Properties of $\Gamma^2$ defined by a modulus function

C. Murugesan and N. Subramanian

Abstract: In this article, we introduce the generalized difference paranormed double sequence spaces $\Gamma^2(\Delta^m_\gamma, f, p, q, s)$ and $\Lambda^2(\Delta^m_\gamma, f, p, q, s)$ defined over a semi-normed sequence space $(X, q)$.

Key Words: entire sequence, analytic sequence, modulus function, semi norm, difference sequence, double sequence, duals.

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1. Introduction

Throughout $w, \chi$ and $\Lambda$ denote the classes of all gai and analytic scalar valued single sequences, respectively. We write $w^2$ for the set of all complex sequences $(x_{mn})$, where $m, n \in \mathbb{N}$, the set of positive integers. Then, $w^2$ is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are due to Bromwich [4]. Later, the double sequence spaces were studied by Hardy [5], Moricz [9], Moricz and Rhoades [10], Basarir and Solankan [2], Tripathy [17], Turkmenoglu [19], and many others.

Let us define the following sets of double sequences:

$$M_u(t) := \left\{ (x_{mn}) \in w^2 : \sup_{m,n \in \mathbb{N}} |x_{mn}|^{t_{mn}} < \infty \right\},$$

$$C_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn} - t_{mn}| = 1 \text{ for some } t \in \mathbb{C} \right\},$$

$$C_0_p(t) := \left\{ (x_{mn}) \in w^2 : p - \lim_{m,n \to \infty} |x_{mn}|^{t_{mn}} = 1 \right\},$$

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\[ \mathcal{L}_u(t) := \left\{ \{x_{mn}\} \in \omega^2 : \sum_{m=1}^\infty \sum_{n=1}^\infty |x_{mn}|^{t_{mn}} < \infty \right\}, \]

\[ \mathcal{C}_{bp}(t) := \mathcal{C}_p(t) \cap \mathcal{M}_u(t) \text{ and } \mathcal{C}_{bop}(t) = \mathcal{C}_0(t) \cap \mathcal{M}_u(t); \]

where \( t = (t_{mn}) \) is the sequence of strictly positive reals \( t_{mn} \) for all \( m,n \in \mathbb{N} \) and \( p = \lim_{m,n \to \infty} \) denotes the limit in the Pringsheim’s sense. In the case \( t_{mn} = 1 \) for all \( m,n \in \mathbb{N}; \mathcal{M}_u(t), \mathcal{C}_p(t), \mathcal{C}_{bop}(t), \mathcal{L}_u(t), \mathcal{C}_{bp}(t) \) and \( \mathcal{C}_{bop}(t) \) reduce to the sets \( \mathcal{M}_u, \mathcal{C}_p, \mathcal{C}_{bop}, \mathcal{L}_u, \mathcal{C}_{bp} \) and \( \mathcal{C}_{bop} \), respectively. Now, we may summarize the knowledge given in some document related to the double sequence spaces. Göklan and Colak[21,22] have proved that \( \mathcal{M}_u(t) \) and \( \mathcal{C}_p(t), \mathcal{C}_{bop}(t) \) are complete paranormed spaces of double sequences and gave the \( \alpha-\), \( \beta-\), \( \gamma-\) duals of the spaces \( \mathcal{M}_u(t) \) and \( \mathcal{C}_{bp}(t) \). Quite recently, in her PhD thesis, Zeltser [23] has essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely [24] have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Cesàro summable double sequences. Nextly, Mursaleen [25] and Mursaleen and Edely [26] have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the \( M- \) core for double sequences and determined those four dimensional matrices transforming every bounded double sequences \( x = (x_{ij}) \) into one whose core is a subset of the \( M- \) core of \( x \). More recently, Altay and Basar [27] have defined the spaces \( \mathcal{B}_k, \mathcal{B}_k(t), \mathcal{C}_k, \mathcal{C}_{bop}, \mathcal{C}_r, \mathcal{L}_u, \mathcal{C}_{bp} \) and \( \mathcal{B}_v \) of double sequences consisting of all double series whose sequence of partial sums are in the spaces \( \mathcal{M}_u, \mathcal{C}_p(t), \mathcal{C}_{bop}, \mathcal{C}_r \) and \( \mathcal{L}_u \), respectively, and also examined some properties of those sequence spaces and determined the \( \alpha-\) duals of the spaces \( \mathcal{B}_k, \mathcal{B}_k(t), \mathcal{C}_k, \mathcal{C}_{bop}, \mathcal{C}_r \) and the \( \beta(\theta) - \) duals of the spaces \( \mathcal{C}_{bop} \) and \( \mathcal{C}_r \) of double series. Quite recently Basar and Sever [28] have introduced the Banach space \( \mathcal{L}_q \) of double sequences corresponding to the well-known space \( \ell_q \) of single sequences and examined some properties of the space \( \mathcal{L}_u \). Quite recently Subramanian and Misra [29] have studied the space \( \chi_{2q}^0(p,q,n) \) of double sequences and gave some inclusion relations.

We need the following inequality in the sequel of the paper. For \( a, b \geq 0 \) and \( 0 < p < 1 \), we have

\[ (a + b)^p \leq a^p + b^p \quad (1) \]

The double series \( \sum_{m,n=1}^\infty x_{mn} \) is called convergent if and only if the double sequence \( (s_{mn}) \) is convergent, where \( s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}(m,n \in \mathbb{N}) \) (see [1]).

A sequence \( x = (x_{mn}) \) is said to be double analytic if \( \sup_{m,n} |x_{mn}|^{1/m+n} < \infty \). The vector space of all double analytic sequences will be denoted by \( \Lambda^2 \). A sequence \( x = (x_{mn}) \) is called double gai sequence if \( |x_{mn}|^{1/m+n} \to 0 \) as \( m,n \to \infty \). The double entire sequences will be denoted by \( \Gamma^2 \). By \( \phi \), we denote the set of all finite sequences.

Consider a double sequence \( x = (x_{ij}) \). The \( (m,n)^{th} \) section \( x_{[m,n]} \) of the sequence
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is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,n} x_{ij} \mathcal{I}_{ij}$ for all $m, n \in \mathbb{N}$; where $\mathcal{I}_{ij}$ denotes the double sequence whose only non zero term is 1 in the $(i,j)^{th}$ place.

An FK-space (or a metric space) $X$ is said to have AK property if $(\mathcal{I}_{mn})$ is a Schauder basis for $X$. Or equivalently $x^{[m,n]} \to x$.

An FDK-space is a double sequence space endowed with a complete metrizable; locally convex topology under which the coordinate mappings $x = (x_k) \to (x_{mn})(m, n \in \mathbb{N})$ are also continuous.

Orlicz [13] used the idea of Orlicz function to construct the space $(L^M)$, Lindenstrauss and Tzafriri [7] investigated Orlicz sequence spaces in more detail, and they proved that every Orlicz sequence space $\ell_M$ contains a subspace isomorphic to $\ell_p, (1 \leq p < \infty)$. Subsequently, different classes of sequence spaces were defined by Parashar and Choudhary [14], Mursaleen et al. [11], Bektas and Altin [3], Tripathy et al. [18], Rao and Subramanian [15], and many others. The Orlicz sequence spaces are the special cases of Orlicz spaces studied in [6].

Recalling [13] and [6], an Orlicz function is a function $M : [0, \infty) \to [0, \infty)$ which is continuous, non-decreasing, and convex with $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function $M$ is replaced by subadditivity of $M$, then this function is called modulus function, defined by Nakano [12] and further discussed by Ruckle [16] and Maddox [8], and many others.

An Orlicz function $M$ is said to satisfy the $\Delta_2-$condition for all values of $u$ if there exists a constant $K > 0$ such that $M(2u) \leq KM(u)$ $(u \geq 0)$. The $\Delta_2-$condition is equivalent to $M(\ell u) \leq K\ell M(u)$, for all values of $u$ and for $\ell > 1$.

Lindenstrauss and Tzafriri [7] used the idea of Orlicz function to construct Orlicz sequence space

$$\ell_M = \left\{ x \in w : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) < \infty, \text{ for some } \rho > 0 \right\} ,$$

The space $\ell_M$ with the norm

$$||x|| = \inf \left\{ \rho > 0 : \sum_{k=1}^{\infty} M \left( \frac{|x_k|}{\rho} \right) \leq 1 \right\} ,$$

becomes a Banach space which is called an Orlicz sequence space. For $M(t) = t^\alpha (1 \leq p < \infty)$, the spaces $\ell_M$ coincide with the classical sequence space $\ell_p$.

If $X$ is a sequence space, we give the following definitions:

(i)$X' = \text{the continuous dual of } X$;

(ii)$X^\alpha = \left\{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} |a_{mn}x_{mn}| < \infty, \text{ for each } x \in X \right\}$;
(iii) $X^\beta = \{ a = (a_{mn}) : \sum_{m,n=1}^{\infty} a_{mn} x_{mn} is convergent, for each x \in X \}$;

(iv) $X^\gamma = \{ a = (a_{mn}) : \sup_{M,N} \geq 1 | \sum_{m,n=1}^{M,N} a_{mn} x_{mn} | < \infty, for each x \in X \}$;

(v) let $X be a Banach space with the norm $\Phi$; then $X^f = \{ f(\beta_{mn}) : f \in X^f \}$;

(vi) $X^s = \{ a = (a_{mn}) : \sup_{m,n} |a_{mn} x_{mn}|^{1/m+n} < \infty, for each x \in X \}$.

$X^\alpha, X^\beta, X^\gamma$ and $X^s$ are called $\alpha$–(or Köthe–Toeplitz) dual of $X$, $\beta$–(or generalized Köthe–Toeplitz) dual of $X$, $\gamma$–dual of $X$, and $s$–dual of $X$ respectively. $X^\alpha$ is defined by Gupta and Kamptan [20]. It is clear that $X^\alpha \subset X^\beta$ and $X^\alpha \subset X^\gamma$, but $X^\alpha \subset X^\gamma$ does not hold, since the sequence of partial sums of a double convergent series need not to be bounded.

The notion of difference spaces of single sequences was introduced by Kizmaz [30] as follows

$$Z(\Delta) = \{ x = (x_k) \in w : (\Delta x_k) \in Z \}$$

for $Z = c, c_0$ and $\ell_\infty$, where $\Delta x_k = x_k - x_{k+1}$ for all $k \in \mathbb{N}$. Here $w, c, c_0$ and $\ell_\infty$ denote the classes of all, convergent, null and bounded scalar valued single sequences respectively. The above difference spaces are Banach spaces normed by

$$||x|| = |x_1| + \sup_{k \geq 1} |\Delta x_k|$$

Later on the notion was further investigated by many others. We now introduce the following difference double sequence spaces defined by

$$Z(\Delta) = \{ x = (x_{mn}) \in w^2 : (\Delta x_{mn}) \in Z \}$$

where $Z = \Lambda^2, \Gamma^2$ and $\Delta x_{mn} = (x_{m,n} - x_{m+n+1}) - (x_{m,n+1} - x_{m+n}) = x_{mn} - x_{mn+1} - x_{m+n} + x_{m+n+1}$ for all $m, n \in \mathbb{N}$

2. Definitions and Preliminaries

$\Gamma^2_3$ and $\Lambda^2_3$ denote the Pringsheim’s sense of double Orlicz space of entire sequences and Pringsheim’s sense of double Orlicz space of bounded sequences respectively.

The notion of a modulus function was introduced by Nakano [12]. We recall that a modulus $f$ is a function from $[0, \infty) \rightarrow [0, \infty)$, such that

(1) $f(x) = 0$ if and only if $x = 0$

(2) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,

(3) $f$ is increasing,

(4) $f$ is continuous from the right at 0. Since $|f(x) - f(y)| \leq f(|x - y|)$, it follows from condition (4) that $f$ is continuous on $[0, \infty)$.

Let $p = (p_{mn})$ be a sequence of strictly positive real numbers and $s \geq 0$. Let $X$ be semi normed space over the field $\mathbb{C}$ of complex numbers with the semi norm $q$. The symbol $w^2(X)$ denotes the space of all sequences defined over $X$, such that
Define the sets:

\[ \Gamma^2_M = \left\{ x \in \mathbb{R}^2 : \left( M \left( \frac{(|x_m|^{1/m+n})}{\rho} \right) \right) \to 0 \text{ as } m, n \to \infty \text{ for some } \rho > 0 \right\} \]

and

\[ \Lambda^2_M = \left\{ x \in \mathbb{R}^2 : \sup_{m,n \geq 1} \left( M \left( \frac{|x_m|^{1/m+n}}{\rho} \right) \right) < \infty \text{ for some } \rho > 0 \right\} . \]

The space \( \Gamma^2_M \) and \( \Lambda^2_M \) is a metric space with the metric

\[ d(x, y) = \inf \left\{ \rho > 0 : \sup_{m,n \geq 1} \left( M \left( \frac{|x_m - y_n|}{\rho} \right) \right)^{1/m+n} \leq 1 \right\} \]

Now we define the following sequence spaces:

\[ \Gamma^2(\Delta^m_w, f, p, q, s) = \left\{ x \in \mathbb{R}^2(X) : (mn)^{-s} \left( f(\Delta^m_w x_m) \right)^{m+n} \to 0 \text{ as } m, n \to \infty , s \geq 0 \right\} \]

\[ \Lambda^2(\Delta^m_w, f, p, q, s) = \left\{ x \in \mathbb{R}^2(X) : \sup_{m,n} (mn)^{-s} \left( f(\Delta^m_w x_m) \right)^{m+n} < \infty , s \geq 0 \right\} \]

where

\[ \Delta^0_w x_m = (v_m x_m) , \Delta^w x_m = (v_m x_m - v_{m+1} x_{m+1} - v_{m+1} x_{m+1} + v_{m+1} x_{m+1}) \]

\[ \Delta^m_w x_m = \Delta^m x_m - \Delta^{m-1} x_m = \Delta^m x_m - \Delta^m x_{m+1} - \Delta^{m-1} x_{m+1} + \Delta^{m-1} x_{m+1} \]

where \( f \) is a modulus function. The following inequality will be used through this article. Let \( p = (p_{mn}) \) be a sequence of positive real numbers with \( 0 < p_{mn} \leq \sup_{mn} p_{mn} = H , D = \max \{ 1, 2^H \} \) . Then, for \( a_{mn}, b_{mn} \in \mathbb{C} \), we have

\[ |a_{mn} + b_{mn}|^{p_{mn}} \leq D \{ |a_{mn}|^{p_{mn}} + |b_{mn}|^{p_{mn}} \} \]

Some well-known spaces are obtained by specializing \( f, s, q, v, \) and \( m \).

(1) If \( f(x) = x \), \( m = 0 \), \( v = (v_{mn}) = \)

\[
\begin{pmatrix}
1, & 1, & \ldots, & 1, & 0, \\
1, & 1, & \ldots, & 1, & 0, \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
1, & 1, & \ldots, & 1, & 0, \\
0, & 0, & \ldots, & 0, & 0,
\end{pmatrix}
\]

with 1 in the upto \( (m, n)^{th} \) position and zero other wise and \( q(x) = |x| \), then

\( \Gamma^2(\Delta^0_w, f, p, q, s) = \Gamma^2(p, s) \) and \( \Lambda^2(\Delta^0_w, f, p, q, s) = \Lambda^2(p, s) \).
Definition 3.3. (1) A sequence space \( x \in X \) whenever \((m, n) \) all

\[
\begin{pmatrix}
1, 1, \ldots, 1, 1, 0, \\
1, 1, \ldots, 1, 1, 0, \\
\quad \\
1, 1, \ldots, 1, 1, 0, \\
0, 0, \ldots, 0, 0, 0, \\
\end{pmatrix}
\]

\( v \) with \( 1 \) in the upto \((m, n)\)th position and zero other wise and \( q(x) = |x|, s = 0, \) then \( \Gamma^2 (\Delta^m, f, p, q, s) = \Gamma^2 (p) \) and \( \Lambda^2 (\Delta^m, f, p, q, s) = \Lambda^2 (p) \).

(2) Symmetric if

(3) Sequence algebra if

Definition 3.4. Let \( p, q \) be semi norms on a linear space \( X. \) Then \( p \) is said to be stronger than \( q \) if \( (x_{mn}) \) is a sequence such that \( p(x_{mn}) \to 0, \) whenever \( q(x_{mn}) \to 0. \) If each is stronger than the other, then the \( p \) and \( q \) are said to be equivalent.

Lemma 3.2. Let \( p \) and \( q \) be semi norms on a linear space \( X. \) Then \( p \) is stronger than \( q \) if and only if there exists a constant \( M \) such that \( q(x) \leq Mp(x) \) for all \( x \in X. \)

Definition 3.3. (1) A sequence space \( X \) is said to be solid or normal if \( (\alpha_{mn}x_{mn}) \in X \) whenever \( (x_{mn}) \in X \) and for all sequences of scalars \( (\alpha_{mn}) \) with \( |\alpha_{mn}| \leq 1, \) for all \( m, n \in \mathbb{N}. \)

(2) Symmetric if \( (x_{mn}) \in X \) implies \( (x_{\pi(mn)}) \in X, \) where \( \pi (mn) \) is a permutation of \( \mathbb{N} \times \mathbb{N}; \)

(3) Sequence algebra if \( x \cdot y \in X \) whenever \( x, y \in X. \)

Definition 3.4. A sequence space \( X \) is said to be monotone if it contains the canonical pre-images of all its step spaces.

Remark 3.5. From Definition 3.3 and 3.4, it is clear that if a sequence space \( X \) is solid then \( X \) is monotone.

Definition 3.6. A sequence space \( X \) is said to be convergence free if \( (y_{mn}) \in X \) whenever \( (x_{mn}) \in X \) and \( x_{mn} = 0 \) implies that \( y_{mn} = 0. \)
4. Main Results

**Theorem 4.1.** Let \( p = (p_{mn}) \) be a analytic sequence. Then \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) are linear spaces.

**Proof:** Let \( x, y \in \Gamma^2 (\Delta^m_v, f, p, q, s) \). For \( \lambda, \mu \in \mathbb{C} \), there exists positive integers \( M_\lambda \) and \( N_\mu \), such that \( |\lambda| \leq M_\lambda \) and \( |\mu| \leq N_\mu \). Since \( f \) is subadditive, \( q \) is a seminorm, and \( \Delta^m_v \) is linear, we have

\[
(\sum mn (\lambda x_{mn} + \mu y_{mn}))^{-s} \left[ f \left( q \left( |\Delta^m_v (\lambda x_{mn} + \mu y_{mn})| \right) \right) \right] \leq D \left( \max (1, |M_\lambda|^H) \right) (\sum mn (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v x_{mn}| \right) \right) \right])^{p_{mn}} + \\
D \left( \max (1, |N_\mu|^H) \right) (\sum mn (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v y_{mn}| \right) \right) \right])^{p_{mn}} \rightarrow 0 \text{ as } m, n \rightarrow \infty.
\]

This means that \( \lambda x + \mu y \in \Gamma^2 (\Delta^m_v, f, p, q, s) \). Hence, \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) is a linear space.

**Theorem 4.2.** The space \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) is a paranormed space, paranormed by \( g(x) = \sum_{i=1}^n \sum_{j=1}^n f (q(v_{ij} w_{ij} + \sup_m (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v x_{mn}| \right) \right) \right])^{p_{mn}/M} \)

where \( M = \max (1, \sup_m p_{mn}) \)

**Proof:** Clearly \( g(x) = g(-x) \) for all \( x \in \Gamma^2 (\Delta^m_v, f, p, q, s) \). It is trivial that \( (|\Delta^m_v x_m|)^{-s} = \theta \) for \( x_m = \theta \),

\[
\begin{pmatrix}
\theta, \theta, \ldots, \theta, \\
\theta, \theta, \ldots, \theta,
\end{pmatrix}
\]

and is the zero element of \( X \). Since \( q(\theta) = 0 \) and \( f(0) = 0 \), we get \( g(\theta) = 0 \). Since \( t_{mn} = p_{mn}/M \leq 1 \), if \( a_{mn} \) and \( b_{mn} \) are complex numbers, then we have

\[
|a_{mn} + b_{mn}|_{t_{mn}} \leq D \left( |a_{mn}|_{t_{mn}} + |b_{mn}|_{t_{mn}} \right)
\]

Since \( M \geq 1 \), the above inequality implies that

\[
\sum_{i=1}^n \sum_{j=1}^n f (q(v_{ij} x_{ij} + \sup_m (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v x_{mn}| \right) \right) \right])^{p_{mn}/M} + \\
\sum_{i=1}^n \sum_{j=1}^n f (q(v_{ij} y_{ij} + \sup_m (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v y_{mn}| \right) \right) \right])^{p_{mn}/M}.
\]

Now, it follows that \( g \) is subadditive. Next, let \( \lambda \) be a non-zero scalar. The continuity of scalar multiplication follows from the inequality

\[
g(\lambda x) \leq K_{\lambda} \sum_{i=1}^n \sum_{j=1}^n f (q(v_{ij} x_{ij} + \sup_m (mn)^{-s} \left[ f \left( q \left( |\Delta^m_v x_{mn}| \right) \right) \right])^{p_{mn}/M} \leq \max (1, K_{\lambda}^{H/M}) \quad g(x),
\]

where \( K_{\lambda} \) is an integer such that \( |\lambda| < K_{\lambda} \). This completes the proof.
Theorem 4.3. Let $f, f_1$ and $f_2$ be modulus functions, $q, q_1$ and $q_2$ be seminorms, and $s, s_1$ and $s_2 ≥ 0$. Then,

1. $f_2^2 (Δ_m^q, f_1, p, q, s) ⊆ f^2 (Δ_m^q, f_1, p, q, s)\) ,
2. $f_2^2 (Δ_m^q, f_1, p, q, s) \cap f^2 (Δ_m^q, f_2, p, q, s) \subseteq f^2 (Δ_m^q, f_1 + f_2, p, q, s)\) ,
3. $f_2^2 (Δ_m^q, f_1, p, q, s) \cap f^2 (Δ_m^q, f, p, q_2, s) \subseteq f^2 (Δ_m^q, f_1, p, q_1 + q_2, s)\) ,
4. If $q_1 < q_2$, then $f^2 (Δ_m^q, f, p, q_1, s) \subseteq f^2 (Δ_m^q, f, p, q_2, s)\) ,
5. If $s_1 \leq s_2$, then $f_2^2 (Δ_m^q, f, p, q, s_1) \subseteq f_2^2 (Δ_m^q, f, p, q, s_2)\) .

Proof: Let $S_{mn} = (mn)^{-s} \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \right]^{p_{mn}} \to 0, (m, n \to \infty)$

Let $\epsilon > 0$ and choose $\delta$ with $0 < \delta < 1$ such that $f (t) < \epsilon$ for $0 \leq t \leq \delta$. Now we write

$I_1 = \{ (m, n) \in \mathbb{N} : f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \leq \delta \}$,

$I_2 = \{ (m, n) \in \mathbb{N} : f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) > \delta \}$,

If $x \in f_2^2 (Δ_m^q, f_1, p, q, s)$, then for $f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) > \delta$, we have

$f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) < f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \delta^{-1} < 1 + \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \delta^{-1} \right]$

where $m, n \in I_2$ and $[u]$ denotes the integer part of $u$. Given $\epsilon > 0$, by the definition of $f$, we have for $f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) > \delta$, $f \left( f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \right) \leq (1 + \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \delta^{-1} \right] f (1) \leq 2 f (1) \left( f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \delta^{-1} \right)$ and hence,

$\left( mn \right)^{-s} \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \right]^{p_{mn}} \leq \left( 2 f (1) \delta^{-1} \right)^{s} S_{mn} < \epsilon, (m, n \in I_2)$

and $m, n > m_2 n_2$. If $x \in f_2^2 (Δ_m^q, f_1, p, q, s)$, for $f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) < \epsilon$, where $(m, n) \in I_1$. Therefore, given $\epsilon > 0$ if $m, n \in I_2$, we have

$\left( mn \right)^{-s} \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \right]^{p_{mn}} \leq \left( mn \right)^{-s} \max \left( \epsilon^{inf p_{mn}}, \epsilon^{sup p_{mn}} \right) < \epsilon$

$(m, n \in I_1), mn > m_1 n_1$

From (2) and (3) for every $m, n > \max \{ (m_1 n_1), (m_2 n_2) \}$,

$\left( mn \right)^{-s} \left[ f_1 \left( q \left( |Δ_m^q x_{mn}| \right) \right) \right]^{p_{mn}} < \epsilon.$
Hence, \( x \in \Gamma^2 (\Delta^m_v, f \circ f_1, p, q, s) \). Thus, \( \Gamma^2 (\Delta^m_v, f_1, p, q, s) \subseteq \Gamma^2 (\Delta^m_v, f \circ f_1, p, q, s) \).

(2) It follows from the inequality

\[
(mn)^{-s} \left[ (f_1 + f_2) \left( \left| \Delta^m_v x_{mn} \right| \right)^{\frac{1}{p+q}} \right] \leq \frac{D}{(mn)^{-s}} \left[ f_1 \left( \left| \Delta^m_v x_{mn} \right| \right)^{\frac{1}{p+q}} \right] + \frac{D}{(mn)^{-s}} \left[ f_2 \left( \left| \Delta^m_v x_{mn} \right| \right)^{\frac{1}{p+q}} \right].
\]

Since (3), (4) and (5) can be established by the same way, we omit the detail. \( \square \)

**Proposition 4.4.** The following inclusion relations hold:

1. \( \Gamma^2 (\Delta^m_v, p, q, s) \subseteq \Gamma^2 (\Delta^m_v, f \circ f_1, p, q, s) \),
2. \( \Gamma^2 (\Delta^m_v, f, p, q) \subseteq \Gamma^2 (\Delta^m_v, f, p, q, s) \),
3. \( \Gamma^2 (\Delta^m_v, p, q) \subseteq \Gamma^2 (\Delta^m_v, f, p, q, s) \).

The proof of the inclusions in (1)-(3) is routine verification. So, we leave it to the reader.

**Proposition 4.5.** If \( q_1 \geq q_2 \), then \( \Gamma^2 (\Delta^m_v, f, p, q_1, s) = \Gamma^2 (\Delta^m_v, f, p, q_2, s) \).

**Theorem 4.6.** For any two sequences \( p = (p_{mn}) \) and \( t = (t_{mn}) \) of strictly positive real numbers and for any two semi norms \( q_1 \) and \( q_2 \) on \( X \), the spaces \( \Gamma^2 (\Delta^m_v, f, p, q_1, s) \) and \( \Gamma^2 (\Delta^m_v, f, p, q_2, s) \) are not disjoint.

**Proof:** Since the zero element belongs to each of the above classes of double sequences, the intersection is non empty. \( \square \)

**Theorem 4.7.** For any two sequences \( (p_{mn}) \) and \( (t_{mn}) \), we have \( \Gamma^2 (\Delta^m_v, f, t, q) \subset \Gamma^2 (\Delta^m_v, f, p, q) \) if and only if \( \liminf \frac{p_{mn}}{t_{mn}} > 0 \).

**Proof:** If we take \( y_{mn} = f \left( q \left( \left| \Delta^m_v x_{mn} \right| \right) \right) \) for all \( m, n \in \mathbb{N} \). \( \square \)

**Theorem 4.8.** For any two sequences \( (p_{mn}) \) and \( (t_{mn}) \), the spaces \( \Gamma^2 (\Delta^m_v, f, t, q) \) and \( \Gamma^2 (\Delta^m_v, f, p, q) \) are identical if and only if \( \liminf \frac{p_{mn}}{t_{mn}} > 0 \) and if and only if \( \liminf \frac{t_{mn}}{p_{mn}} > 0 \).

**Theorem 4.9.** \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) is not solid for \( m > 0 \)

To prove that the space \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) is not solid, in general, we give the following counter-example: Let \( X = \mathbb{C} \), \( f(x) = x, q(x) = |x|, \alpha_{mn} = (-1)^{p_{mn}}, s = 0, v = (v_{mn}) \) with \( v = (v_{mn}) = p_{mn} = 1 \) for all \( m, n \in \mathbb{N} \). Then, \( |x_{mn}|^{\frac{1}{p_{mn}}} = (mn)^{m-1} \in \Gamma^2 (\Delta^m_v, f, p, q, s) \), but \( \alpha_{mn} x_{mn} \notin \Gamma^2 (\Delta^m_v, f, p, q, s) \).

**Theorem 4.10.** \( \Gamma^2 (\Delta^m_v, f, p, q, s) \) is not sequence algebra.
Example 4.13. Let \( q(x) = |x|, f(x) = x, s = 0, v = (v_{mn}) = \begin{pmatrix} 1, & 1, & \ldots & 1 \\ 1, & 1, & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1, & 1, & \ldots & 1 \end{pmatrix} \) and \( p_{mn} = 1 \) for all \( m, n \in \mathbb{N} \).

Consider \( |x_{mn}|^{\frac{1}{m+n}} = (mn)^{m-1} \) and \( |y_{mn}|^{\frac{1}{m+n}} = (mn)^{m-1} \), then \( x, y \in \Gamma^2(\Delta_v^m, f, p, q, s) \) and \( x \cdot y \notin \Gamma^2(\Delta_v^m, f, p, q, s) \).

**Theorem 4.11.** The space \( \Gamma^2(\Delta_v^m, f, p, q, s) \) is not convergence free in general.

**Proof:** To prove that the space \( \chi^2(\Delta_v^m, f, p, q, s) \) is not convergence free, in general, we give the following counter-example: Consider the sequences \((\Delta_v^m x_{mn})\), \((\Delta_v^m y_{mn})\) \(\in \Gamma^2(\Delta_v^m, f, p, q, s)\) defined by \((\Delta_v^m x_{mn}) = \left(\frac{1}{m+n}\right)^{m+n}\) and \((\Delta_v^m y_{mn}) = \left(\frac{m-n}{m+n}\right)^{m+n}\) for all \( m, n \in \mathbb{N} \). Then,

\[
(mn)^{-s} \left[ f \left( q \left( \frac{1}{m+n} \right) \right) \right]^{p_{mn}} \rightarrow 0, \quad \text{as} \quad m, n \rightarrow \infty,
\]

which implies that \((\Delta_v^m x_{mn}) \rightarrow 0\) as \( m, n \rightarrow \infty \). Similarly, \((mn)^{-s} \left[ f \left( q \left( \frac{m-n}{m+n} \right) \right) \right]^{p_{mn}} \rightarrow 0 \) as \( m, n \rightarrow \infty \). But, \( \{\Delta_v^m y_{mn}\} \) does not tends to zero, as \( m, n \rightarrow \infty \). This step completes the proof. \( \square \)

**Theorem 4.12.** Let \( f \) be a modulus function. Then \( \Gamma^2(\Delta_v^m, f, p, q, s) \subseteq \Lambda^2(\Delta_v^m, f, p, q, s) \) and the inclusions are strict.

**Proof:**

\[
(mn)^{-s} \left[ f \left( q \left( |\Delta_v^m x_{mn}| \right) \right) \right]^{p_{mn}} \leq D(mn)^{-s} \left[ f \left( q \left( |\Delta_v^m x_{mn}| \right) \right) \right]^{p_{mn}} \]

Then, there exists an integer \( K \) such that

\[
(mn)^{-s} \left[ f \left( q \left( |\Delta_v^m x_{mn}| \right) \right) \right]^{p_{mn}} \leq D(mn)^{-s} \left[ f \left( q \left( |\Delta_v^m x_{mn}| \right) \right) \right]^{p_{mn}} + \max \left[ 1, (K)^H \right].
\]

Therefore, \( x \in \Lambda^2(\Delta_v^m, f, p, q, s) \). \( \square \)

**Example 4.13.** Let \( q(x) = |x|, f(x) = 0, s = 0, v = (v_{mn}) = \begin{pmatrix} 1, & 1, & \ldots & 1 \\ 1, & 1, & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1, & 1, & \ldots & 1 \end{pmatrix} \) and \( p_{mn} = 1 \) for all \( m, n \in \mathbb{N} \). Then \( x = (mn)^{m+n} = (mn)^{m^2+mn} \in \Lambda^2(\Delta_v^m, f, p, q, s) \), but \( x \notin \Gamma^2(\Delta_v^m, f, p, q, s) \). Since \( |\Delta_v^m (mn)^{m-n} = (-1)^m \cdot m! \).
Properties of $\Gamma^2$ defined by a modulus function

References

22. A.Gökhan and R.Colak, Double sequence spaces $\ell_2^p$, ibid., 160(1), (2005), 147-153.


