$–closed set $F$ such that $A \cap F \neq \emptyset$ and $F \cap M_\lambda \subset G$.

(c) If $G \neq \emptyset$ is $\lambda$–open, then $G = \cup \{F \cap M_\lambda \mid F \cap M_\lambda \subset G \text{ and } F \text{ is } \lambda$–closed$\}$.

(d) If $F$ is $\lambda$–closed, then $F = \cap \{G \cup (X - M_\lambda) \mid F \subset G \cup (X - M_\lambda) \text{ and } G \text{ is } \lambda$–open$\}$.

(e) For every $x \in M_\lambda$, $c_\lambda(\{x\}) \cap M_\lambda \subset \lambda_\lambda(\{x\})$.

**Proof:** (a)$\Rightarrow$(b). Suppose $(X, \lambda)$ is a $\lambda^* - R_0$ space. Let $A$ be a nonempty subset of $M_\lambda$ and $G$ be a $\lambda$–open set such that $A \cap G \neq \emptyset$. If $x \in A \cap G$, then $x \in G$ and so by hypothesis, $c_\lambda(\{x\}) \cap M_\lambda \subset G$. If $F = c_\lambda(\{x\})$, then $F$ is the required $\lambda$–closed set such that $A \cap F \neq \emptyset$ and $F \cap M_\lambda \subset G$.

(b)$\Rightarrow$(c). Let $G$ be $\lambda$–open. Clearly, $G \supset \cup \{F \cap M_\lambda \mid F \cap M_\lambda \subset G \text{ and } F \text{ is } \lambda$–open$\}$. $F$ is
λ−closed}. If \( x \in G \), then \( \{ x \} \cap G \neq \emptyset \) and so by (b), there is a \( \lambda \)-closed set \( F \) such that \( \{ x \} \cap \bar{F} \neq \emptyset \) and \( F \cap M_{\lambda} \subset G \) which implies that \( x \in \bigcup \{ F \cap M_{\lambda} \mid F \cap M_{\lambda} \subset G \text{ and } F \text{ is } \lambda \)-closed}]. Therefore, \( G \subset \bigcup \{ F \cap M_{\lambda} \mid F \cap M_{\lambda} \subset G \text{ and } F \text{ is } \lambda \)-closed}]. This completes the proof.

(c)⇒(d). Let \( F \) be \( \lambda \)-closed. By (c), \( X - F = \bigcup \{ K \cap M_{\lambda} \mid F \subset (X - K) \cup (X - M_{\lambda}) \) and \( K \) is \( \lambda \)-closed} and so \( F = \bigcap \{ (X - K) \cup (X - M_{\lambda}) \mid F \subset (X - K) \cup (X - M_{\lambda}) \) and \( X - K \) is \( \lambda \)-open} = \( \bigcap \{ G \cup (X - M_{\lambda}) \mid F \subset G \cup (X - M_{\lambda}) \) and \( G \) is \( \lambda \)-open}.

(d)⇒(e). Let \( x \in M_{\lambda} \). If \( y \notin \bigwedge_{\lambda} \{ \{ x \} \} \), then by Lemma 3.1(g), \( \{ x \} \cap c_{\lambda} \{ \{ y \} \} = \emptyset \). By (d), \( c_{\lambda} \{ \{ y \} \} = \bigcap \{ G \cup (X - M_{\lambda}) \mid c_{\lambda} \{ \{ y \} \} \subset G \cup (X - M_{\lambda}) \) and \( G \) is \( \lambda \)-open}.

Therefore, there is a \( \lambda \)-open set \( G \) such that \( c_{\lambda} \{ \{ y \} \} \subset G \cup (X - M_{\lambda}) \) and \( x \notin G \) which implies that \( y \notin c_{\lambda} \{ \{ x \} \} \). Therefore, \( c_{\lambda} \{ \{ x \} \} \subset \bigwedge_{\lambda} \{ \{ x \} \} \).

(e)⇒(a). Let \( G \) be a \( \lambda \)-open set such that \( x \in G \). If \( y \in c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} \), then by (e), \( y \in \bigwedge_{\lambda} \{ \{ x \} \} \). Since \( \bigwedge_{\lambda} \{ \{ x \} \} \subset \bigwedge_{\lambda} \{ \{ y \} \} \) and \( y \in \bigwedge_{\lambda} \{ \{ y \} \} \) implies that \( \bigwedge_{\lambda} \{ \{ y \} \} \subset \bigwedge_{\lambda} \{ \{ x \} \} \). Hence \( \bigwedge_{\lambda} \{ \{ x \} \} \cap M_{\lambda} = \bigwedge_{\lambda} \{ \{ x \} \} \).

Theorem 4.4. For a generalized space \((X, \mu)\), the following are equivalent.

(a) \((X, \lambda)\) is a \( \lambda^* - R_0 \) space.

(b) For every \( x \in M_{\lambda} \), \( c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} = \bigwedge_{\lambda} \{ \{ x \} \} \).

Proof: (a)⇒(b). Let \( x \in M_{\lambda} \). By Theorem 4.3, \( c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} \subset \bigwedge_{\lambda} \{ \{ x \} \} \). To prove the converse, assume that \( y \in \bigwedge_{\lambda} \{ \{ x \} \} \). By Lemma 4.1(b), \( x \in c_{\lambda} \{ \{ y \} \} \cap M_{\lambda} \) and so \( c_{\lambda} \{ \{ x \} \} \subset c_{\lambda} \{ \{ y \} \} \) which implies that \( c_{\lambda} \{ \{ x \} \} \cap c_{\lambda} \{ \{ y \} \} \neq X - M_{\lambda} \). By Theorem 4.1, \( c_{\lambda} \{ \{ x \} \} = c_{\lambda} \{ \{ y \} \} \) and so \( y \in c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} \). Hence \( c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} = \bigwedge_{\lambda} \{ \{ x \} \} \).

(b)⇒(a). The proof follows from Theorem 4.3.

Theorem 4.5. For a generalized space \((X, \mu)\), the following are equivalent.

(a) \((X, \lambda)\) is a \( \lambda^* - R_0 \) space.

(b) If \( F \) is a \( \lambda \)-closed set, then \( F \cap M_{\lambda} = \bigwedge_{\lambda} \{ F \cap M_{\lambda} \} \).

(c) If \( F \) is a \( \lambda \)-closed set and \( x \in F \cap M_{\lambda} \), then \( \bigwedge \{ \{ x \} \} \subset F \cap M_{\lambda} \).

(d) If \( x \in M_{\lambda} \), then \( \bigwedge \{ \{ x \} \} \subset c_{\lambda} \{ \{ x \} \} \cap M_{\lambda} \).

Proof: (a)⇒(b). If \((X, \lambda)\) is \( \lambda^* - R_0 \) and \( F \) is \( \lambda \)-closed, by Theorem 4.3, \( F = \bigcap \{ G \cup (X - M_{\lambda}) \mid F \subset G \cup (X - M_{\lambda}) \) and \( G \) is \( \lambda \)-open} and so \( F \cap M_{\lambda} = \bigwedge_{\lambda} \{ F \cap M_{\lambda} \} \).
intersection of \( F \cap M_\lambda \cap G \cap M_\lambda \subset G \) and \( G \) is \( \lambda \)-open = \( \wedge_\lambda (F - M_\lambda) \).

(b) \( \Rightarrow \) (c). Let \( z \in \wedge_\lambda (\{x\}) \). Then \( z \) is in every \( \lambda \)-open set containing \( x \). Since \( x \in F \cap M_\lambda \), \( x \) is in every \( \lambda \)-open set containing \( F \cap M_\lambda \) and so \( z \) is in every \( \lambda \)-open set containing \( F \cap M_\lambda \). Therefore, \( z \in \wedge_\lambda (F \cap M_\lambda) = F \cap M_\lambda \) and so \( \wedge (\{x\}) \subset F \cap M_\lambda \).

(c) \( \Rightarrow \) (d). The proof is clear.

(d) \( \Rightarrow \) (a). Let \( x \in c_\lambda(\{y\}) \cap M_\lambda \). By Lemma 4.1(b), \( y \in \wedge_\lambda (\{x\}) \) and so by hypothesis, \( y \in c_\lambda(\{x\}) \cap M_\lambda \). By Theorem 4.4, \( (X, \lambda) \) is a \( \lambda^* - R_0 \) space. \( \square \)

The following Theorem 4.6 gives a characterization of \( \lambda^* - R_1 \) space.

**Theorem 4.6.** For a generalized space \((X, \mu)\), the following are equivalent.

(a) \((X, \lambda)\) is a \( \lambda^* - R_1 \) space.

(b) For \( x, y \in M_\lambda \) such that \( \wedge_\lambda (\{x\}) \neq \wedge_\lambda (\{y\}) \), there exist disjoint \( \lambda \)-open sets \( G \) and \( H \) such that \( c_\lambda(\{x\}) \cap M_\lambda \subset G \) and \( c_\lambda(\{y\}) \cap M_\lambda \subset H \).

**Proof.** (a) \( \Rightarrow \) (b). Let \( x, y \in M_\lambda \) such that \( \wedge_\lambda (\{x\}) \neq \wedge_\lambda (\{y\}) \). Then, by Lemma 4.1(i), \( c_\lambda(\{x\}) \neq c_\lambda(\{y\}) \). Since \((X, \lambda)\) is a \( \lambda^* - R_1 \) space, there exist disjoint \( \lambda \)-open sets \( G \) and \( H \) such that \( c_\lambda(\{x\}) \cap M_\lambda \subset G \) and \( c_\lambda(\{y\}) \cap M_\lambda \subset H \).

(b) \( \Rightarrow \) (a). Let \( x, y \in M_\lambda \) such that \( c_\lambda(\{x\}) \neq c_\lambda(\{y\}) \). By Lemma 4.1(i), \( \wedge_\lambda (\{x\}) \neq \wedge_\lambda (\{y\}) \). By hypothesis, there exist disjoint \( \lambda \)-open sets \( G \) and \( H \) such that \( c_\lambda(\{x\}) \cap M_\lambda \subset G \) and \( c_\lambda(\{y\}) \cap M_\lambda \subset H \) and so \((X, \lambda)\) is a \( \lambda^* - R_1 \) space.

5. \( G_\mu \)-regular generalized spaces

In [11], \( \mu g \)-regular spaces are defined as follows. Let \((X, \tau)\) be a topological space and \( \mu \) be a generalized topology on \( X \). \((X, \tau)\) is said to be a \( \mu g \)-regular space, if for each closed set \( F \) and a point \( x \notin F \), there exist disjoint \( \mu \)-open sets \( U \) and \( V \) such that \( x \in U, F \subset V \). The space \((X, \tau)\) of Example 3.1 with the family of all generalized open sets \( \mu \), which is not strong, is not \( \mu g \)-regular and the space \((X, \tau)\) of Example 3.2 (resp. Example 3.3) with the family of all generalized open sets \( \mu \), which is strong, is also not \( \mu g \)-regular. Example 2.4(a) of [11] gives an example of a \( \mu g \)-regular space. A space \((X, \lambda)\) is said to be a \( \lambda \)-regular space [10], if for each \( x \in M_\lambda \) and \( \lambda \)-closed set \( F \) such that \( x \notin F \), there exist disjoint \( \lambda \)-open sets \( U \) and \( V \) such that \( x \in U, F \cap M_\lambda \subset V \). The space \((X, \mu)\) in Example 3.2 is not a \( \mu \)-regular space. Spaces \((X, \mu)\) in Examples 5.1(a) and (b) below are \( \mu \)-regular spaces. The following Lemma 5.1 is due to Min [10] where (c) follows from (b).

**Lemma 5.1.** Let \((X, \mu)\) be a generalized space. Then the following hold.

(a) \((X, \lambda)\) is \( \lambda \)-regular if and only if for each \( x \in M_\lambda \) and \( \lambda \)-open set \( U \) containing \( x \), there is a \( \lambda \)-open set \( V \) containing \( x \) such that \( x \in V \subset c_\lambda(V) \cap M_\lambda \subset U \) [10, Theorem 3.12].

(b) If \((X, \mu)\) is \( \mu \)-regular, then every \( \mu \)-open set is a \( \delta(\mu) \)-open set [10, Theorem 3.13].

(c) If \((X, \mu)\) is \( \mu \)-regular, then \( \alpha(\mu) = \nu(\delta), \sigma(\mu) = \xi(\delta), \pi(\mu) = \eta(\delta), b(\mu) = \varepsilon(\delta) \) and \( \beta(\mu) = \psi(\delta) \).
Let $X$ be a nonempty set and $\mu$ be a generalized topology on $X$. The space $(X, \mu)$ is said to be $g_{\mu} -$regular if for each pair consisting of a point $x \in M_\lambda$ and a $g_{\mu} -$closed set $F$ not containing $x$, there exist disjoint $\mu -$open sets $U$ and $V$ such that $x \in U$ and $F \subset V$. By Theorem 3.4(a), every $g_{\mu} -$regular space is a $\mu -$regular space and the following Example 5.1(b) shows that the converse is not true. Example 5.1(c) gives an example of a $g_{\lambda} -$regular space. Theorem 5.1 below gives a characterization of $g_{\lambda} -$regular spaces.

**Example 5.1.** (a) Let $X = \mathbb{R}$, the set of all real numbers and $\mathbb{Z}$ be the set of all integers. Then $\mu = \varphi (\mathbb{R} - \mathbb{Z})$ is a GT on $X$. Clearly, a subset $G$ of $X$ is $\mu -$open if and only if $G \subset \mathbb{R} - \mathbb{Z}$ and a subset $F$ of $X$ is $\mu -$closed if and only if $F \supset \mathbb{Z}$. Note that $X \notin \mu$, $c_\mu (A) = A \cup \mathbb{Z}$ for every subset $A$ of $X$. Then $(X, \mu)$ is $\mu -$regular.

(b) Let $X = \{a, b, c, d\}$ and $\mu = \{\emptyset, \{a\}, \{b, c\}, \{a, b, c\}\}$. The space $(X, \mu)$ is $\mu -$regular. If $A = \{a, c\}$, then $A$ is $g_{\mu} -$closed. Since $b$ and $A$ are not separated by disjoint $\mu -$open sets, $(X, \mu)$ is not $g_{\mu} -$regular.

(c) Consider the space $(X, \mu)$ of Example 3.5. Then $(X, \mu)$ is a $g_{\mu} -$regular space. Note that this space is not strong.

**Theorem 5.1.** Let $(X, \mu)$ be a generalized space. Then the following are equivalent.

(a) $(X, \lambda)$ is $g_{\lambda} -$regular.

(b) For each $g_{\lambda} -$open set $G$ and $x \in G$, there exists a $\lambda -$open set $U$ such that $x \in U \subset c_\lambda (U) \cap M_\lambda \subset G$.

**Proof:** (a)$\Rightarrow$(b) Suppose $(X, \lambda)$ is $g_{\lambda} -$regular. Let $G$ be a $g_{\lambda} -$open set containing $x$. Then $M_\lambda \setminus G$ is a $g_{\lambda} -$closed set such that $x \notin M_\lambda \setminus G$. By hypothesis, there exists disjoint $\lambda -$open sets $U$ and $V$ such that $x \in U$ and $M_\lambda \setminus G \subset V$. Since $U \cap V = \emptyset$, $c_\lambda (U) \cap V = \emptyset$ and so $c_\lambda (U) \cap M_\lambda \subset (X - V) \cap M_\lambda = M_\lambda \setminus V \subset G$. Thus, there exists a $\lambda -$open set $U$ such that $x \in U \subset c_\lambda (U) \cap M_\lambda \subset G$.

(b)$\Rightarrow$(a) Suppose the condition holds. Let $x \in X$ and $F$ be a $g_{\lambda} -$closed set such that $x \notin F$. Then $U = M_\lambda \setminus F$ is a $g_{\lambda} -$open set such that $x \in U$. By hypothesis, there exists a $\lambda -$open set $V$ such that $x \in V \subset c_\lambda (V) \cap M_\lambda \subset U$. Since $c_\lambda (V) \cap M_\lambda \subset U = M_\lambda \setminus F$, we have $F = M_\lambda \setminus (M_\lambda \setminus F) \subset M_\lambda \setminus c_\lambda (V) \cap M_\lambda = M_\lambda \setminus c_\lambda (V) = G$. Then $V$ and $G$ are the required $\lambda -$open sets such that $x \in V$ and $F \subset G$. Therefore, $(X, \lambda)$ is $g_{\lambda} -$regular. \[\square\]

The following Theorem 5.2 gives another characterization of $g_{\mu} -$regular spaces.

**Theorem 5.2.** Let $(X, \mu)$ be a generalized space. Then the following are equivalent.

(a) $(X, \lambda)$ is a $g_{\lambda} -$regular space.

(b) For each $g_{\lambda} -$closed set $F$ and $x \notin F$, there exists $\lambda -$open sets $U$ and $V$ such that $x \in U$, $F \subset V$ and $c_\lambda (U) \cap c_\lambda (V) = X - M_\lambda$.

**Proof:** (a)$\Rightarrow$(b). Let $F$ be a $g_{\lambda} -$closed set and $x \notin F$. Then there exists disjoint $\lambda -$open sets $U$ and $V$ such that $x \in U$ and $F \subset V$. Clearly, $(X - M_\lambda) \subset c_\lambda (U) \cap c_\lambda (V)$. Moreover, $c_\lambda (U) \cap c_\lambda (V) = (c_\lambda (U) \cap c_\lambda (V)) \cap M_\lambda \cup (X - M_\lambda)$, by Lemma 3.1(d) and so $c_\lambda (A) \cap c_\lambda (B) \supset ((U \cap V) \cap M_\lambda) \cup (X - M_\lambda) = \emptyset \cup (X - M_\lambda) = X - M_\lambda$. 

V. Pankajam and D. Sivaraj.
Hence \( c_\lambda(A) \cap c_\lambda(B) = X - M_\lambda \).

(b)\( \Rightarrow \) (a). Enough to prove that if \( A \) and \( B \) are \( \lambda \)-open set such that \( c_\lambda(A) \cap c_\lambda(B) = X - M_\lambda \), then \( A \cap B = \emptyset \). Now \( \emptyset = (X - M_\lambda) \cap M_\lambda = (c_\lambda(A) \cap c_\lambda(B)) \cap M_\lambda \supset (A \cap B) \cap M_\lambda = A \cap B \) and so \( A \cap B = \emptyset \). Therefore, the proof follows. \( \square \)

The following Lemma 5.2 follows from the definitions. Corollary 5.2A below follows from Theorem 5.2 and Lemma 5.2.

**Lemma 5.2.** Let \((X, \mu)\) be a generalized space. Then \((X, \lambda)\) is \(\lambda^* - R_0\) if and only if every point of \(M_\lambda\) is \(g_\lambda^*\)-closed.

**Corollary 5.2A.** Let \((X, \lambda)\) be an \(\lambda^* - R_0\), \(g_\lambda\)-regular space. Then the following hold.

(a) For distinct points \(x\) and \(y\) of \(M_\lambda\), there exist \(\lambda\)-open sets \(U\) and \(V\) such that \(x \in U\), \(y \in V\) and \(c_\lambda(U) \cap c_\lambda(V) = X - M_\lambda\).

(b) For distinct points \(x\) and \(y\) of \(M_\lambda\), there exist disjoint \(\lambda\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(y \in V\).

Let \(X\) be a nonempty set and \(\mu\) be a generalized topology on \(X\). A point \(x\) is said to be in the \(\theta\)-closure of \(A\) [6], denoted by \(c_{\theta(\mu)}(A)\), if \(A \cap c_{\mu}(U) \neq \emptyset\) for every \(x \in U \in \mu\). The following Theorem 5.3 gives characterizations of \(g_\lambda\)-regular spaces in terms of the \(\theta\)-closure operator.

**Theorem 5.3.** Let \(X\) be a nonempty set, \(\mu\) be a generalized topology on \(X\). Then the following are equivalent.

(a) \(X\) is a \(g_\lambda\)-regular space.

(b) \(c_{\theta(\lambda)}(A) \cap M_\lambda = \cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed for every subset \(A\) of \(M_\lambda\).

(c) \(c_{\theta(\lambda)}(A) \cap M_\lambda = A\) for every \(g_\lambda^*\)-closed set \(A\).

**Proof:** (a)\( \Rightarrow \) (b). Clearly, \(A \subset \cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed. We first prove that \(\cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed. Let \(x \in \cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed. Suppose \(x \notin c_{\theta(\lambda)}(A)\). Then there is a \(\lambda\)-open set \(U\) containing \(x\) such that \(A \cap c_\lambda(U) = \emptyset\) and so \(A \cap U = \emptyset\). Since \(X - U\) is a \(\lambda\)-closed set and hence a \(g_\lambda^*\)-closed set containing \(A\), \(x \in X - U\), a contradiction. Hence \(x \in c_{\theta(\lambda)}(A)\) which implies that \(\cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed \(\subset c_{\theta(\lambda)}(A)\). Conversely, suppose \(x \notin \cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed. Then, there exists a \(g_\lambda^*\)-closed set \(F\) such that \(A \subset F\) and \(x \in X - F\). Then there exists disjoint \(\lambda\)-open sets \(U\) and \(V\) such that \(x \in U \subset c_\lambda(U) \subset X - V \subset X - F \subset X - A\). Hence \(A \cap c_\lambda(U) = \emptyset\) which implies that \(x \notin c_{\theta(\lambda)}(A)\). Hence it follows that \(A \subset \cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed. Hence \(\cap \{F \mid A \subset F\} \cap g_\lambda^*\)-closed \(= c_{\theta(\lambda)}(A) \cap M_\lambda\).

(b)\( \Rightarrow \) (c). The proof is clear.

(c)\( \Rightarrow \) (a). Let \(F\) be a \(g_\lambda^*\)-closed set not containing \(x\). Then \(x \notin c_{\theta(\lambda)}(F)\). Then there exists a \(\lambda\)-open set \(U\) containing \(x\) such that \(F \cap c_\lambda(U) = \emptyset\). Then \(U\) and \(X - c_\lambda(U)\) are the required disjoint \(\lambda\)-open sets such that \(x \in U\) and \(F \subset X - c_\lambda(U)\). Therefore, \(X\) is a \(g_\lambda\)-regular space. \(\square\)
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