Solidity and some double sequence spaces

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ABSTRACT: In this paper we investigate the solidity (normality) of the sequence spaces $c^2_A$, $\ell^2_A$, $\Lambda^2_A$ and $\Gamma^2_A$.

Key Words: entire sequence, analytic sequence, double sequence.

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1. Introduction

Throughout $\omega, \Gamma$ and $\Lambda$ denote the classes of all, entire and analytic scalar valued single sequences respectively.

We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$ the set of positive integers. Then $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces is found in Bromwich[2]. Later on it was investigated by Hardy[3], Moricz[4], Moricz and Rhoades[5], Basarir and Solanku[1], Tripathy[6], Colak and Turkmenoglu[7], Turkmenoglu[8], and many others.

We need the following inequality in the sequel of the paper. For $a, b \geq 0$ and $0 < p < 1$, we have

$$ (a + b)^p \leq a^p + b^p \tag{1} $$

The double series $\sum_{m,n=1}^{\infty} x_{mn}$ is called convergent if and only if the double sequence $(s_{mn})$ is called convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij} (m, n = 1, 2, 3, \ldots)$ (see[9]). A sequence $x = (x_{mn})$ is said to be double analytic if

$$ \sup_{m>n} |x_{mn}|^{1/(m+n)} < \infty. $$

The vector space of all double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called double entire sequence if $|x_{mn}|^{1/(m+n)} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by $\Gamma^2$.

Let $\phi = \{ all\ finite\ sequences \}$. Consider a double sequence $x = (x_{ij})$. The $(m, n)^{th}$

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section $x^{[m,n]}$ of the sequence is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \delta_{ij}$ for all $m, n \in \mathbb{N}$,

\[
\delta_{mn} = \begin{pmatrix}
0, & 0, & \ldots, & 0, & \ldots \\
0, & 0, & \ldots, & 0, & \ldots \\
\vdots & \vdots & \ddots & \vdots & \ddots \\
0, & 0, & \ldots, 1, & 0, & \ldots \\
0, & 0, & \ldots, 0, & 0, & \ldots
\end{pmatrix}
\]

with 1 in the $(m, n)^{th}$ position and zero otherwise. An FK-space (or a metric space) $X$ is said to have AK property if $(\delta_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$. An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous. If $X$ is a sequence space, we give the following definitions:

(i) $X' = $ the continuous dual of $X$;

(ii) $X^\alpha = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \}$

(iii) $X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn}x_{mn} \text{ is convergent, for each } x \in X \}$

(iv) $X^\gamma = \{ a = (a_{mn}) : m, n \geq 1, \left( \sum_{m,n=1}^{M,N} a_{mn}x_{mn} \right) < \infty, \text{ for each } x \in X \}$;

(v) let $X$ be an FK - space $\supset \Phi$; then $X^f = \{ f(\delta_{mn}) : f \in X' \}$;

(vi) $X^\Lambda = \{ a = (a_{mn}) : \sup_{m,n} |a_{mn}x_{mn}|^{1/m+n} < \infty, \text{ for each } x \in X \}$;

\[X^\alpha, X^\beta, X^\gamma\] are called $\alpha = \text{(or Köthe – Toeplitz) dual of } X, \beta = \text{(or generalized – Köthe – Toeplitz) dual of } X, \gamma = \text{ dual of } X, \Lambda = \text{ dual of } X$ respectively.

### 2. Definitions and Preliminaries

Let $w^2$ denote the set of all complex double sequences. A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{m,n} |x_{mn}|^{1/m+n} < \infty$. The vector space of all prime sense double analytic sequences will be denoted by $\Lambda^2$. A sequence $x = (x_{mn})$ is called prime sense double entire sequence if $|x_{mn}|^{1/m+n} \to 0$ as $m, n \to \infty$. The double entire sequences will be denoted by $\Gamma^2$. The space $\Lambda^2$ and $\Gamma^2$ is a metric space with the metric

\[
d(x, y) = \sup_{m,n} \left\{ |x_{mn} - y_{mn}|^{1/m+n} : m, n : 1, 2, 3, \ldots \right\}
\]

for all $x = \{x_{mn}\}$ and $y = \{y_{mn}\} \in \Gamma^2$.

$c^2 = \text{the space of all double convergent sequences.}$

$\ell^2$ = the space of all sequences $x = \{x_{mn}\}$ such that $\sum_{m,n=1}^{\infty} |x_{mn}|$ converges.

Let $A = (a_{mn}^{jk}) (m, n, j, k = 1, 2, 3, \ldots)$ be an infinite matrix. Given a sequence $x = \{x_{mn}\}$ we write formally...
$$y_{mn} = A_{mn}(x) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} a_{mn}^{j} x_{mn} (j, k = 1, 2, \cdots)$$

The sequence \{\(y_{mn}\)\} = \{\(A_{mn}(x)\)\} will be denoted by \(Ax\) or \(y\). Let \(X\) be a sequence space and let \(X_A\) be the set of all those sequences \(x = \{x_{mn}\}\) for which \(Ax \in X\).

The set of all matrices transforming \(X\) into \(X\) will be denoted by \((X, X)\).

We recall the following:

A sequence space \(X\) is called solid (or normal) if and only if \(\Lambda^2_X \subset X\).

Any matrix in \((c^2, c^2)\) is called a conservative matrix. A conservative matrix which preserves the limit is said to be a Toeplitz matrix.

3. Main Results

**Proposition 3.1** If \(A\) is a conservative matrix, which fails to sum an analytic sequence, then \(c^2_A\) is not solid.

**Proof:** The constant sequence

$$e = \begin{pmatrix}
1, & 1, & \ldots, & 1, & 0, \\
1, & 1, & \ldots, & 1, & 0, \\
\vdots, & \vdots, & \ddots, & \vdots, & \vdots \\
1, & 1, & \ldots, & 1, & 0, \\
0, & 0, & \ldots, & 0, & 0, \\
\end{pmatrix}$$

is in \(c^2_A\) with 1 in the \((m, n)\)th position and zero otherwise.

By our hypothesis, there exists a analytic sequence \(b\) such that \(b \notin c^2_A\). That is \(b \cdot e \notin c^2_A\). Therefore, \(\Lambda^2 \cdot c^2_A \not\subset c^2_A\). Showing that \(c^2_A\) is not solid. This completes the proof. \(\square\)

**Corollary 3.2** If \(A\) is a Toeplitz matrix, then \(c^2_A\) is not solid.

**Proposition 3.3** If \(A \in (\ell^2, \ell^2)\), then \(\ell^2_A\) is in general not solid.

**Proof:** Let

$$A = \begin{pmatrix}
1, & 1, & 0, & 0, & \ldots, & 0, & 0, \\
0, & 0, & 1, & 1, & \ldots, & 0, & 0, \\
\vdots, & \vdots, & \ddots, & \ddots, & \ddots, & \vdots, & \vdots, \\
0, & 0, & \ldots, 1, & 1, & \ldots, & 0, & 0, \\
0, & 0, & \ldots, 0, & 0, & \ldots, & 0, & 0, \\
\end{pmatrix}$$

with 1 in the \((m, n)\)th position and 1 in the \((m+1, n+1)\)th position and zero other wise.
That

\[ a_{m,2n-1}a_{m,2n} - a_{3m-1,n}a_{2m,2n} = 1, (m, n = 1, 2, 3, \ldots) \]

\[ a_{m,2n}a_{m,2n} - a_{3m,n}a_{2m,2n} = 1, (m, n = 1, 2, 3, \ldots) \]

\[ a_{mn} = 0, \text{ Otherwise.} \]

Then \( \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |a_{mn}^j| = 1 \) for each fixed \( j, k \)

Showing that \( A \in (\ell^2, \ell^2) \) We note that

\[ x \in \ell^2_A \text{ if and only if } \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |x_{m,2n-1}x_{m,2n} - x_{3m-1,n}x_{2m,2n}| \text{ converges. Take} \]

\[ x = \begin{pmatrix} 1, & -1, & \ldots, & -1, & 0, & \ldots \\ 1, & -1, & \ldots, & -1, & 0, & \ldots \\ \vdots & & & & & \vdots \\ 1, & -1, & \ldots, & -1, & 0, & \ldots \\ 0, & 0, & \ldots, & 0, & 0, & \ldots \end{pmatrix} \]

so that \( x \in \ell^2_A \).

with 1 in the \( (m, n) \)th position and -1 in the up to \( (m+1, n+1) \)th, zero otherwise. Take

\[ b = x = \begin{pmatrix} 1, & -1, & \ldots, & -1, & 0, & \ldots \\ 1, & -1, & \ldots, & -1, & 0, & \ldots \\ \vdots & & & & & \vdots \\ 1, & -1, & \ldots, & -1, & 0, & \ldots \\ 0, & 0, & \ldots, & 0, & 0, & \ldots \end{pmatrix} \]

Then \( b \) is in \( \Lambda^2 \).

with 1 and -1 alternatively up to \( (m, n) \)th position and zero otherwise. Now

\[ y = bx = \begin{pmatrix} 1, & 1, & \ldots, & 1, & 0, & \ldots \\ 1, & 1, & \ldots, & 1, & 0, & \ldots \\ \vdots & & & & & \vdots \\ 1, & 1, & \ldots, & 1, & 0, & \ldots \\ 0, & 0, & \ldots, & 0, & 0, & \ldots \end{pmatrix} = e \]

with 1 up to \( (m, n) \)th position and zero other wise.
For $c$, we have
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} |y_{m,2n-1}y_{m,2n} + y_{3m-1,n}y_{2m,2n}| = 2 + 2 + \cdots
\]
which is a divergent series. Thus $\Lambda^2 \cdot \ell^2_A \subset \ell^2_A$. Hence $\ell^2_A$ is not solid. This completes the proof. \qed

**Proposition 3.4** If $A \in (\Lambda^2, \Lambda^2)$, then $\Lambda^2_A$ is in general, not solid.

**Proof:** Let
\[
A = \begin{pmatrix}
-1, & 1, & 0, & 0, & 0, & 0, & \ldots \\
0, & 0, & -1, & 1, & 0, & 0, & \ldots \\
\vdots & & & \ddots & & & \ddots \\
0, & 0, & \ldots -1, & 1, & 0, & 0, & \ldots \\
0, & 0, & \ldots 0, & 0, & 0, & 0, & \ldots \\
\end{pmatrix}
\]
with $-1$ in the $(m,n)^{th}$ position and $1$ in the $(m+1,n+1)^{th}$ position and zero otherwise.

Another words, $A = (a_{mn}^{jk})$ is defined by
\[
a_{m,2n-1}a_{m,2n} = a_{3m-1,n}a_{2m,2n} = 1, (m,n = 1, 2, 3, \ldots)
\]
\[
a_{m,2n}a_{m,2n} = a_{3m,n}a_{2m,2n} = 1, (m,n = 1, 2, 3, \ldots)
\]
\[
a_{mn}^{jk} = 0, \text{ Otherwise.}
\]

Then
\[
\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| a_{mn}^{jk} \right|^{1/m+n} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \left| (a_{m,2n-1}a_{m,2n} - a_{3m-1,n}a_{2m,2n}) \right| + \left| (a_{m,2n}a_{m,2n} - a_{3m,n}a_{2m,2n}) \right| = 2 \text{ for each fixed } j,k.
\]

Consequently, $A \in (\Lambda^2, \Lambda^2)$.

Note that $x \in \Lambda^2_A$ if and only if
\[
Ax = \begin{pmatrix}
-a_{11} + a_{12}, & -a_{13} + a_{14}, & \ldots \\
-a_{21} + a_{22}, & -a_{23} + a_{24}, & \ldots \\
\vdots & & \ddots \\
\end{pmatrix} \in \Lambda^2
\]
we take
\[ A = \begin{pmatrix} 1, & 2, & 3, & 4, & \ldots \\ 1, & 2, & 3, & 4, & \ldots \\ \vdots \\ \vdots \\ \end{pmatrix} \text{ so that} \]

\[ Ax = \begin{pmatrix} 1, & 1, & \ldots, & 1, & 0, \ldots \\ 1, & 1, & \ldots, & 1, & 0, \ldots \\ \vdots \\ \vdots \\ 1, & 1, & \ldots, & 1, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \end{pmatrix} \text{ and } x \in \Lambda^2_A. \]

\[ b = \begin{pmatrix} -1, & 1, & \ldots, & -1, & 1, & 0, \ldots \\ -1, & 1, & \ldots, & -1, & 1, & 0, \ldots \\ \vdots \\ \vdots \\ -1, & 1, & \ldots, & -1, & 1, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \end{pmatrix} \]

Then

\[ bx = \begin{pmatrix} 0, & 0, & \ldots, & 0, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \\ \vdots \\ \vdots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \end{pmatrix} \]

and

\[ A(bx) = \begin{pmatrix} 0, & 0, & \ldots, & 0, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \\ \vdots \\ \vdots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \\ 0, & 0, & \ldots, & 0, & 0, \ldots \end{pmatrix} \notin \Lambda^2. \]

Thus \( \Lambda^2 \cdot \Lambda^2_A \not\subseteq \Lambda^2_A \). Hence \( \Lambda^2_A \) is not a solid space. This completes the proof. \( \Box \)

**Proposition 3.5** If \( A \in (\Gamma^2, \Gamma^2) \), the \( \Gamma_A^2 \) is not necessarily solid.
Proof: Let

\[
A = \begin{pmatrix}
-1, & 1, & 0, & 0, & 0, & 0, & ... \\
0, & 0, & -1, & 1, & 0, & 0, & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0, & 0, & \ldots, -1, & 1, & 0, & 0, & ... \\
0, & 0, & \ldots, 0, & 0, & 0, & 0, & ...
\end{pmatrix}
\]

writing \( \{t_{mn}\} \) for the transform of \( \{x_{mn}\} \), so that

\[
t_{mn} = -(x_{m,2n-1}x_{m,2n} - x_{3m-1,n}x_{2m,2n}) + (x_{m,2n}x_{m,2n} - x_{3m,n}x_{2m,2n}), \quad (m, n = 1, 2, 3, \ldots)
\]

We can verify directly that

\[
|x_{mn}|^{1/m+n} \to 0 \Rightarrow |t_{mn}|^{1/m+n} \to 0 \quad (m, n \to \infty)
\]

For if \( \eta < 1 \), then \( |a + b|^\eta < |a|^\eta + |b|^\eta \) so that

\[
|t_{mn}|^{1/m+n} < |x_{m,2n-1}x_{m,2n} - x_{3m-1,n}x_{2m,2n}|^{1/m+n} + |x_{m,2n}x_{m,2n} - x_{3m,n}x_{2m,2n}|^{1/m+n}
\]

Since \( |x_{mn}|^{1/m+n} \to 0 \quad (m, n \to \infty) \), we have \( |x_{mn}| < 1 \) for sufficiently large \( m, n \).

Supposing that \( m, n \) is large enough for

\[
|x_{m,2n-1}x_{m,2n} - x_{3m-1,n}x_{2m,2n}| < 1, \quad |x_{m,2n}x_{m,2n} - x_{3m,n}x_{2m,2n}| < 1.
\]

Hence if \( |x_{mn}|^{1/m+n} \to 0 \), then \( |t_{mn}|^{1/m+n} \to 0 \quad (m, n \to \infty) \).

It is now trivial that

\[
\begin{pmatrix}
1, & 1, & \ldots, 1, & 1, & 0, & 0, & ... \\
1, & 1, & \ldots, 1, & 1, & 0, & 0, & ... \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
1, & 1, & \ldots, 1, & 1, & 0, & 0, & ... \\
0, & 0, & \ldots, 0, & 0, & 0, & 0, & ...
\end{pmatrix}
\]
belongs to $\Gamma^2_A$ but
\[
\begin{pmatrix}
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
\cdot & & & & & & \\
\cdot & & & & & & \\
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
0, & 0, & \ldots & 0, & 0, & 0, & \ldots
\end{pmatrix}
\]
does not.

Here we have
\[
b = \begin{pmatrix}
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
\cdot & & & & & & \\
\cdot & & & & & & \\
-1, & 1, & \ldots & -1, & 1, & 0, & \ldots \\
0, & 0, & \ldots & 0, & 0, & 0, & \ldots
\end{pmatrix}
\]
in $\Lambda^2$

So $\Gamma^2_A$ is not solid. This completes the proof.

References


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