Some New Properties of $b$-closed spaces

N. Rajesh

Abstract: In [5], the authors introduced the notion of $b$-closed spaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

Key Words: Topological spaces, $b$-open sets, $b$-$\theta$-open sets.

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1. Introduction and Preliminaries

Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of continuity, separation axioms etc. by utilizing generalized open sets. For a subset $A$ of a topological space $(X, \tau)$, $\text{Cl}(A)$ and $\text{Int}(A)$ denote the closure of $A$ and the interior of $A$, respectively. A subset $A$ of a topological space $(X, \tau)$ is called a $b$-open [1] (= $\gamma$-open [4]) set if $A \subset \text{Int}(\text{Cl}(A)) \cup \text{Cl}(\text{Int}(A))$. The complement of a $b$-open set is called a $b$-closed set. The intersection of all $b$-closed sets of $X$ containing $A$ is called the $b$-closure [1] of $A$ and is denoted by $b\text{Cl}(A)$. For each $x \in X$, the family of all $b$-open sets of $(X, \tau)$ containing a point $x$ is denoted by $BO(X, x)$. The $b$-interior of $A$ is the union of all $b$-open sets contained in $A$ and is denoted by $b\text{Int}(A)$. A set $A$ is called a $b$-regular set [5] if it is both $b$-open and $b$-closed. The $b$-$\theta$-closure [5] of a subset $A$, denoted by $b\text{Cl}_{\theta}(A)$, is the set of all $x \in X$ such that $b\text{Cl}(U) \cap A \neq \emptyset$ for every $U \in BO(X, x)$. A subset $A$ is called $b$-$\theta$-closed [5] if $A = b\text{Cl}_{\theta}(A)$. By [5], it is proved that, for a subset $A$, $b\text{Cl}_{\theta}(A)$ is the intersection of all $b$-$\theta$-closed sets containing $A$. The complement of a $b$-$\theta$-closed set is called a $b$-$\theta$-open set. In [5], the authors introduced the notion of $b$-closed spaces and investigated its fundamental properties. In this paper, we investigate some more properties of this type of closed spaces.

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2. $b(\theta)$-convergence and $b(\theta)$-adherence

**Definition 2.1** [2] A grill $\mathcal{G}$ on a topological space $X$ is defined to be a collection of nonempty subsets of $X$ such that (i) $A \in \mathcal{G}$ and $A \subseteq B \subseteq X \Rightarrow B \in \mathcal{G}$ and (ii) $A, B \subseteq X$ and $A \cup B \in \mathcal{G} \Rightarrow A \in \mathcal{G}$ or $B \in \mathcal{G}$.

**Definition 2.2** A grill $\mathcal{G}$ on a topological space $X$ is said to be:

(i) $b(\theta)$-adhere at $x \in X$ if for each $U \in BO(X, x)$ and $G \in \mathcal{G}$, $b\text{Cl}(U) \cap G \neq \emptyset$.

(ii) $b(\theta)$-converge to a point $x \in X$ if for each $U \in BO(X, x)$, there is some $G \in \mathcal{G}$, such that $G \subseteq b\text{Cl}(U)$.

**Remark 2.3** A grill $\mathcal{G}$ is $b(\theta)$-convergent to a point $x \in X$ if and only if $\mathcal{G}$ contains the collection $\{b\text{Cl}(U) : U \in BO(X, x)\}$.

**Definition 2.4** A filter $\mathcal{F}$ on a topological space $X$ is said to be $b(\theta)$-adhere at $x \in X$ (resp. $b(\theta)$-converge to $x \in X$) if for each $F \in \mathcal{F}$ and each $U \in BO(X, x), F \cap b\text{Cl}(U) \neq \emptyset$ (resp. to each $U \in BO(X, x)$, there corresponds $F \in \mathcal{F}$ such that $F \subseteq b\text{Cl}(U)$).

**Definition 2.5** [6] If $\mathcal{G}$ is a grill (or a filter) on a topological space $X$, then the section of $\mathcal{G}$, denoted by $\text{sec}\mathcal{G}$, is given by,

$$\text{sec}\mathcal{G} = \{A \subseteq X : A \cap G \neq \emptyset \text{ for all } G \in \mathcal{G}\}.$$

**Theorem 2.6** [6] Let $X$ be a topological space. Then we have

(i) For any grill (filter) $\mathcal{G}$ on $X$, $\text{sec}\mathcal{G}$ is a filter (resp. grill) on $X$.

(ii) If $\mathcal{F}$ and $\mathcal{G}$ are respectively a filter and a grill on $X$ with $\mathcal{F} \subseteq \mathcal{G}$, then there is an ultrafilter $\mathcal{U}$ on $X$ such that $\mathcal{F} \subseteq \mathcal{U} \subseteq \mathcal{G}$.

**Theorem 2.7** If a grill $\mathcal{G}$ on a topological space $X$, $b(\theta)$-adheres at some point $x \in X$, then $\mathcal{G}$ is $b(\theta)$-converges to $x$.

**Proof:** Let a grill $\mathcal{G}$ on $X$, $b(\theta)$-adheres at some point $x \in X$. Then for each $U \in BO(X, x)$ and each $G \in \mathcal{G}$, $b\text{Cl}(U) \cap G \neq \emptyset$ so that $b\text{Cl}(U) \in \text{sec}\mathcal{G}$ for each $U \in BO(X, x)$, and hence $X \setminus b\text{Cl}(U) \notin \mathcal{G}$. Then $b\text{Cl}(U) \in \mathcal{G}$ (as $\mathcal{G}$ is a grill and $X \in \mathcal{G}$) for each $U \in BO(X)$. Hence $\mathcal{G}$ must $b(\theta)$-converge to $x$. \(\square\)

**Example 2.8** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{b\}, \{c\}, \{b, c\}, \{a, c\}, X\}$ and $\mathcal{G} = \{\{b\}, \{c\}, \{a, b\}, \{b, c\}, \{a, c\}, X\}$. Then the grill $\mathcal{G}$ is $b(\theta)$-convergent but not $b(\theta)$-adheres.

**Definition 2.9** Let $X$ be a topological space. Then for any $x \in X$, we adopt the following notation:

$\mathcal{G}_{(b(\theta), x)} = \{A \subseteq X : x \in b\text{Cl}(A)\}$,

$\text{sec}\mathcal{G}_{(b(\theta), x)} = \{A \subseteq X : A \cap G \neq \emptyset, \text{ for all } G \in \mathcal{G}_{(b(\theta), x)}\}$.

**Theorem 2.10** A grill $\mathcal{G}$ on a topological space $X$, $b(\theta)$-adheres to a point $x$ of $X$ if and only if $\mathcal{G} \subseteq \text{sec}\mathcal{G}_{(b(\theta), x)}$. 
Proof: A grill $\mathcal{G}$ on a topological space $X$, $b(\theta)$-adheres to a point $x$ of $X$, we have $b\text{Cl}(U) \cap G \neq \emptyset$ for all $U \in BO(X, x)$ and all $G \in \mathcal{G}$; hence $b\text{Cl}_G(G)$ for all $G \in \mathcal{G}$. Then $G \in \mathcal{G}(b(\theta), x)$, for all $G \in \mathcal{G}$; hence $\mathcal{G} \subseteq \mathcal{G}(b(\theta), x)$. Conversely, let $\mathcal{G} \subseteq \mathcal{G}(b(\theta), x)$. Then for all $G \in \mathcal{G}$, $b\text{Cl}(U) \cap G \neq \emptyset$, so that for all $U \in BO(X, x)$ and for all $G \in \mathcal{G}$, $b\text{Cl}(U) \cap G \neq \emptyset$. Hence $\mathcal{G}$ $b(\theta)$-adheres at $x$. \hfill $\Box$

**Theorem 2.11** A grill $\mathcal{G}$ on a topological space $X$, $b(\theta)$-convergent to a point $x$ of $X$ if and only if $\text{sec}(b(\theta), x) \subseteq \mathcal{G}$.

**Proof:** Let $\mathcal{G}$ be a grill on a topological space $X$, $b(\theta)$-convergent to a point $x \in X$. Then for each $U \in BO(X, x)$ there exists $G \in \mathcal{G}$ such that $G \subseteq b\text{Cl}(U)$, and hence $b\text{Cl}(U) \subseteq \mathcal{G}$ for each $U \in BO(X, x)$. Now, $B \in \text{sec}(b(\theta), x) \Rightarrow X \setminus B \notin \mathcal{G}(b(\theta), x) \Rightarrow x \notin b\text{Cl}(X \setminus B) \Rightarrow$ there exists $U \in BO(X, x)$ such that $b\text{Cl}(U) \cap (X \setminus B) = \emptyset \Rightarrow b\text{Cl}(U) \subseteq B$, where $U \in BO(X, x) \Rightarrow B \in \mathcal{G}$. Conversely, let if possible, $\mathcal{G}$ not $b(\theta)$-convergent to $x$. Then for some $U \in BO(X, x)$, $b\text{Cl}(U) \notin \mathcal{G}$ and hence $b\text{Cl}(U) \notin \text{sec}(b(\theta), x)$. Thus for some $A \in \mathcal{G}(b(\theta), x)$, $A \cap b\text{Cl}(U) = \emptyset$. But $A \in \mathcal{G}(b(\theta), x) \Rightarrow x \in b\text{Cl}(A) \Rightarrow b\text{Cl}(A) \cap U \neq \emptyset$. \hfill $\Box$

3. $b$-closedness and grills

**Definition 3.1** A nonempty subset $A$ of a topological space $X$ is called $b$-closed relative to $X$ [5] if for every cover $\mathcal{U}$ of $A$ by $b$-open sets of $X$, there exists a finite subset $\mathcal{U}_0$ of $\mathcal{U}$ such that $A \subseteq \cup \{b\text{Cl}(U) : U \in \mathcal{U}_0\}$. If, in addition, $A = X$, then $X$ is called a $b$-closed space.

**Theorem 3.2** For a topological space $X$, the following statements are equivalent:

(i) $X$ is $b$-closed;

(ii) Every maximal filter base $b(\theta)$-converges to some point of $X$;

(iii) Every filter base $b(\theta)$-adhere to some point of $X$;

(iv) For every family $\{V_\alpha : \alpha \in I\}$ of $b$-closed sets that $\cap \{V_i : i \in I\} = \emptyset$, there exists a finite subset $I_0$ of $I$ such that $\cap \{b\text{Int}(V_i) : i \in I_0\} = \emptyset$.

**Proof:** (i) $\Rightarrow$ (ii): Let $\mathcal{F}$ be a maximal filter base on $X$. Suppose that $\mathcal{F}$ does not $b$-converge to any point of $X$. Since $\mathcal{F}$ is maximal, $\mathcal{F}$ does not $b$-$\theta$-accumulate at any point of $X$. For each $x \in X$, there exist $F_x \in \mathcal{F}$ and $V_x \in BO(X, x)$ such that $b\text{Cl}(V_x) \cap F_x = \emptyset$. The family $\{V_x : x \in X\}$ is a cover of $X$ by $b$-open sets of $X$. By (i), there exists a finite number of points $x_1, x_2, \ldots, x_n$ of $X$ such that $X = \cup \{b\text{Cl}(V_x) : i = 1, 2, \ldots, n\}$. Since $\mathcal{F}$ is a filter base on $X$, there exists $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \cap \{F_x : i = 1, 2, \ldots, n\}$. Therefore, we obtain $F_0 = \emptyset$. This is a contradiction. (ii) $\Rightarrow$ (iii): Let $\mathcal{F}$ be any filter base on $X$. Then, there exists a maximal filter base $\mathcal{F}_0$ such that $\mathcal{F} \subseteq \mathcal{F}_0$. By (ii), $\mathcal{F}_0$ $b$-$\theta$-converges to some point $x \in X$. For every $F \in \mathcal{F}$ and every $V \in BO(X, x)$, there exists $F_0 \in \mathcal{F}_0$.
such that \( F_0 \subseteq b\text{Cl}(V) \); hence \( \emptyset \neq F_0 \cap F \subseteq b\text{Cl}(V) \cap F \). This shows that \( \mathcal{F} \) does not \( b\theta \)-accumulate at any point. \( \Box \)

**Theorem 3.3** A topological space \( X \) is \( b\theta \)-closed if and only if every grill on \( X \) is \( b(\theta) \)-convergent in \( X \).

**Proof:** Let \( \mathcal{G} \) be any grill on a \( b\theta \)-closed space \( X \). Then by Theorem 2.6, sec\( \mathcal{G} \) is a filter on \( X \). Let \( B \in \text{sec}\mathcal{G} \), then \( X \setminus B \notin \mathcal{G} \) and hence \( B \in \mathcal{G} \) (as \( \mathcal{G} \) is a grill). Thus sec\( \mathcal{G} \) is a filter on \( X \). Then by Theorem 2.6(ii), there exists an ultrafilter \( \mathcal{U} \) on \( X \) such that sec\( \mathcal{G} \) is not a subset of \( \mathcal{U} \) and \( \mathcal{U} \subseteq \mathcal{G} \). Now as \( X \) is \( b\theta \)-closed, in view of Theorem 3.2, the ultrafilter \( \mathcal{U} \) is \( b\theta \)-convergent to some point \( x \in X \). Then for each \( U \in \text{BO}(X,x) \), there exists \( F \in \mathcal{U} \) such that \( F \subseteq \text{Cl}(U) \). Consequently, \( \text{bCl}(U) \in \mathcal{U} \subseteq \mathcal{G} \). That is \( \text{bCl}(U) \in \mathcal{G} \), for each \( U \in \text{BO}(X,x) \). Hence \( \mathcal{G} \) is \( b\theta \)-convergent to \( x \). Conversely, if every grill on \( X \) be \( b\theta \)-convergent to some point of \( X \). By virtue of Theorem 3.2 it is enough to show that every ultrafilter on \( X \) is \( b\theta \)-converges in \( X \), which is immediate from the fact that an ultrafilter on \( X \) is also a grill on \( X \). \( \Box \)

**Theorem 3.4** A topological space \( X \) is \( b\theta \)-closed relative to \( X \) if and only if every grill \( \mathcal{G} \) on \( X \) with \( A \in \mathcal{G} \), \( b\theta \)-converges to a point in \( A \).

**Proof:** Let \( A \) be \( b\theta \)-closed relative to \( X \) and \( \mathcal{G} \) a grill on \( X \) satisfying \( A \in \mathcal{G} \) such that \( \mathcal{G} \) does not \( b\theta \)-converges to any point in \( A \). Then to each \( a \in A \), there corresponds some \( U_a \in \text{BO}(X,a) \) such that \( \text{bCl}(U_a) \notin \mathcal{G} \). Now \( \{U_a : a \in A \} \) is a cover of \( A \) by \( b\theta \)-open sets of \( X \). Then for each \( a \in A \), let \( U_a \) be \( b\theta \)-open set of \( X \). Then \( A \subseteq \bigcup_{i=1}^{n} \text{Cl}(U_n) = U \) (say) for some positive integer \( n \). Since \( \mathcal{G} \) is a grill, \( U \notin \mathcal{G} \); hence \( A \notin \mathcal{G} \), which is a contradiction. Consequently, let \( A \) be not \( b\theta \)-closed relative to \( X \). Then for some cover \( \mathcal{U} = \{U_n : a \in I \} \) of \( A \) by \( b\theta \)-open sets of \( X \), \( \mathcal{F} = \{A \setminus \bigcup_{a \in I} \text{bCl}(U_a) : I_0 \text{ is finite subset of } I \} \) is a filterbase on \( X \). Then the family \( \mathcal{F} \) can be extended to an ultrafilter \( \mathcal{F}^{*} \) on \( X \). Then \( \mathcal{F}^{*} \) is a grill on \( X \) with \( A \in \mathcal{F}^{*} \) (as each \( F \) of \( \mathcal{F} \) is a subset of \( A \)). Now for each \( x \in A \), there must exists \( \beta \in I \) such that \( x \in U_\beta \), as \( \mathcal{U} \) is a cover of \( A \). Then for any \( G \in \mathcal{F}^{*} \), \( G \cap (A \setminus \text{bCl}(U_\beta)) \neq \emptyset \), so that \( G \supset \text{bCl}(U_\beta) \) for all \( G \in \mathcal{G} \). Hence \( \mathcal{F}^{*} \) cannot \( b\theta \)-converges to any point of \( A \). The contradiction proves the desired result. \( \Box \)
Theorem 3.5 If $X$ is any topological space such that every grill $\mathcal{G}$ on $X$ with the property that $\bigcap_{i=1}^{n} b\text{Cl}_{b}(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in $X$, then $X$ is a $b$-closed space.

**Proof:** Let $\mathcal{U}$ be an ultrafilter on $X$. Then $\mathcal{U}$ is a grill on $X$ and also for each finite subcollection $\{U_1, U_2, \ldots, U_n\}$ of $\mathcal{U}$, $\bigcap_{i=1}^{n} b\text{Cl}_{b}(U_i) \supseteq \bigcap_{i=1}^{n} U_i \neq \emptyset$, so that $\mathcal{U}$ is a grill on $X$ with the given condition. Hence by hypothesis, $\mathcal{U}$, $b(\theta)$-adheres. Consequently, by Theorem 3.2, $X$ is $b$-closed.

**Theorem 3.6** [5] For any $A \subseteq X$, $b\text{Cl}_{b}(A) = \cap\{b\text{Cl} : A \subseteq U \in BO(X)\}$.

**Definition 3.7** A grill $\mathcal{G}$ on a topological space $X$ is said to be:

(a) $b(\theta)$-linked if for any two members $A, B \in \mathcal{G}$, $b\text{Cl}_{b}(A) \cap b\text{Cl}_{b}(B) \neq \emptyset$,

(b) $b(\theta)$-conjoint if for every finite subfamily $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$, $b\text{Int}(\bigcap_{i=1}^{n} b\text{Cl}_{b}(A_i)) \neq \emptyset$.

It is clear that every $b(\theta)$-conjoint grill is $b(\theta)$-linked. The following example shows that the converse is need not be true in general.

**Example 3.8** Let $X = \{a, b, c\}$, $\tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, X\}$ and $\mathcal{G} = \{\{c\}, \{b, c\}, \{a, c\}, X\}$. Then the grill $\mathcal{G}$ is $b(\theta)$-linked but not $b(\theta)$-conjoint.

**Theorem 3.9** In a $b$-closed space $X$, every $b(\theta)$-conjoint grill $b(\theta)$-adheres in $X$.

**Proof:** Consider any $b(\theta)$-conjoint grill $\mathcal{G}$ on a $b$-closed space $X$. We first note from Theorem 3.5 that for $A \subseteq X$, $b\text{Cl}_{b}(A)$ is $b$-closed (as an arbitrary intersection of $b$-closed sets is $b$-closed). Thus $\{b\text{Cl}_{b}(A) : A \in \mathcal{G}\}$ is a collection of $b$-closed sets in $X$ such that $b\text{Int}(\bigcap_{i=1}^{n} b\text{Cl}_{b}(A_i)) \neq \emptyset$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Then $b\text{Int}(\bigcap_{i=1}^{n} (b\text{Cl}_{b}(A_i))) \neq \emptyset$ for any finite subcollection $A_1, A_2, \ldots, A_n$ of $\mathcal{G}$. Thus by Theorem 3.2, $\cap_{A \in \mathcal{G}} \{b\text{Cl}_{b}(A) : A \in \mathcal{G}\} \neq \emptyset$. That is there exists $x \in X$ such that $x \in b\text{Cl}_{b}(A)$ for all $A \in \mathcal{G}$. Hence $\mathcal{G} \subseteq \mathcal{G}(b(\theta), x)$ so that by Theorem 2.10, $\mathcal{G}$, $b(\theta)$-adheres at $x \in X$.

**Definition 3.10** A subset $A$ of a topological space $X$ is called $b$-regular open if $A = b\text{Int}(b\text{Cl}(A))$. The complement a $b$-regular open set is called a $b$-regular closed set.

**Definition 3.11** A topological space $X$ is called $b$-almost regular if for each $x \in X$ and each $b$-regular open set $V$ in $X$ with $x \in V$, there is a $b$-regular open set $U$ in $X$ such that $x \in U \subseteq b\text{Cl}(U) \subseteq V$.

**Theorem 3.12** In a $b$-almost regular $b$-closed space $X$, every grill $\mathcal{G}$ on $X$ with the property $\bigcap_{i=1}^{n} b\text{Cl}_{b}(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, \ldots, G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in $X$. 

Proof: Let $X$ be a $b$-almost regular $b$-closed space and $\mathcal{G} = \{G_\alpha : \alpha \in I\}$ a grill on $X$ with the property that $\bigcap_{\alpha \in I_0} b \text{Cl}_b(G_\alpha) \neq \emptyset$ for every finite subset $I_0$ of $I$. We consider $\mathcal{F} = \{\bigcap_{\alpha \in I_0} b \text{Cl}_b(G_\alpha) : I_0$ is a finite subfamily of $I\}$. Then $\mathcal{F}$ is a filterbase on $X$. By the $b$-closedness of $X$, $\mathcal{F}$, $b(\theta)$-adheres at some $x \in X$, that is, $x \in b \text{Cl}_b(b \text{Cl}_b(G))$ for all $G \in \mathcal{G}$, that is, $\mathcal{G} \subseteq (b(\theta), x)$. Hence by Theorem 2.10, $\mathcal{G}$ $b(\theta)$-adheres at $x \in X$.

Corollary 3.13 In a $b$-almost regular space $X$, the following statements are equivalent:

(i) Every grill $\mathcal{G}$ on $X$ with the property that $\bigcap_{i=1}^n b \text{Cl}_b(G_i) \neq \emptyset$ for every finite subfamily $\{G_1, G_2, ..., G_n\}$ of $\mathcal{G}$, $b(\theta)$-adheres in $X$.

(ii) $X$ is $b$-closed.

(iii) Every $b(\theta)$-conjugate grill $b(\theta)$-adheres in $X$.

Theorem 3.14 Every grill $\mathcal{G}$ on a topological space $X$ with the property that $\bigcap \{b \text{Cl}_b(G) : G \in \mathcal{G}_0\} \neq \emptyset$ for every finite subsets $\mathcal{G}_0$ of $\mathcal{G}$, $b(\theta)$-adheres in $X$ if and only if for every family $\mathcal{F}$ of subsets of $X$ for which the family $\{b \text{Cl}_b(F) : F \in \mathcal{F}\}$ has the finite intersection property, we have $\bigcap \{b \text{Cl}_b(F) : F \in \mathcal{F}\} \neq \emptyset$.

Proof: Let every grill on a topological space $X$ satisfying the given condition, $b(\theta)$-adhere in $X$, and suppose that $\mathcal{F}$ is a family of subsets of $X$ such that the family $\mathcal{G}^* = \{b \text{Cl}_b(F) : F \in \mathcal{F}\}$ has the finite intersection property. Let $\mathcal{U}$ be the collection of all those families $\mathcal{G}$ of subsets of $X$ for which $\mathcal{G}^* = \{b \text{Cl}_b(G) : G \}$ has the finite intersection property and $\mathcal{F} \subseteq \mathcal{G}$. Then $\mathcal{F} \in \mathcal{U}$ is a partially ordered set under set inclusion in which every chain clearly has an upper bound. By Zorn’s lemma, $\mathcal{F}$ is then contained in a maximal family $\mathcal{U}^* \in \mathcal{U}$. It is easy to verify that $\mathcal{U}^*$ is a grill with the stipulated property. Hence $\bigcap \{b \text{Cl}_b(F) : F \in \mathcal{F}\} \supseteq \bigcap \{b \text{Cl}_b(U) : F \in \mathcal{U}\} \neq \emptyset$. Conversely, if $\mathcal{F}$ is a grill on $X$ with the given property, then for every finite subfamily $\mathcal{G}_0$ of $\mathcal{F}$, $\bigcap \{b \text{Cl}_b(F) : F \in \mathcal{F} \neq \emptyset\}$. So, by hypothesis, $\bigcap \{b \text{Cl}_b(F) : F \in \mathcal{F}\} \neq \emptyset$. Hence $\mathcal{F}$, $b(\theta)$-adheres in $X$.

Definition 3.15 A topological space $X$ is called $b(\theta)$-linkage $b$-closed if every $b(\theta)$-linked grill on $X$, $b(\theta)$-adheres.

Theorem 3.16 Every $b(\theta)$-linkage $b$-closed space is $b$-closed.

Proof: The proof is clear.

Proposition 3.17 [5] Let $A$ be a subset of a topological space $(X, \tau)$. Then:

(i) If $A \in BO(X)$, then $b \text{Cl}(A) = b \text{Cl}_b(A)$.

(ii) If $A$ is $b$-regular, then $A$ is $b(\theta)$-closed.
Theorem 3.18 In the class of \( b \)-almost regular spaces, the concept of \( b \)-closedness and \( b(\theta) \)-linkage \( b \)-closedness become identical.

Proof: In view of Theorem 3.16, it is enough to show that a \( b \)-almost regular \( b \)-closed space is \( b(\theta) \)-linkage \( b \)-closed. Let \( \mathcal{G} \) be any \( b(\theta) \)-linked grill on a \( b \)-almost regular \( b \)-closed space \( X \) such that \( \mathcal{G} \) does not \( b(\theta) \)-adhere in \( X \). Then for each \( x \in X \), there exists \( G_x \in \mathcal{G} \) such that \( x \notin b\text{Cl}_b(G_x) = b\text{Cl}_b(b\text{Cl}_b(G_x)) \). Then there exists \( U_x \in BO(X, x) \) such that \( b\text{Cl}(U_x) \cap b\text{Cl}_b(G_x) = \emptyset \), which gives \( b\text{Cl}_b(U_x) \cap b\text{Cl}_b(G_x) = \emptyset \) by Proposition 3.17. Since \( b\text{Cl}_b(G_x) \in \mathcal{G} \) and \( \mathcal{G} \) is a \( b(\theta) \)-linked grill on \( X \), \( b\text{Cl}_b(U_x) = b\text{Cl}(U_x) \notin \mathcal{G} \). Now, \( \{U_x : x \in X\} \) is a cover of \( X \) by \( b \)-open sets of \( X \). So by \( b \)-closedness of \( X \), \( X = \cup \{b\text{Cl}(U_x_i) : i = 1, 2, ..., n\} \), for a finite subset \( \{x_1, x_2, ..., x_n\} \) of \( X \). It is then follows that \( x \notin \mathcal{G} \) for \( i = 1, 2, ..., n \), which is a contradiction. Hence \( \mathcal{G} \) must \( b(\theta) \)-adhere in \( X \), proving \( X \) to be \( b(\theta) \)-linkage \( b \)-closed.

Definition 3.19 A topological space \( X \) is said to be \( b \)-compact \([3]\) if every cover \( U \) of \( X \) by \( b \)-open sets of \( X \) has a finite subcover.

Definition 3.20 A topological space \( X \) is \( b(\theta) \)-regular if every grill on \( X \) which \( b(\theta) \)-converges must \( b \)-converge (not necessarily to the same point), where \( b \)-convergence of a grill is defined in the usual way. That is a grill \( \mathcal{G} \) on \( X \) is said to \( b \)-converge to \( x \in X \) if \( BO(X, x) \subseteq \mathcal{G} \).

Theorem 3.21 A topological space \( X \) is \( b \)-compact if and only if every grill \( b \)-converges.

Proof: Let \( \mathcal{G} \) be a grill on a \( b \)-compact space such that \( \mathcal{G} \) does not \( b \)-converge to any point \( x \in X \). Then for each \( x \in X \), there exists \( U_x \in BO(X, x) \) with (⋆) \( U_x \notin \mathcal{G} \). As \( \{U_x : x \in X\} \) is a cover of the \( b \)-compact space \( X \) by \( b \)-open sets, there exist finitely many points \( x_1, x_2, ..., x_n \) in \( X \) such that \( X = \bigcap_{i=1}^{n} U_{x_i} \). Since \( X \in \mathcal{G} \) for some \( i \), \( 1 \leq i \leq n \), \( U_{x_i} \notin \mathcal{G} \), which goes against (⋆). Conversely, let every grill on \( X \) \( b \)-converge and if possible, let \( X \) be not \( b \)-compact. Then there exists a cover \( \mathcal{U} \) of \( X \) by \( b \)-open sets of \( X \) having no finite subcover. Then \( \mathcal{F} = \{X \cup \bigcup_{\mathcal{U}_0} : \mathcal{U}_0 \text{ is a finite subcollection of } \mathcal{U} \} \) is a filterbase on \( X \). Then \( \mathcal{F} \) is contained in an ultrafilter \( \mathcal{G} \), and then \( \mathcal{G} \) \( b \)-converges to some point \( x \in X \). Then for some \( U \in \mathcal{U} \), \( x \in U \), and hence \( U \in \mathcal{G} \). But \( X \cup U \in \mathcal{F} \subseteq \mathcal{U} \). Thus \( U \) and \( X \cup U \) both belong to \( \mathcal{U} \), which is a filter, so giving a contradiction.

Theorem 3.22 A \( b \)-compact space \( X \) is \( b \)-closed, while the converse is also true if \( X \) is \( b(\theta) \)-regular.

Proof: The proof is clear.
Definition 3.23 A topological space \((X, \tau)\) is said to be \(b\)-regular [5] if for any closed set \(F \subseteq X\) and any point \(x \in X \setminus F\), there exists disjoint \(b\)-open sets \(U\) and \(V\) such that \(x \in U\) and \(F \subseteq V\).

Theorem 3.24 A topological space \(X\) is \(b\)-regular [5] if and only if for each \(x \in X\) and each \(U \in BO(X, x)\), there exists \(V \in BO(X, x)\) such that \(b\text{Cl}(V) \subseteq U\).

Theorem 3.25 Every \(b\)-regular space is \(b(\theta)\)-regular.

Proof: Let \(\mathcal{G}\) be a grill on a \(b\)-regular \(X\), \(b(\theta)\)-converging to a point \(x\) of \(X\). For each \(U \in BO(X, x)\), there exists, by \(b\)-regularity of \(X\), a \(V \in BO(X, x)\) such that \(b\text{Cl}(V) \subseteq U\). By hypothesis, \(b\text{Cl}(V) \in \mathcal{G}\). Hence \(\mathcal{G}\) \(b\)-converges to \(x\), proving \(X\) to be \(b(\theta)\)-regular. \(\square\)

Example 3.26 Let \(X = \{a, b, c\}, \tau = \{\emptyset, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}\). Clearly, \(X\) is \(b\)-compact. Hence by Theorem 3.21, every grill on \(X\) must \(b\)-converge in \(X\). Thus, \(X\) is \(b(\theta)\)-regular. But it is easy to check that \(X\) is not \(b\)-regular.

Theorem 3.27 If a topological space \(X\) is \(b\)-closed \(b\)-regular, then \(X\) is \(b\)-compact.

Proof: Let \(X\) be a \(b\)-closed and \(b\)-regular space. Let \(\{V_\alpha : \alpha \in I\}\) be any open cover of \(X\). For each \(x \in X\), there exists an \(\alpha(x) \in I\) such that \(x \in V_{\alpha(x)}\). Since \(X\) is \(b\)-regular, there exists \(U(x) \in BO(X, x)\) such that \(U(x) \subseteq b\text{Cl}(U(x)) \subseteq V_{\alpha(x)}\). Then, \(\{U(x) : x \in X\}\) is a \(b\)-open cover of the \(b\)-closed space \(X\) and hence there exists a finite amount of points, say, \(x_1, x_2, \ldots, x_n\) such that \(X = \bigcup_{i=1}^{n} b\text{Cl}(U(x_i)) = \bigcup_{i=1}^{n} V_{\alpha(x_i)}\). This shows that \(X\) is compact. \(\square\)

4. Sets which are \(b\)-closed relative to a space

Theorem 4.1 For a topological space \(X\), the following statements are equivalent:

(i) \(A\) is \(b\)-closed relative to \(X\);

(ii) Every maximal filter base \(b(\theta)\)-converges to some point of \(X\);

(iii) Every filter base \(b(\theta)\)-adhere to some point of \(X\);

(iv) For every family \(\{V_\alpha : \alpha \in I\}\) of \(b\)-closed sets such that \(\bigcap\{V_\alpha : i \in I\} \cap A = \emptyset\), there exists a finite subset \(I_0\) of \(I\) such that \(\bigcap\{b\text{Int}(V_i) : i \in I_0\} \cap A = \emptyset\).

Proof: The proof is clear. \(\square\)

Theorem 4.2 If \(X\) is a \(b\)-closed space, then every cover of \(X\) by \(b\)-\(\theta\)-open set has a finite subcover.
Proof: Let \( \{V_\alpha : \alpha \in I \} \) be any cover of \( X \) by \( b\theta \)-open subsets of \( X \). For each \( x \in X \), there exists \( \alpha(x) \in I \) such that \( x \in V_{\alpha(x)} \) is \( b\theta \)-open, there exists \( V_x \in BO(X, x) \) such that \( V_x \subseteq b\text{Cl}(V_x) \subseteq V_{\alpha(x)} \). The family \( \{V_x : x \in X\} \) is a \( b\)-open cover of \( X \). Since \( X \) is \( b\)-closed, there exists a finite number of points, say, \( x_1, x_2, \ldots, x_n \) such that \( X = \bigcup_{i=1}^{n} b\text{Cl}(V_{x_i}) \). Therefore, we obtain that \( X = \bigcup_{i=1}^{n} V_{x_i} \). \( \square \)

**Theorem 4.3** Let \( A, B \) be subsets of a topological space \( X \). If \( A \) is \( b\theta\)-closed and \( B \) is \( b\)-closed relative to \( X \), then \( A \cap B \) is \( b\)-closed relative to \( X \).

**Proof:** Let \( \{V_\alpha : \alpha \in I\} \) be any cover of \( A \cap B \) by \( b \)-open subsets of \( X \). Since \( X \setminus A \) is \( b\theta \)-open, for each \( x \in B \setminus A \) there exists \( W_x \in BO(X, x) \) such that \( b\text{Cl}(W_x) \subseteq X \setminus A \). The family \( \{W_x : x \in B \setminus A \} \cup \{V_\alpha : \alpha \in I\} \) is a cover of \( B \) by \( b \)-open sets of \( X \). Since \( B \) is \( b\)-closed relative to \( X \), there exists a finite number of points, say, \( x_1, x_2, \ldots, x_n \) in \( B \setminus A \) and a finite subset \( I_0 \) of \( I \) such that \( B \subseteq \bigcup_{i=1}^{n} b\text{Cl}(W_{x_i}) \cup \bigcup_{\alpha \in I_0} b\text{Cl}(V_\alpha) \). Since \( b\text{Cl}(W_{x_i}) \cap A = \emptyset \) for each \( i \), we obtain that \( A \cap B \subseteq \bigcup_{\alpha \in I_0} \{b\text{Cl}(V_\alpha) : \alpha \in I_0\} \). This shows that \( A \cap B \) is \( b\)-closed relative to \( X \). \( \square \)

**Corollary 4.4** If \( K \) is \( b\theta\)-closed of a \( b\)-closed space \( X \), then \( K \) is \( b\)-closed relative to \( X \).

**Definition 4.5** A topological space \( X \) is called \( b\)-connected [5] if \( X \) cannot be expressed as the union of two disjoint \( b \)-open sets. Otherwise, \( X \) is \( b \)-disconnected.

**Theorem 4.6** Let \( X \) be a \( b \)-disconnected space. Then \( X \) is \( b \)-closed if and only if every \( b \)-regular subset of \( X \) is \( b \)-closed relative to \( X \).

**Proof:** Necessity: Every \( b \)-regular set is \( b\theta \)-closed by Proposition 3.17. Since \( X \) is \( b \)-closed, the proof is completed by Corollary 4.4.

Sufficiency: Let \( \{V_\alpha : \alpha \in I\} \) be any cover of \( X \) by \( b \)-open subsets of \( X \). Since \( X \) is \( b \)-disconnected, there exists a proper \( b \)-regular subset \( A \) of \( X \). By our hypothesis, \( A \) and \( X \setminus A \) are \( b \)-closed relative to \( X \). There exist finite subsets \( A_1 \) and \( A_2 \) of \( A \) such that \( A \subseteq \bigcup_{\alpha \in A_1} b\text{Cl}(V_\alpha) \), \( X \setminus A \subseteq \bigcup_{\alpha \in A_2} b\text{Cl}(V_\alpha) \). Therefore, we obtain that \( X = \bigcup \{b\text{Cl}(V_\alpha) : \alpha \in A_1 \cup A_2\} \). \( \square \)

**Theorem 4.7** If there exists a proper \( b \)-regular subset \( A \) of a topological space \( X \) such that \( A \) and \( X \setminus A \) are \( b \)-closed relative to \( X \), then \( X \) is \( b \)-closed.

**Proof:** This proof is similar to the Theorem 4.6 and hence omitted. \( \square \)

**Definition 4.8** A function \( f : (X, \tau) \to (Y, \sigma) \) is called \( b \)-irresolute [4] if \( f^{-1}(V) \) is \( b \)-open in \( X \) for every \( b \)-open subset \( V \) of \( Y \).
Lemma 4.9 A function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $b$-irresolute if and only if for each subset $A$ of $X$, $f(b\text{Cl}(A)) \subseteq b\text{Cl}(f(A))$.

Theorem 4.10 If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $b$-irresolute surjection and $K$ is $b$-closed relative to $X$, then $f(K)$ is $b$-closed relative to $Y$.

Proof: Let $\{V_\alpha : \alpha \in I\}$ be any cover of $f(K)$ by $b$-open subsets of $Y$. Since $f$ is $b$-irresolute, $\{f^{-1}(V_\alpha) : \alpha \in I\}$ is a cover of $K$ by $b$-open subsets of $X$, where $K$ is $b$-closed relative to $X$. Therefore, there exists a finite subset $I_0$ of $I$ such that $K \subseteq \bigcup_{\alpha \in A_0} b\text{Cl}(f^{-1}(V_\alpha))$. Since $f$ is $b$-irresolute surjective, by Lemma 4.9, we have $f(K) \subseteq \bigcup_{\alpha \in A_0} f(b\text{Cl}(f^{-1}(V_\alpha))) \subseteq \bigcup_{\alpha \in A_0} f(b\text{Cl}(V_\alpha))$. \qed

Corollary 4.11 If a function $f : (X, \tau) \rightarrow (Y, \sigma)$ is $b$-irresolute surjection and $X$ is $b$-closed, then $Y$ is $b$-closed.

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N. Rajesh
Department of Mathematics
Rajah Serfoji Govt. College
Thanjavur-613005
Tamilnadu, India.
E-mail address: nrajeshtopology@yahoo.co.in