Global behavior of the difference equation

\[ x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}} \]

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**ABSTRACT:** The aim of this work is to investigate the global stability, periodic nature, oscillation and the boundedness of all admissible solutions of the difference equation

\[ x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \quad n = 0, 1, 2, \ldots \]

where \( A, B, C \) are positive real numbers.

Key Words: difference equation, periodic solution, globally asymptotically stable.

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1. Introduction and Preliminaries

Recently there has been a great interest in studying the qualitative properties of rational difference equations. For the systematical studies of rational and non-rational difference equations, one can refer to the monographs [1,3,4,5,6] and references therein.

M. Aloqeili in [2] discussed the stability properties and semi-cycle behavior of the solutions of the difference equation

\[ x_{n+1} = \frac{x_{n-1}}{a - x_nx_{n-1}}, \quad n = 0, 1, \ldots \]

with real initial conditions and positive real number \( a \).

In this paper, we study the global asymptotic stability of the difference equation

\[ x_{n+1} = \frac{Ax_{n-1}}{B - Cx_nx_{n-2}}, \quad n = 0, 1, \ldots \]  

(1)

2000 Mathematics Subject Classification: 39A11
where $A, B, C$ are nonnegative real numbers.

Consider the difference equation

$$x_{n+1} = f(x_n, x_{n-1}, \ldots, x_{n-k}), \quad n = 0, 1, \ldots$$

(2)

where $f : R^{k+1} \rightarrow R$.

**Definition 1.1** [4].

An equilibrium point for equation (2) is a point $\bar{x} \in R$ such that $\bar{x} = f(\bar{x}, \bar{x}, \ldots, \bar{x})$.

**Definition 1.2** [4].

1. An equilibrium point $\bar{x}$ for equation (2) is called locally stable if for every $\epsilon > 0$, $\exists \delta > 0$ such that every solution $\{x_n\}$ with initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in [\bar{x} - \delta, \bar{x} + \delta]$ is such that $x_n \in [\bar{x} - \epsilon, \bar{x} + \epsilon], \forall n \in N$. Otherwise $\bar{x}$ is said to be unstable.

2. The equilibrium point $\bar{x}$ of equation (2) is called locally asymptotically stable if it is locally stable and there exists $\gamma > 0$ such that for any initial conditions $x_{-k}, x_{-k+1}, \ldots, x_0 \in [\bar{x} - \gamma, \bar{x} + \gamma]$, the corresponding solution $\{x_n\}$ tends to $\bar{x}$.

3. An equilibrium point $\bar{x}$ for equation (2) is called global attractor if every solution $\{x_n\}$ converges to $\bar{x}$ as $n \rightarrow \infty$.

4. The equilibrium point $\bar{x}$ for equation (2) is called globally asymptotically stable if it is locally asymptotically stable and global attractor.

The linearized equation associated with equation (2) is

$$y_{n+1} = \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})y_{n-i}, \quad n = 0, 1, 2, \ldots$$

(3)

The characteristic equation associated with equation (3) is

$$\lambda^{k+1} - \sum_{i=0}^{k} \frac{\partial f}{\partial x_{n-i}}(\bar{x}, \ldots, \bar{x})\lambda^{k-i} = 0.$$

(4)

**Theorem 1.3** [4]. Assume that $f$ is a $C^1$ function and let $\bar{x}$ be an equilibrium point of equation (2). Then the following statements are true:

1. If all roots of equation (4) lie in the open disk $|\lambda| < 1$, then $\bar{x}$ is locally asymptotically stable.

2. If at least one root of equation (4) has absolute value greater than one, then $\bar{x}$ is unstable.

The change of variables $x_n = \sqrt{BC}y_n$ reduces equation (1) to the difference equation

$$y_{n+1} = \frac{ry_{n-1}}{1-y_{n-1}y_{n-2}}, \quad n = 0, 1, 2, \ldots$$

(5)

where $r = \frac{A}{B}$. 

2. Linearized stability analysis

In this section we study the local asymptotic stability of the equilibrium points of equation (5). We can see that equation (5) has the equilibrium points \( \bar{y} = 0 \) and \( \bar{y}_1 = \sqrt{1 - r}, \bar{y}_2 = -\sqrt{1 - r} \) when \( r < 1 \) and the zero equilibrium only when \( r \geq 1 \).

The linearized equation associated with equation (5) about \( \bar{y} \) is

\[
z_{n+1} - \frac{r}{1 - \bar{y}^2} z_{n-1} - \frac{r\bar{y}^2}{(1 - \bar{y}^2)^2} (z_n + z_{n-2}) = 0, \quad n = 0, 1, 2, \ldots \quad (6)
\]

The characteristic equation associated with this equation is

\[
\lambda^3 - \frac{r}{1 - \bar{y}^2} \lambda - \frac{r\bar{y}^2}{(1 - \bar{y}^2)^2} (\lambda^2 + 1) = 0. \quad (7)
\]

We summarize the results of this section in the following theorem.

**Theorem 2.1**

1. If \( r > 1 \), then the zero equilibrium point is a saddle point.

2. If \( r < 1 \), then the equilibrium points \( \bar{y} = 0 \) is locally asymptotically stable and \( \bar{y}_1 = \sqrt{1 - r}, \bar{y}_2 = -\sqrt{1 - r} \) are unstable.

**Proof:** The linearized equation associated with equation (5) about \( \bar{y} = 0 \) is

\[
z_{n+1} - rz_{n-1} = 0, \quad n = 0, 1, 2, \ldots
\]

The characteristic equation associated with this equation is

\[
\lambda^3 - r\lambda = 0.
\]

That is \( \lambda = 0, \pm \sqrt{r} \).

1. If \( r < 1 \), then \( |\lambda| < 1 \) for all roots and \( \bar{y} = 0 \) is locally asymptotically stable.

2. If \( r > 1 \), it follows that \( \bar{y} = 0 \) is unstable (saddle point).

The linearized equation (6) about \( \bar{y} = \pm \sqrt{1 - r} \) becomes

\[
z_{n+1} - z_{n-1} - (\frac{1}{r})(1 - r)(z_n + z_{n-2}) = 0, \quad n = 0, 1, 2, \ldots
\]

The associated characteristic equation is

\[
\lambda^3 - \lambda - (\frac{1}{r})(1 - r)(\lambda^2 + 1) = 0.
\]

Let \( f(\lambda) = \lambda^3 - \lambda - (\frac{1}{r})(1 - r)(\lambda^2 + 1) \).

We can see that \( f(\lambda) \) has a root in \((1, \infty)\). Then \( \bar{y}_1 = \sqrt{1 - r}, \bar{y}_2 = -\sqrt{1 - r} \) are unstable.
3. Oscillation

Theorem 3.1 Assume that \( r < 1 \). Then the interval \( (-\sqrt{1-r}, \sqrt{1-r}) \) is an invariant interval for equation \((5)\).

Proof: The proof is by induction. Suppose that \( y_{-2}, y_{-1}, y_0 \in (-\sqrt{1-r}, \sqrt{1-r}) \). Hence \( |y_{-i}| < \sqrt{1-r}, i = 0, 1, 2 \). This implies that \( |y_{-2y_0}| < 1 - r \).

Then

\[
|y_1| = \frac{r |y_{-1}|}{1 - y_0y_2} < \frac{r |y_{-1}|}{1 - |y_0y_2|} < |y_{-1}|,
\]

\[
|y_2| = \frac{r |y_0|}{1 - y_1y_{-1}} < \frac{r |y_0|}{1 - |y_1y_{-1}|} < |y_0|,
\]

where \(|y_1| < |y_{-1}| < \sqrt{1-r}\).

Now if for a certain \( n_0 \in \mathbb{N} \) we have \( y_{n_0-2}, y_{n_0-1}, y_{n_0} \in (-\sqrt{1-r}, \sqrt{1-r}) \), then

\[
|y_{n_0+1}| = \frac{r|y_{n_0-1}|}{1 - y_{n_0}y_{n_0-2}} < \frac{r|y_{n_0-1}|}{1 - |y_{n_0}y_{n_0-2}|} < |y_{n_0-1}| < \sqrt{1-r}. \]

This completes the proof. \( \square \)

Corollary 3.2 Assume that \{\( y_n \)\}_{n=-2}^{\infty} be a solution of equation \((5)\) such that either \( y_{-2}, y_{-1}, y_0 \in (0, \sqrt{1-r}) \) (or \( (-\sqrt{1-r}, 0) \)). Then \{\( y_n \)\}_{n=-2}^{\infty} is positive (or negative). Moreover, \{\( y_n \)\}_{n=-2}^{\infty} decreases (or increases) to zero equilibrium point.

Theorem 3.3 Let \{\( y_n \)\}_{n=-2}^{\infty} be a nontrivial solution of equation \((5)\) and let \( \bar{y}_1 = \sqrt{1-r}, \bar{y}_2 = -\sqrt{1-r} \) denote the nonzero equilibrium points of equation \((5)\) such that either,

\[(C_1) \quad \bar{y}_2 = -\sqrt{1-r} < y_{-1} < 0 < y_{-2}, y_0 < \sqrt{1-r} = \bar{y}_1 \]

or

\[(C_2) \quad \bar{y}_2 = -\sqrt{1-r} < y_{-2}, y_0 < 0 < y_{-1} < \sqrt{1-r} = \bar{y}_1 \]

is satisfied. Then \{\( y_n \)\}_{n=-2}^{\infty} oscillates about \( y = 0 \) with semicycles of length one.

Moreover \( y_{2n+2} < (>) y_{2n} \) and \( y_{2n+1} > (>) y_{2n-1}, n = 0, 1, 2, \ldots \).

Proof: Assume that condition \((C_1)\) is satisfied. Then we have

\[
y_1 = \frac{r|y_{-1}|}{1 - y_0y_{-2}} > y_{-1}, y_2 = \frac{r|y_0|}{1 - y_{-1}y_1} < y_0.
\]

Now suppose that for a fixed \( n_0 \in \mathbb{N} \) we have

\[-\sqrt{1-r} < y_{2n_0-1} < y_{2n_0+1} < 0 \quad \text{and} \quad 0 < y_{2n_0} < y_{2n_0-2} < \sqrt{1-r}. \]

Then

\[
0 > y_{2n_0+3} = \frac{r|y_{2n_0+1}|}{1 - y_{2n_0+2}y_{2n_0}} > y_{2n_0+1},
\]

and

\[
0 < y_{2n_0+2} = \frac{r|y_{2n_0}|}{1 - y_{2n_0+1}y_{2n_0-1}} < y_{2n_0}.
\]

Therefore, \( y_{2n} > y_{2n+2} > 0 \) and \( y_{2n-1} < y_{2n+1} < 0 \) for all \( n \geq -1 \) and the result follows.

For condition \((C_2)\), the result is similar and will be omitted. \( \square \)
4. Global behavior of equation (5)

Theorem 4.1 The following statements are true.

1. If $r < 1$, then the zero equilibrium point is a global attractor with basin $(-\sqrt{1-r}, \sqrt{1-r})$.

2. If $r = 1$, then equation (5) has prime period two solutions of the form $\ldots, 0, \varphi, 0, \varphi, 0, \ldots$, where $\varphi \in \mathbb{R}$.

3. If $r > 1$, then there exist solutions which are neither bounded nor persist.

Proof:

1. Suppose that $y_{-2}, y_{-1}, y_0 \in (-\sqrt{1-r}, \sqrt{1-r})$. Then using theorem (3.1) we have that $y_n \in (-\sqrt{1-r}, \sqrt{1-r}), n \geq 1$.

Moreover, we have $| y_{n+1} | < | y_{n-1} |, n = 0, 1, \ldots$.

That is $| y_{2n+1} | < | y_{2n-1} |$ and $| y_{2n+2} | < | y_{2n} |, n = 0, 1, \ldots$.

From equation (5) we have

$$| y_{2n+1} | = \frac{r | y_{2n-1} |}{1 - y_{2n+1}y_{2n-1}} \leq \frac{r | y_{2n-1} |}{1 - | y_{2n+1}y_{2n-1} |}$$

and

$$| y_{2n+2} | = \frac{r | y_{2n} |}{1 - y_{2n+1}y_{2n-1}} \leq \frac{r | y_{2n} |}{1 - | y_{2n+1}y_{2n-1} |}.$$ 

Now suppose that $\lim_{n \to \infty} | y_{2n+1} | = L$ and $\lim_{n \to \infty} | y_{2n} | = M$. Then

$$L \leq \frac{rL}{1 - M^2} \quad \text{and} \quad M \leq \frac{rM}{1 - L^2}.$$ 

If $L \neq 0$, then $|1 - M^2| \leq r$. This implies that $\sqrt{1-r} \leq M \leq \sqrt{1+r}$, which is a contradiction. Hence we have $L = 0$. The second inequality gives $M \leq rM$, from which $M = 0$ where $r < 1$. Therefore, $\{y_n\}_{n=-2}^{\infty}$ converges to zero. This completes the proof.

2. Clear!

3. Let $\{y_n\}_{n=-2}^{\infty}$ be a solution of equation (5) with the initial conditions, $| y_{-i} | < \sqrt{r-1} (> \sqrt{r+1}), i = 0, 2$ and $| y_{-i} | > \sqrt{r+1} (< \sqrt{r-1}), i = 1$. We consider only the case $| y_{-i} | < \sqrt{r-1}, i = 0, 2$ and $| y_{-i} | > \sqrt{r+1}, i = 1$.

It follows that $| y_{-2y_0} | = | y_{-2} | y_0 < r - 1$. That is $-r + 1 < y_{2y_0} < r - 1$. This implies that $-r + 2 < 1 - y_{2y_0} < r$. Hence we have

$$| y_1 | = \frac{| ry_{-1} |}{1 - y_0y_{-2}} > \frac{r | y_{-1} |}{r} = | y_{-1} | > \sqrt{r+1}.$$ 

It follows that $| y_1y_{-1} | = | y_1 | y_{-1} > r + 1$, which implies that $r + 2 < 1 - y_1y_{-1} < -r$ and so

$$| y_2 | = \frac{| ry_0 |}{1 - y_1y_{-1}} < \frac{r | y_0 |}{r} = | y_0 | < \sqrt{r-1}.$$
By induction we get $|y_{2n+1}| > |y_{2n-1}| > \sqrt{r+1}$ and $|y_{2n+2}| < |y_{2n}| < \sqrt{r-1}$, $n \geq 0$.

Now suppose that $|y_{2n}| \to L$ and $|y_{2n+1}| \to M$ as $n \to \infty$. But

$$
|y_{2n+2}| = \left| \frac{r |y_{2n}|}{1 - y_{2n+1}y_{2n-1}} \right| \leq \left| \frac{r |y_{2n}|}{1 - M} \right|.
$$

Then $L \leq \frac{rL}{1-M}$. If $L \neq 0$, then $|1 - M^2| \leq r$. This implies that $M \leq \sqrt{r+1}$. This is a contradiction and so $L = 0$. Now, as

$$
|y_{2n+1}| = \left| \frac{r |y_{2n-1}|}{1 - y_{2n}y_{2n-2}} \right| \geq \left| \frac{r |y_{2n-1}|}{1 + |y_{2n}||y_{2n-2}|} \right|
$$

then we have $M \geq \frac{rM}{1+M} = rM$ and therefore, $M = \infty$.

The case when $|y_{-i}| > \sqrt{r+1}$, $i = 0, 2$ and $|y_{-i}| < \sqrt{r-1}$, $i = 1$ is similar and will be omitted.

**Conjecture** Assume that $r < 1$. Then the zero equilibrium point is global asymptotically stable (in the set of all admissible solutions).

5. **Numerical examples**

**Example 5.1** Figure 1. shows that if $r = 0.6$, then the solution $\{y_n\}_{n=-2}^\infty$ with initial conditions $y_{-2} = 0.3$, $y_{-1} = -0.2$, $y_0 = 0.35$ converges to zero.

**Example 5.2** Figure 2. shows that if $r = 1.9$, then the solution $\{y_n\}_{n=-2}^\infty$ with initial conditions $y_{-2} = 0.9$, $y_{-1} = -1.7$, $y_0 = 0.9$ is neither bounded nor persist.

![Figure 1: The difference equation $y_{n+1} = \frac{0.6y_{n-1}}{1-y_{n-2}y_n}$](image-url)
Figure 2: The difference equation $y_{n+1} = \frac{1.9y_n - 1}{1 - y_n - 2y_n}$

References


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