A Decomposition of Pairwise Continuity via Ideals

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Abstract: In this paper, we introduce and study the notions of (i, j)-regular-I-closed sets, (i, j)-A-I-sets, (i, j)-I-locally closed sets, \( p\)-A-I-continuous functions and \( p\)-I-LC-continuous functions in ideal bitopological spaces and investigate some of their properties. Also, a new decomposition of pairwise continuity is obtained using these sets.

Key Words: (j, i)-regular closed sets, (i, j)-A-I-sets, (i, j)-I-locally closed sets, \( (i, j)\)-regular-I-closed sets, (i, j)-A-I-sets, \( (i, j)\)-I-locally closed sets, \( p\)-A-I-continuous functions and \( p\)-I-LC-continuous functions.

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1. Introduction and Preliminaries

In 1963, J. C. Kelly [9] introduced the notion of bitopological spaces. Such spaces are equipped with two arbitrary topologies. B. Dvalishvili [3] introduced the concept of \((i, j)\)-regular closed sets in bitopological spaces. In [8], M. Jelic introduced the concepts of \((i, j)\)-A-I-sets, \((i, j)\)-locally closed sets, \( p\)-A-I-continuity and \( p\)-I-LC-continuity in bitopological spaces. Throughout this paper, \( \tau_j\)-cl(A) and \( \tau_i\)-int(A) denote the closure of A with respect to \( \tau_j \) and the interior of A with respect to \( \tau_i \) and the spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) are bitopological spaces on which no separation axioms are assumed unless explicitly stated.

An ideal topological space is a topological space \((X, \tau)\) with an ideal \( J \) on X and is denoted by \((X, \tau, J)\). The subject of ideals in topological spaces has been introduced and studied by Kuratowski [11] and Vaidyanathaswamy [17].

Let \((X, \tau_1, \tau_2)\) be a bitopological space and let \( J \) be an ideal of subsets of X. An ideal bitopological space is a bitopological space \((X, \tau_1, \tau_2)\) with an ideal \( J \) on X and is denoted by \((X, \tau_1, \tau_2, J)\). For a subset \( A \) of X and \( j = 1, 2 \), \( A^*_j(J) = \{x \in X : A \cap U \notin J \text{ for every } U \in \tau_j(X, x)\} \) is called the local function [11] of \( A \) with respect to \( J \) and \( \tau_j \). We simply write \( A^*_j \) instead of \( A^*_j(J) \) in case there is no chance for confusion. For every ideal bitopological space \((X, \tau_1, \tau_2, J)\), there exists...
a topology $\tau_j^*$, finer than $\tau_j$. Additionally, $\tau_j$-cl$^*(A)$=A∪$A_j^*$ defines a Kuratowski closure operator \[18\] for $\tau_j^*$. Also, $\tau_j$-cl$^*(A) \subseteq \tau_j$-cl$(A)$ for any subset $A$ of $X$.

The hypothesis $X = X_j^*$ is equivalent to the hypothesis $\tau_j \cap J = \emptyset$. In an ideal topological spaces $(X, \tau, J)$, a space is called Hayashi-Samuels space if $\tau \cap J = \emptyset$.

In an ideal bitopological spaces $(X, \tau_1, \tau_2, J)$, we call a space is a Hayashi-Samuels space if $\tau_j \cap J \neq \emptyset$, $j = 1$ or 2.

For every ideal bitopological space $(X, \tau_1, \tau_2, J)$, there exists a topology $\tau_j^*$, finer than $\tau_j$, generated by $\beta(\tau, \tau_j)$=\{U-I : U \in \tau_j \text{ and } I \in J\}, but in general $\beta(\tau, \tau_j)$ is not always a topology. If $J = \emptyset$, then $A_j^* = \tau_j$-cl$(A)$. Hence in this case $\tau_j$-cl$^*(A) = \tau_j$-cl$(A)$ and $\tau_j^* = \tau_j$. If $J = \mathcal{P}(X)$, then $A_j^* = \emptyset$ for every $A \subseteq X$. The family of all nowhere dense subsets of a bitopological space $(X, \tau_1, \tau_2)$ is defined by ij-N = \{A \subseteq X : \tau_i$-int$(\tau_j$-cl$(A)) = \emptyset\}, where $i, j = 1, 2$ and $i \neq j$.

Recently, M. Rajamani et al. \[14\] introduced the notions of $(i, j)$-B3-sets, $(i, j)$-C3-sets, $(i, j)$-$S_3$-sets, $(i, j)$-$\alpha$-$J$-open sets, $(i, j)$-semi-$J$-open sets, $(i, j)$-pre-$J$-open sets and $(i, j)$-$\beta$-$J$-open sets and obtained decompositions of pairwise continuity. In this paper, we introduce the notions of $(i, j)$-$\alpha$-$J$-closed sets, $(i, j)$-$A_3$-locally closed sets, $p$-$A_3$-continuous functions and $p$-$J$-LC-continuous functions to obtain another decomposition of pairwise continuity in ideal bitopological spaces.

**Definition 1.1.** A subset $A$ of a space $(X, \tau_1, \tau_2)$ is said to be

1. $(i, j)$-regular closed \[3\] if $A = \tau_i$-cl$(\tau_j$-int$(A))$,
2. $(i, j)$-semi-open \[12\] if $A \subseteq \tau_j$-cl$(\tau_i$-int$(A))$,
3. $(i, j)$-pre-open \[8\] if $A \subseteq \tau_i$-int$(\tau_j$-cl$(A))$,
4. $(i, j)$-$\alpha$-open \[15\] if $A \subseteq \tau_i$-int$(\tau_j$-cl$(\tau_i$-int$(A)))$,
5. $(i, j)$-$\alpha^*$-set \[16\] if $\tau_i$-int$(A) = \tau_i$-int$(\tau_j$-cl$(\tau_i$-int$(A)))$,
6. $(i, j)$-$A$-set \[8\] if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is $(j, i)$-regular closed,
7. $(i, j)$-locally closed set (briefly $(i, j)$-LC-set) \[8\] if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is $\tau_j$-closed,
8. $(i, j)$-C-set \[16\] if $A = U \cap V$, where $U$ is $\tau_i$-open and $V$ is an $(i, j)$-$\alpha^*$-set, where $i \neq j$, $i, j = 1, 2$.

**Definition 1.2.** A subset $A$ of an ideal topological space $(X, \tau, J)$ is said to be

1. $\tau^*$-dense-in-itself \[5\] if $A \subseteq A^*$,
2. $\tau^*$-closed \[6\] if $A^* \subseteq A$,
3. $\tau^*$-perfect \[5\] if $A = A^*$,
4. $\alpha$-$J$-open \[4\] if $A \subseteq \text{int}(cl^*(\text{int}(A)))$,
5. semi-$J$-open \[4\] if $A \subseteq cl^*(\text{int}(A))$,
6. pre-$J$-open \[2\] if $A \subseteq \text{int}(cl^*(A))$,
7. $J$-open \[7\] if $A \subseteq \text{int}(A)$,
8. $\alpha^*$-$J$-set \[4\] if $\text{int}(A) = \text{int}(cl^*(\text{int}(A)))$,
9. regular-$J$-closed \[10\] if $A = (\text{int}(A))^*$,
10. $J$-locally closed \[2\] if $A = U \cap V$, where $U \in \tau$ and $V$ is $\tau^*$-perfect,
11. $C_3$-set \[4\] if $A = U \cap V$, where $U \in \tau$ and $V$ is an $\alpha^*$-$J$-set,
12. $A_3$-set \[10\] if $A = U \cap V$, where $U \in \tau$ and $V$ is a regular-$J$-closed set.

**Definition 1.3.** \[14\] A subset $A$ of an ideal bitopological space $(X, \tau_1, \tau_2, J)$ is said to be
1. \((i, j)\)-pre-J-open if \(A \subseteq \tau_i\text{-}\text{int}(\tau_j\text{-}\text{cl}^*(A))\),
2. \((i, j)\)-J-open if \(A \subseteq \tau_i\text{-}\text{int}(A^*)\),
3. \((i, j)\)-semi-J-open if \(A \subseteq \tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A))\),
4. \((i, j)\)-\(\alpha\)-J-open if \(A \subseteq \tau_i\text{-}\text{int}(\tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A)))\),
5. \((i, j)\)-\(\alpha^*\)-J-set if \(\tau_i\text{-}\text{int}(\tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A))) = \tau_i\text{-}\text{int}(A)\),
6. \((i, j)\)-\(C_J\)-set if \(A = U \cap V\), where \(U \in \tau_i\) and \(V\) is an \((i, j)\)-\(\alpha^*\)-J-set, where \(i \neq j\), \(i, j = 1, 2\).

Lemma 1.4. [6] Let \((X, \tau)\) be a topological space with ideals \(I\) and \(J\) and \(A, B\) subsets of \(X\). Then the following properties hold:
1. If \(A \subseteq B\), then \(A^* \subseteq B^*\),
2. If \(\exists \subseteq J\Rightarrow A^*(\exists) \subseteq B^*(\exists)\),
3. \(A^* = \text{cl}(A^*) \subseteq \text{cl}(A)\),
4. \((A^*)^* \subseteq A^*\),
5. \((A \cup B)^* = A^* \cup B^*\).

Lemma 1.5. [14] Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and let \(I = ij\text{-}N\) be the family of all nowhere dense subsets of \((X, \tau_1, \tau_2)\). Then \(A^*\{\exists\} = \tau_1\text{-}\text{cl}(\tau_1\text{-}\text{int}(\tau_j\text{-}\text{cl}(A)))\).

Lemma 1.6. [14] Let \((X, \tau_1, \tau_2, I)\) be an ideal bitopological space and \(I = \{\emptyset\}\) or \(I = ij\text{-}N\). Then a subset \(A\) of \(X\) is \((i, j)\)-pre-J-open if and only if \(A\) is \((i, j)\)-pre open.

2. \((i, j)\)-Regular-J-closed sets

Definition 2.1. A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, I)\) is said to be \((i, j)\)-regular-J-closed if \(A = (\tau_i\text{-}\text{int}(A))^*_j\).

Proposition 2.2. For a subset of an ideal bitopological space \((X, \tau_1, \tau_2, I)\), the following hold:
1. Every \((i, j)\)-regular-J-closed set is \((i, j)\)-\(\alpha^*\)-J-set and \((i, j)\)-semi-J-open.
2. Every \((i, j)\)-regular-J-closed set is \(\tau_j\)-\(s\)-perfect.

Proof: (1) Let \(A\) be an \((i, j)\)-regular-J-closed set. Then, we have \(\tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A)) = \tau_i\text{-}\text{int}(A) \cup \tau_i\text{-}\text{int}(A)^*_j = \tau_i\text{-}\text{int}(A) \cup A = A\). Thus, \(\tau_i\text{-}\text{int}(\tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A))) = \tau_i\text{-}\text{int}(A)\) and \(A \subseteq \tau_j\text{-}\text{cl}^*(\tau_i\text{-}\text{int}(A))\). Therefore, \(A\) is \((i, j)\)-\(\alpha^*\)-J-set and \((i, j)\)-semi-J-open.

2. Let \(A\) be an \((i, j)\)-regular-J-closed set. Then, \(A = (\tau_i\text{-}\text{int}(A))^*_j\). Since \(\tau_i\text{-}\text{int}(A) \subseteq A\), \((\tau_i\text{-}\text{int}(A))^*_j \subseteq A^*_j\) by (1) of Lemma 1.4. Then we have \(A = (\tau_i\text{-}\text{int}(A))^*_j \subseteq A^*_j\). On the other hand, by (5) of Lemma 1.4, \(A^*_j = (\tau_i\text{-}\text{int}(A))^*_j \subseteq (\tau_i\text{-}\text{int}(A))^*_j = A\). Therefore, we obtain \(A = A^*_j\). Thus, \(A\) is \(\tau_j\)-\(s\)-perfect. \(\Box\)

Remark 2.3. The converses of Proposition 2.2 need not be true as the following examples show.

Example 2.4. Let \(X = \mathbb{R}, \tau_1 = \tau_2 = \text{usual topology on } \mathbb{R}\) and \(I\) be the ideal of finite subsets of \(\mathbb{R}\). Let \(A = \mathbb{Q}\). Then \(A\) is \((1, 2)\)-\(\alpha^*\)-J-set but it is not \((1, 2)\)-regular-J-closed.
Example 2.5. Let \( X = \mathbb{R} \), \( \tau_1 = \tau_2 \) be usual topology on \( \mathbb{R} \) and \( I \) be the ideal of finite subsets of \( (0, 1] \). Let \( A = (0, 1] \). Then \( A \) is \((1, 2)\)-semi-\( I \)-open but it is not \((1, 2)\)-regular-\( I \)-closed.

Example 2.6. Let \( X = \mathbb{N} \), \( \tau_1 = \{\emptyset, \{1\}, \{1, 2\}, \{1, 2, 3\}, \ldots\} \), \( \tau_2 = \mathcal{P}(\mathbb{N}) \) and \( I = \mathcal{P}(2\mathbb{N} - 1) \), where \( 2\mathbb{N} - 1 \) denotes the set of all odd natural numbers. Let \( A = \{2, 4, 6, 8, \ldots\} \). Then \( A \) is \( \tau_2 \)-\( \ast \)-perfect but it is not \((1, 2)\)-regular-\( I \)-closed.

Corollary 2.7. Every \((i, j)\)-regular-\( I \)-closed set is \( \tau_j^* \)-closed and \( \tau_j \)-\( \ast \)-dense-in-itself.

Proof: The proof is obvious from \((2)\) of Proposition 2.2. \( \square \)

Proposition 2.8. In an ideal bitopological space \((X, \tau_1, \tau_2, I)\), every \((i, j)\)-regular-\( I \)-closed set is \((j, i)\)-regular closed.

Proof: Let \( A \) be any \((i, j)\)-regular-\( I \)-closed set. Then, we have \( A = (\tau_i \text{-int}(A))^*_I \). Thus, \( \tau_j \text{-cl}(A) = \tau_j \text{-cl}((\tau_i \text{-int}(A))^*_I) = (\tau_i \text{-int}(A))^*_I \cap A \) by \((3)\) of Lemma 1.4. Also from \((3)\) of Lemma 1.4, we have \((\tau_i \text{-int}(A))^*_I \subseteq \tau_j \text{-cl}(\tau_i \text{-int}(A)) \) and hence \( A = (\tau_i \text{-int}(A))^*_I \subseteq \tau_j \text{-cl}(\tau_i \text{-int}(A)) \subseteq \tau_j \text{-cl}(A) = A \). Thus we have \( A = \tau_j \text{-cl}(\tau_i \text{-int}(A)) \) and hence \( A \) is \((j, i)\)-regular closed set. \( \square \)

Remark 2.9. The converse of Proposition 2.8 need not be true as the following example shows.

Example 2.10. Let \( X = \mathbb{R} \), \( \tau_1 = \tau_2 \) be usual topology on \( \mathbb{R} \) and \( I = \mathcal{P}(\mathbb{R}) \). Let \( A = [0, 1] \). Then \( A \) is \((2, 1)\)-regular closed which is not \((1, 2)\)-regular-\( I \)-closed.

Proposition 2.11. Every \( \tau_j^* \)-closed set is an \((i, j)\)-\( \ast \)-\( I \)-set in an ideal bitopological space \((X, \tau_1, \tau_2, I)\).

Proof: Let \( A \) be \( \tau_j^* \)-closed. Then we have \( \tau_i \text{-int}(\tau_j \text{-cl}(\tau_i \text{-int}(A))) \subseteq \tau_i \text{-int}(\tau_i \text{-cl}(\ast_I A)) \). Clearly, \( \tau_i \text{-int}(A) \subseteq \tau_i \text{-int}(\tau_j \text{-cl}(\ast_I A)) \). This shows that \( A \) is \((i, j)\)-\( \ast \)-\( I \)-open. \( \square \)

Example 2.12. The converse of the above proposition need not be true. In Example 2.4, the set \( A = \mathbb{Q} \) is \((1, 2)\)-\( \ast \)-\( I \)-set but it is not \( \tau_j^* \)-closed.

Proposition 2.13. Let \( (X, \tau_1, \tau_2, I) \) be an ideal bitopological space and \( I = \{\emptyset\} \) or \( I = ij \mathcal{N} \), where \( ij \mathcal{N} \) is the ideal of all nowhere dense sets in \((X, \tau_1, \tau_2)\). Then a subset \( A \) of \( X \) is \((i, j)\)-regular-\( I \)-closed if and only if \( A \) is \((i, j)\)-\( \ast \)-\( I \)-closed.

Proof: By Proposition 2.8, we need to show only sufficiency in both cases.

If \( I = \{\emptyset\} \), then \( A^*_I(\emptyset) = \tau_j \text{-cl}(A) \). If \( A \) is \((i, j)\)-\( \ast \)-\( I \)-closed, we have \((\tau_i \text{-int}(A))^*_I = \tau_j \text{-cl}(\tau_i \text{-int}(A)) = A \). Thus \( A \) is \((i, j)\)-regular-\( I \)-closed.

If \( I = ij \mathcal{N} \), then \( A^*_I(ij \mathcal{N}) = \tau_j \text{-cl}(\tau_i \text{-int}(\tau_j \text{-cl}(A))) \), for any subset \( A \) of \( X \). If \( A \) is \((i, j)\)-\( \ast \)-\( I \)-closed, we obtain \( (\tau_i \text{-int}(A))^*_I = \tau_j \text{-cl}(\tau_i \text{-int}(\tau_j \text{-cl}(\tau_i \text{-int}(A))) \) = \( \tau_j \text{-cl}(\tau_i \text{-int}(A)) \) = \( A \). This shows that \( A \) is \((i, j)\)-regular-\( I \)-closed. \( \square \)
Remark 2.14. Following examples show that the notion of \((i,j)\)-regular-J-closedness is independent with the notions of \(\tau_i\)-openness and \((i,j)\)-\(\alpha\)-J-openness.

Example 2.15. In Example 2.5, the set \(A = [0,1]\) is \((1,2)\)-regular-J-closed which is not \(\tau_1\)-open and \((1,2)\)-\(\alpha\)-J-open.

Example 2.16. Let \(X = \mathbb{R}\), \(\tau_1 = \{0, \mathbb{Q}, X\}\), \(\tau_2 = \) usual topology on \(\mathbb{R}\) and \(\mathcal{I} = \mathcal{P}(\mathbb{Q})\). Let \(A = \mathbb{Q}\). Then the set \(A\) is \(\tau_1\)-open and \((1,2)\)-\(\alpha\)-J-open but it is not \((1,2)\)-\(\mathbb{R}\)-closed.

Remark 2.17. From the above definitions and results, we have the following diagram. None of them is reversible.

\[
\begin{array}{ccc}
\tau_j^*\text{-closed} & \rightarrow & (i,j)\text{-}\alpha^*\text{-J-open} \\
\uparrow & & \uparrow \\
\tau_j\text{-}\ast\text{-perfect} & \leftarrow & (i,j)\text{-}\text{regular-J-closed} \\
\downarrow & & \downarrow \\
\tau_j\text{-}\ast\text{-dense-in-itself} & \rightarrow & (j,i)\text{-}\text{regular closed} \\
\downarrow & & \downarrow \\
\end{array}
\]

Remark 2.18. We can say that \((j,i)\)-regular closed and \(\tau_j\text{-}\ast\text{-dense-in-itself are independent. In Example 2.10, the set } A = [1,2] \text{ is } (2,1)\text{-regular closed but not } \tau_2\text{-}\ast\text{-dense-in-itself. Also, in Example 2.4 the set } A = \mathbb{Q} \text{ is } \tau_2\text{-}\ast\text{-dense-in-itself but not } (2,1)\text{-regular closed.}

3. \((i,j)\)-\(A_2\)-sets and \((i,j)\)-\(\mathcal{J}\)-locally closed sets

Definition 3.1. A subset \(A\) of an ideal bitopological space \((X, \tau_1, \tau_2, \mathcal{J})\) is called an
1. \((i,j)\)-\(A_2\)-set if \(A = U \cap V\), where \(U \in \tau_i\) and \(V\) is an \((i,j)\)-\text{regular-J-closed set},
2. \((i,j)\)-\mathcal{J}\text{-locally closed set (briefly \((i,j)\)-\mathcal{J}\text{-LC set}) if } A = U \cap V\), where \(U \in \tau_i\) and \(V\) is \(\tau_j\text{-}\ast\text{-perfect.}

Proposition 3.2. Let \((X, \tau_1, \tau_2, \mathcal{J})\) be an ideal bitopological space and \(A\) a subset of \(X\). Then the following hold:
1. If \(A\) is a \(\tau_1\text{-open set and } (X, \tau_1, \tau_2, \mathcal{J})\) is a Haššis-Samuels space, then \(A\) is an \((i,j)\)-\(A_2\)-set.
2. If \(A\) is an \((i,j)\)-\text{regular-J-closed set, then } A\) is an \((i,j)\)-\(A_2\)-set.

Proof: Since \(X \in \tau_i\) and \(X\) is an \((i,j)\)-\text{regular-J-closed set, the proof is obvious.}

Remark 3.3. The converses of Proposition 3.2 need not be true as the following example shows.

Example 3.4. Let \(X = \mathbb{R}\), \(\tau_1 = \{\emptyset, \mathbb{Q}, X\}\), \(\tau_2 = \) usual topology on \(\mathbb{R}\) and \(\mathcal{I}\) be the ideal of finite subsets of \(\mathbb{R}\). Let \(A = \mathbb{Q}\). Then \(A\) is a \((1,2)\)-\(A_2\)-set but it is not \((1,2)\)-\text{regular-J-closed. In Example 2.5, the set } A = [0,1] \text{ is a } (1,2)\text{-}\(A_2\)-set but it is not } \tau_1\text{-open.}
Proposition 3.5. Let \((X, \tau_1, \tau_2, J)\) be an ideal bitopological space and \(A\) a subset of \(X\). Then the following hold:
1. If \(A\) is an \((i, j)-A_1\)-set then \(A\) is an \((i, j)-C_2\)-set and an \((i, j)-J\)-locally-closed set.
2. If \(A\) is an \((i, j)-A_2\)-set then \(A\) is an \((i, j)-A\)-set.

Proof: This is an immediate consequence of Proposition 2.2 and 2.8. \(\square\)

Remark 3.6. The converses of Proposition 3.5 need not be true. In Example 2.10, the set \(A = [1, 2]\) is a \((1, 2)-A\)-set but it is not a \((1, 2)-A_1\)-set. In Example 2.16, the set \(A = \mathbb{Q} \cup \{\sqrt{2}\}\) is a \((1, 2)-C_2\)-set but it is not a \((1, 2)-A_1\)-set. In Example 2.6, the set \(A = \{2, 4, 6, 8, \ldots\}\) is \((1, 2)-J\)-locally-closed but it is not a \((1, 2)-A_1\)-set.

Proposition 3.7. For a subset \(A\) of a Hayashi-Samuels space \((X, \tau_1, \tau_2, J)\), the following are equivalent:
1. \(A\) is a \(\tau_i\)-open set.
2. \(A\) is an \((i, j)-\alpha-J\)-open set and an \((i, j)-A_3\)-set.
3. \(A\) is an \((i, j)-\alpha-J\)-open set and an \((i, j)-A\)-set.

Proof: 1 \(\Rightarrow\) 2. Let \(A\) be a \(\tau_i\)-open set. Hence \(A\) is an \((i, j)-\alpha-J\)-open set. On the other hand, \(A = A \cap X\), where \(A \in \tau_i\) and \(X\) is an \((i, j)\)-regular-3-closed set. Hence \(A\) is an \((i, j)-A\)-set.
2 \(\Rightarrow\) 3. This is obvious since every \((i, j)-\alpha-J\)-open set is \((i, j)-\alpha-J\)-open.
3 \(\Rightarrow\) 1. Let \(A\) be an \((i, j)-\alpha-J\)-open set and an \((i, j)-A_3\)-set. Then \(A = U \cap V\), where \(U \in \tau_i\) and \(V\) is an \((i, j)\)-regular-3-closed set. Now, \(A = U \cap A \subseteq U \cap \tau_i\)-int\((\tau_j-\text{cl}^*(A)) = U \cap \tau_i\)-int\((\tau_j-\text{cl}^*(U \cap V)) \subseteq U \cap \tau_i\)-int\((\tau_j-\text{cl}^*(U) \cap \tau_j-\text{cl}^*(V)) = U \cap \tau_i\)-int\((\tau_j-\text{cl}^*(U) \cap V) = U \cap \tau_i\)-int\((\tau_j-\text{cl}^*(U)) \cap \tau_i\)-int\((V) = U \cap \tau_i\)-int\((V) \subseteq \tau_i\)-int\((U \cap V)\). That is, \(A \subseteq \tau_i\)-int\((U \cap V)\) and also \(U \cap V \supseteq U \cap \tau_i\)-int\((V) \subseteq \tau_i\)-int\((U \cap V)\). Therefore \(A = \tau_i\)-int\((U \cap V) = \tau_i\)-int\((A)\). Hence \(A\) is \(\tau_i\)-open. \(\square\)

4. Decomposition of Pairwise continuity

Definition 4.1. \([13]\) A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is said to be \(p\)-continuous if the induced mappings \(f : (X, \tau_k) \to (Y, \sigma_k), (k = 1, 2)\) are continuous.

Definition 4.2. A function \(f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)\) is said to be \(p\)-continuous \([15]\) (resp. \(p\)-pre-continuous \([8]\), \(p\)-\(A\)-continuous \([8]\)) if for every \(V \in \sigma_1\), \(f^{-1}(V)\) is \((i, j)-\alpha\)-open (resp. \((i, j)-\alpha\)-pre-open, \((i, j)-A\)-set) of \((X, \tau_1, \tau_2)\).

Definition 4.3. \([14]\) A function \(f : (X, \tau_1, \tau_2, J) \to (Y, \sigma_1, \sigma_2)\) is said to be \(p\)-\(\alpha\)-continuous (resp. \(p\)-pre-\(\alpha\)-continuous, \(p\)-\(C_3\)-continuous) if for every \(V \in \sigma_1\), \(f^{-1}(V)\) is \((i, j)-\alpha\)-open (resp. \((i, j)-\alpha\)-pre-open, \((i, j)-C_3\)-set) of \((X, \tau_1, \tau_2, J)\).

Definition 4.4. A function \(f : (X, \tau_1, \tau_2, J) \to (Y, \sigma_1, \sigma_2)\) is said to be \(p\)-\(A_3\)-continuous (resp. \(p\)-\(J\)-\(LC\)-continuous) if for every \(V \in \sigma_1\), \(f^{-1}(V)\) is an \((i, j)-A_3\)-set (resp. \((i, j)-J\)-\(LC\)-set) of \((X, \tau_1, \tau_2, J)\).
Proposition 4.5. 1. Every p-A\(_3\) continuous function is p-C\(_3\) continuous.
2. Every p-A\(_3\) continuous function is p-J-LC-continuous.
3. Every p-A\(_3\) continuous function is p-A-continuous.

Proof: The proof follows from Proposition 3.5. \(\square\)

Remark 4.6. The converses of Proposition 4.5 need not be true as seen from the following examples show.

Example 4.7. Let \(X=\{a, b, c\}\) with topologies \(\tau_1=\{\emptyset, \{a, b\}, X\}\) and \(\tau_2=\{\emptyset, \{a, b\}, X\}\) and an ideal \(J=\{\emptyset, \{a\}, \{a, b\}\}\) and let \(Y=\{p, q, r\}\) with topologies \(\sigma_1=\{\emptyset, \{r\}, Y\}\) and \(\sigma_2=\{\emptyset, \{p, q\}, Y\}\). Let \(f: (X, \tau_1, \tau_2, J) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function defined as \(f(a)=p\) and \(f(b)=q\) and \(f(c)=r\). Then \(f\) is p-C\(_3\)-continuous but not p-A\(_3\)-continuous.

Example 4.8. Let \(X=\{a, b, c\}\) with topologies \(\tau_1=\{\emptyset, \{a, b\}, X\}\) and \(\tau_2=\{\emptyset, \{a, b\}, \{b, c\}, X\}\) and an ideal \(J=\{\emptyset, \{a\}, \{a, c\}\}\) and let \(Y=\{p, q, r\}\) with topologies \(\sigma_1=\{\emptyset, \{q\}, \{q, r\}, Y\}\) and \(\sigma_2=\{\emptyset, \{p\}, Y\}\). Let \(f: (X, \tau_1, \tau_2, J) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function defined as \(f(a)=p\), \(f(b)=q\) and \(f(c)=r\). Then \(f\) is p-J-LC-continuous but not p-A\(_3\)-continuous.

Example 4.9. Let \(X=\{a, b, c, d\}\) with topologies \(\tau_1=\{\emptyset, \{d\}, \{a, c\}, \{a, c, d\}, X\}\) and \(\tau_2=\{\emptyset, \{a\}, \{d\}, \{a, d\}, X\}\) and an ideal \(J=\{\emptyset, \{a\}, \{a, d\}\}\) and let \(Y=\{p, q, r\}\) with topologies \(\sigma_1=\{\emptyset, \{q\}, \{q, r\}, Y\}\) and \(\sigma_2=\{\emptyset, \{p\}, Y\}\). Let \(f: (X, \tau_1, \tau_2, J) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function defined as \(f(a)=p\), \(f(b)=f(c)=q\) and \(f(d)=r\). Then \(f\) is p-A-continuous but not p-A\(_3\)-continuous.

Theorem 4.10. Let \((X, \tau_1, \tau_2, J)\) be a Hayashi-Samuels space. For a function \(f: (X, \tau_1, \tau_2, J) \rightarrow (Y, \sigma_1, \sigma_2)\), the following are equivalent:
1. \(f\) is p-continuous.
2. \(f\) is p-\(\alpha\)-J-continuous and p-A\(_3\)-continuous.
3. \(f\) is p-pre-\(\alpha\)-J-continuous and p-A\(_3\)-continuous.

Proof: The proof is obvious from Proposition 3.7. \(\square\)

Corollary 4.11. Let \((X, \tau_1, \tau_2, J)\) be an ideal bitopological space and \(J=\{\emptyset\}\) or \(ij-N\). For a function \(f: (X, \tau_1, \tau_2, J) \rightarrow (Y, \sigma_1, \sigma_2)\), the following are equivalent:
1. \(f\) is p-continuous.
2. \(f\) is p-\(\alpha\)-continuous and p-A-continuous.
3. \(f\) is p-pre-\(\alpha\)-continuous and p-A-continuous.

Proof: Let \(I=\{\emptyset\}\). Then we have \(A^*\alpha=\tau_j-\text{cl}(A)\) and hence \(\tau_j-\text{cl}^*(A)=\tau_j-\text{cl}(A)\) for any subset \(A\) of \(X\). Therefore, we obtain \(A\) is \((i, j)\)-\(\alpha\)-open if and only if it is \((i, j)\)-\(\alpha\)-open. By Proposition 2.13, \(A\) is an \((i, j)\)-A\(_3\)-set if and only if it is an \((i, j)\)-A-set and \(A\) is \((i, j)\)-pre-A\(_3\)-open if and only if it is \((i, j)\)-pre-open. The proof of the corollary follows immediately from Lemma 1.6 and Theorem 4.10.
Let \( I = ij - N \). Then we have \( A_j^* = \tau_j - cl(\tau_j - int(\tau_j - cl(A))) \) and \( A_j^* = A \cup \tau_j - cl(\tau_j - int(\tau_j - cl(A))) \) for any subset \( A \) of \( X \). Therefore, \( \tau_j - int(\tau_j - cl(\tau_j - int(\tau_j - cl(\tau_j - int(A)))))) = \tau_j - int(\tau_j - cl(\tau_j - int(A))) \). We obtain \( A \) is \((i, j)\)-\( \alpha \)-open if and only if it is \((i, j)\)-\( \alpha \)-open. By Proposition 2.13, \( A \) is an \((i, j)\)-\( \alpha \)-set if and only if it is an \((i, j)\)-\( \alpha \)-open. The proof of the corollary follows immediately from Lemma 1.6 and Theorem 4.10. 

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