A New Characterization of PSL(2, 27)

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Abstract: Let $G$ be a group and $\pi_e(G)$ be the set of element orders of $G$. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. Set $nse(G) := \{m_k | k \in \pi_e(G)\}$. In this paper, we prove if $G$ is a group such that $nse(G) = nse(PSL(2, 27))$, then $G \cong PSL(2, 27)$.

Key Words: Element order, set of the numbers of elements of the same order, Sylow subgroup.

Contents

1. Introduction 43

2. Preliminary Results 44

3. Proof of the Main Theorem 46

1. Introduction

If $n$ is an integer, then we denote by $\pi(n)$ the set of all prime divisors of $n$. Let $G$ be a finite group. Denote by $\pi(G)$ the set of primes $p$ such that $G$ contains an element of order $p$. Also the set of element orders of $G$ is denoted by $\pi_e(G)$. A finite group $G$ is called a simple $K_n$–group, if $G$ is a simple group with $|\pi(G)| = n$. Set $n_i = m_i(G) := |\{g \in G | \text{the order of } g \text{ is } i\}|$. In fact, $n_i$ is the number of elements of order $i$ in $G$, and $nse(G) := \{m_i | i \in \pi_e(G)\}$, the set of numbers of elements with the same order. Throughout this paper, we denote by $\phi$ the Euler totient function.

If $G$ is a finite group, then we denote by $P_q$ a Sylow $q$–subgroup of $G$ and $n_q(G)$ is the number of Sylow $q$–subgroup of $G$, that is, $n_q(G) = |\text{Syl}_q(G)|$. All further unexplained notation is standard and we refer to [1], for example.

The problem of characterizing groups $G$ by the set $nse(G)$ was first studied by Shao et al. [2] where the authors proved that the simple $K_4$–group $G$ are characterized by the set $nse(G)$ and the group order $|G|$. In [3], the authors showed that the alternating group $A_n$ for $4 \leq n \leq 6$ are uniquely determined by only the set of numbers of elements of the same order. Later on, it is proved in [4] that the simple groups $PSL(2, q)$ for $q \in \{7, 8, 11, 13\}$ are also characterized by this set and they

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asked whether \( G \cong \text{PSL}(2, q) \) if \( nse(G) = nse(\text{PSL}(2, q)) \), where \( q \) is a prime power.

In this paper, we give a positive answer to this question and show that the group \( \text{PSL}(2, q) \) is characterizable by only \( nse(G) \) for \( q = 27 \). In fact the main theorem of our paper is as follows:

**Main Theorem:** Let \( G \) be a group. Suppose \( nse(G) = nse(\text{PSL}(2, 27)) \). Then \( G \cong \text{PSL}(2, 27) \).

We note that although we apply the technique used in [4], but by that method, we cannot characterized the group with order more than 2000. Because, they used the GAP program and in the library of GAP, there are only the groups with order less than 2000. In this paper, we use a new technique for the proof of our main result and our method can work for the groups with order more than 2000.

2. Preliminary Results

In this section we present some preliminary lemmas that will be used in the proof of the main theorem.

**Lemma 2.1.** [5, Theorem 9.3.1] Let \( G \) be a finite solvable group and \( |G| = m \cdot n \), where \( m = p_1^{\alpha_1} \cdots p_r^{\alpha_r} \), \( (m, n) = 1 \). Let \( \pi = \{p_1, \ldots, p_r\} \) and \( h_m \) be the number of \( \pi \)-Hall subgroups of \( G \). Then \( h_m = q_1^{\beta_1} \cdots q_s^{\beta_s} \) satisfies the following conditions for all \( i \in \{1, 2, \ldots, s\} \):

1. \( q_i^{\beta_i} \equiv 1 \pmod{p_j} \), for some \( p_j \).
2. The order of some chief factor of \( G \) is divisible by \( q_i^{\beta_i} \).

**Lemma 2.2.** [6] If \( G \) is a simple \( K_3 \)-group, then \( G \) is isomorphic to one of the following groups: \( A_5, A_6, \text{PSL}(2, 7), \text{PSL}(2, 8), \text{PSL}(2, 17), \text{PSL}(3, 3), \text{PSU}(3, 3) \) or \( \text{PSU}(4, 2) \).

**Lemma 2.3.** [7] Let \( G \) be a simple \( K_4 \)-group. Then \( G \) is isomorphic to one of the following groups:

1. \( A_7, A_8, A_9, A_{10} \).
2. \( M_{11}, M_{11}, J_2 \).
3. (a) \( L_2(r) \), where \( r \) is a prime and satisfies \( r^2 - 1 = 2^a3^b \cdot c \) with \( a \geq 1, b \geq 1, c \geq 1 \) and \( v > 3 \) is a prime;
   (b) \( L_2(2^m) \), where \( m \) satisfies \( 2^m - 1 = u, 2^m + 1 = 3t^b \), with \( m \geq 2, u, t \) are primes, \( t > 3, b \geq 1 \).
(c) $L_2(3^m)$, where $m$ satisfies $3^m + 1 = 4t$, $3^m - 1 = 2u$, with $m \geq 2$, $u$, $t$ odd primes, $b \geq 1$, $c \geq 1$;

(d) $L_2(16)$, $L_2(25)$, $L_2(49)$, $L_2(81)$, $L_3(4)$, $L_3(5)$, $L_3(7)$, $L_3(8)$, $L_3(17)$, $L_4(3)$, $S_4(4)$, $S_4(5)$, $S_4(7)$, $S_4(9)$, $S_6(2)$, $O_5^+(2)$, $G_2(3)$, $U_3(4)$, $U_3(5)$, $U_3(7)$, $U_3(8)$, $U_5(9)$, $U_4(3)$, $U_5(2)$, $Sz(8)$, $Sz(32)$, $3D_4(2)$, $2F_4(2)'$.

**Lemma 2.4.** [2] Let $G$ be a finite group and let $P \in \text{Syl}_p(G)$, where $p \in \pi(G)$. Let $G$ have a normal series $K \triangleleft L \triangleleft G$. If $P \leq L$ and $p \nmid |K|$, then the following hold:

1. $N_{G/K}(PK/K) = N_G(P)K/K$;
2. $|G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(G) = n_p(L)$;
3. $|L/K : N_{L/K}(PK/K)t| = |G : N_G(P)| = |L : N_L(P)|$, that is, $n_p(L/K)t = n_p(G) = n_p(L)$ for some positive integer $t$, and $|N_K(P)t| = |K|$.

**Lemma 2.5.** [8] Let $G$ be a finite group and $m$ be a positive integer dividing $|G|$. If $L_m(G) = \{g \in G | g^m = 1\}$, then $m \mid |L_m(G)|$.

**Lemma 2.6.** [3] Let $G$ be a group containing more than two elements. Let $k \in \pi_e(G)$ and $m_k$ be the number of elements of order $k$ in $G$. If $s = \sup \{m_k | k \in \pi_e(G)\}$ is finite, then $G$ is finite and $|G| \leq s(s^2 - 1)$.

**Lemma 2.7.** [9] Let $G$ be a finite group and $p \in \pi(G)$ be odd. Suppose that $P$ is a Sylow $p$-subgroup of $G$ and $n = p^m$, where $(p, m) = 1$. If $P$ is not cyclic and $s > 1$, then the number of elements of order $n$ is always a multiple of $p^s$.

**Lemma 2.8.** [10] Let $G$ be a finite group and $M$ be normal subgroup of $G$. Then both the Sylow $p$-number $n_p(M)$ and the Sylow $p$-number $n_p(G/M)$ of the quotient $G/M$ divide the Sylow $p$-number $n_p(G)$ of $G$ and moreover $n_p(M) \ n_p(G/M) \mid n_p(G)$.

Let $G$ be a group such that $\text{nse}(G) = \text{nse}(\text{PSL}(2, 27))$. By Lemma 2.6, we can assume that $G$ is finite. Let $m_n$ be the number of elements of order $n$. We note that $m_n = k\phi(n)$, where $k$ is the number of cyclic subgroups of order $n$ in $G$. Also we note that if $n > 2$, then $\phi(n)$ is even. If $n \in \pi_e(G)$, then by Lemma 2.5 and the above notation we have:
Theorem 3.6. Let $G$ be a group such that $\text{nse}(G) = \text{nse}(\text{PSL}(2, 27)) = \{1, 351, 728, 2106, 4536\}$. At first, we prove that $\pi(G) \subseteq \{2, 3, 7, 13\}$. Since $351 \in \text{nse}(G)$, it follows from $(\ast)$ that $2 \in \pi(G)$ and $m_2 = 351$. Let $2 \neq p \in \pi(G)$, by $(\ast)$, $p | (1 + m_p)$ and $(p - 1) | m_p$, which implies that $p \in \{3, 7, 13\}$. Therefore, $\pi(G) \subseteq \{2, 3, 7, 13\}$. If $3, 7$ and $13 \in \pi(G)$, then $m_3 = 728, m_7 = 2106$ and $m_{13} = 4536$, by $(\ast)$. Suppose that $13 \in \pi(G)$. Because $\phi(13^2) = 156$ and $\phi(13^3) = 2028$, by $(\ast)$ we can see easily that $G$ does not contain any elements of order $13^2$ and $13^3$. Thus $\exp(P_{13}) = 13$ and $|P_{13}| | (1 + m_{13}) = 4537$ by Lemma 2.5. Hence $|P_{13}| = 13$ and $n_{13} = m_{13}/\phi(13) = 378$ | $|G|$. Therefore if $13 \in \pi(G)$, because $n_{13} | |G|$ this implies that $3$ and $7 \in \pi(G)$. As $\phi(16) = 8, \phi(49) = 42$ and $\phi(729) = 486$, it is easy to check that $G$ does not contain any elements of order $16, 49$ and $729$. If $7 \in \pi(G)$, then $|P_7| | (1 + m_7) = 2107$. Hence $|P_7| | 49$. Also since $16 \notin \pi_e(G)$, we have $|P_2| | 16$. We know that if $3 \in \pi(G)$, then $3$ and $7 \in \pi(G)$. So if we show that $\pi(G)$ could not be the sets $\{2\}$ and $\{2, 3\}$, $\{2, 7\}$ and $\{2, 3, 7\}$, then $\pi(G)$ must be equal to $\{2, 3, 7, 13\}$. We consider the following cases:

Case a. $\pi(G) = \{2\}$. We have $\pi_e(G) \subseteq \{1, 2, 4, 8\}$ and so $|\pi_e(G)| \leq 4$, which is a contradiction since $|\text{nse}(G)| = 5$. Thus this case impossible.

Case b. $\pi(G) = \{2, 3\}$. Since $729 \notin \pi_e(G)$, we have $\exp(P_3) = 3, 9, 27, 81$ or $243$. If $\exp(P_3) = 3$, then $|P_3| | (1 + m_3) = 729$. Hence $|P_3| | 3^6$. Let $|P_3| = 3$. Then $n_3 = m_3/\phi(3) = 364$ | $|G|$ since $7 \notin \pi(G)$, we get a contradiction. So $|G| = 2^m \times 3^n$ where $m \leq 4$ and $2 \leq n \leq 6$, on the other hand, $7722 \leq |G|$ and so $m = 4$ and $n = 6$. Since $\pi_e(G) \subseteq \{1, 2, 4, 8\} \cup \{3, 3 \times 2, 3 \times 4, 3 \times 8\}$ and the sum of all the numbers in $\text{nse}(G)$ is $7722$, we have $|G| = 11664 = 7722 + 2106k_1 + 2106k_2 + 4536k_3$ where $0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 3$. Therefore, $3942 = 728k_1 + 2106k_2 + 4536k_3$. It is easy to check that this equation has no solution.

If $\exp(P_3) = 9$, then $|P_3| | (1 + m_3 + m_9)$ by Lemma 2.5. Since $m_9 \in \{2106, 4536\}$, we have $|P_3| | 3^4$. On the other hand, $|P_2| | 16$ and $7722 \leq |G|$, a contradiction. Similarly if $\exp(P_3) = 27$, then $|P_3| | 3^4$, a contradiction.

In the proof of the main theorem, we apply $(\ast)$ and the above comments.
If \( \exp(P_3) = 81 \), \( |P_3| \mid (1 + m_3 + m_9 + m_{27} + m_{81}) \), so \( |P_3| \mid 3^4 \). It is clear that \( |G| = 11664 = 3^6 \times 16 \). Since \( \pi_v(G) \subseteq \{1, 2, 4, 8\} \cup \{3, 3 \times 2, 3 \times 4, 3 \times 8\} \cup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \cup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \cup \{81, 81 \times 2, 81 \times 4, 81 \times 8\} \), we have \( |G| = 11664 = 7722 + 728k_1 + 2106k_2 + 4536k_3 \) where \( 0 \leq k_1 + k_2 + k_3 = |\pi_v(G)| - |\text{nse}(G)| \leq 15 \). It is easy to check that this equation has no solution.

If \( \exp(P_3) = 243 \), then \( |P_3| = 3^n \) where \( n \geq 5 \). If \( n = 5 \) since \( m_{243} \in \{2106, 4536\} \), we have \( n_3 = m_{243}/\phi(243) = 13 \) or 28. As the group \( P_3 \) is cyclic of order 243, it has two elements of order 3. Since every element of order 3 lies in one or more of Sylow 3-subgroups, \( m_3 \leq 2 \times 28 = 56 \), a contradiction. If \( n > 5 \), then by Lemma 2.7, 243 \( | \) \( m_{243}, \) a contradiction.

**Case c.** \( \pi(G) = \{2, 7\} \). Since 49 \( \not\in \pi_v(G) \), we have \( \exp(P_7) = 7 \). Then \( |P_7| \mid (1 + m_7) = 2107 \). Hence \( |P_7| \mid 49 \). Assume \( |P_7| = 7 \), so \( n_7 = m_7/\phi(7) = 351 \mid |G| \). Since 13 \( \not\in \pi(G) \), we get a contradiction. If \( |P_7| = 49 \), then by \( |P_2| \mid 16 \) and 7722 \( \not\leq |G| \), we get a contradiction.

**Case d.** \( \pi(G) = \{2, 3, 7\} \). With the same argument as in Case c, since 13 \( \not\in \pi(G) \) we obtain \( |P_7| = 49 \). Hence \( |G| = 2^m \times 3^n \times 49 \) where \( m \leq 4 \) and \( n \leq 6 \).

We know that \( \pi_v(G) \subseteq \{1, 2, 4, 8\} \cup \{3, 3 \times 2, 3 \times 4, 3 \times 8\} \cup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \cup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \cup \{81, 81 \times 2, 81 \times 4, 81 \times 8\} \cup \{243, 243 \times 2, 243 \times 4, 243 \times 8\} \cup \{7, 2 \times 7, 4 \times 7, 8 \times 7\} \cup \{3 \times 7, 9 \times 7, 27 \times 7, 81 \times 7, 243 \times 7\} \cup \{2 \times 3 \times 7, 2 \times 9 \times 7, 2 \times 27 \times 7, 2 \times 81 \times 7, 4 \times 3 \times 7, 4 \times 9 \times 7, 4 \times 27 \times 7, 4 \times 81 \times 27, 8 \times 3 \times 7, 8 \times 9 \times 7, 8 \times 27 \times 7, 8 \times 81 \times 7\} \), then \( |\pi_v(G)| \leq 45 \). Therefore, \( |G| = 2^m \times 3^n \times 49 = 7722 + 728k_1 + 2106k_2 + 4536k_3 \) where \( 0 \leq k_1 + k_2 + k_3 = |\pi_v(G)| - |\text{nse}(G)| \leq 40 \), \( m \leq 4 \) and \( n \leq 6 \). By an easy computer calculation we can see that if \( n = 6 \) then this equation has no solution.

If \( n < 6 \), then \( n_7 = 1, 8 \) or \( 2^i \times 3^j \) where \( 1 \leq i \leq 4 \) and \( 1 \leq j \leq 5 \). If \( n_7 = 1, 8 \), since every element of order 7 lies in one or more of Sylow 7-subgroups, we have \( m_7 \leq 48 \times 8 \), a contradiction. So \( n_7 = 2^i \times 3^j \) where \( 1 \leq i \leq 4 \) and \( 1 \leq j \leq 5 \).

We show that \( G \) is a nonsolvable group. Suppose that \( G \) is a solvable group. Then by Lemma 2.1, \( 3^j \equiv 1 \) (mode 7), a contradiction. Hence we conclude that \( G \) is a finite nonsolvable group. Let \( N \) be the solvable radical subgroup of \( G \) and let \( H/N \) be a chief factor of \( G \). Then \( H/N \) is non-abelian and so it is isomorphic to a direct product of isomorphic non-abelian simple groups. We know that \( G \) is a \( K_3 \)-group, thus \( H/N \) is a simple \( K_3 \)-group or \( H/N \) is a direct product of simple \( K_3 \)-groups. By Lemma 2.2, \( H/N \cong \text{PSL}(2, 7) \), \( \text{PSL}(2, 7) \times \text{PSL}(2, 7) \), \( \text{PSL}(2, 8) \) or \( \text{PSL}(2, 8) \times \text{PSL}(2, 8) \). On the other hand, by Lemma 2.8 \( n_p(H/N) \mid n_p(G) \) for
every prime \( p \in \pi(G) \). Hence \( H/N \cong \text{PSL}(2, 7) \) or \( \text{PSL}(2, 8) \). Let \( H/N \cong \text{PSL}(2, 7) \).

Since \( n_{\pi}(\text{PSL}(2, 7)) = 8 \), by Lemma 2.8 we have \( 8 \mid n_{\pi}(G) \), so \( n_{\pi}(G) = 16 \times 81 \).

Therefore, \( |G| = 16 \times 81 \times 49 = 16 \times 243 \times 49 \). On the other hand, if \( |G| = 16 \times 81 \times 49 \) or \( 16 \times 243 \times 49 \), then the equation \( |G| = 7722 + 278k_{1} + 2106k_{2} + 4536k_{3} \) where \( 0 \leq k_{1} + k_{2} + k_{3} = |\pi_{e}(G)| - |\text{nse}(G)| \leq 40 \) has no solution, a contradiction.

Now let \( H/N \cong \text{PSL}(2, 8) \). By Lemma 2.8 \( 36 \mid n_{\pi}(G) \), because \( n_{\pi}(\text{PSL}(2, 8)) = 36 \), so \( n_{\pi}(G) = 36 \) or \( 16 \times 81 \).

Therefore, \( |G| = 4 \times 27 \times 49 \times 8 \times 81 \times 49 \), \( 8 \times 27 \times 49 \times 8 \times 81 \times 49 \). \( 8 \times 234 \times 49 \times 16 \times 9 \times 49 \). \( 16 \times 27 \times 49 \) or \( 16 \times 243 \times 49 \). As \( 7722 \leq |G| \), so \( |G| \neq 4 \times 27 \times 49 \times 8 \times 9 \times 49 \) and \( 16 \times 9 \times 49 \).

Let \( |G| = 4 \times 81 \times 49 \times 8 \times 9 \times 49 \times 8 \times 81 \times 49 \), \( 8 \times 243 \times 49 \), \( 16 \times 9 \times 49 \) or \( 16 \times 243 \times 49 \). Then it is easy to check that the equation \( |G| = 7722 + 278k_{1} + 2106k_{2} + 4536k_{3} \) where \( 0 \leq k_{1} + k_{2} + k_{3} = |\pi_{e}(G)| - |\text{nse}(G)| \leq 40 \) has no solution. Also if \( |G| = 4 \times 243 \times 49 \), then \( \exp(P_{2}) = 2 \) or \( 4 \), so \( |\pi_{e}(G)| \leq 34 \).

Now it is easy to check that the equation \( |G| = 7722 + 278k_{1} + 2106k_{2} + 4536k_{3} \) where \( 0 \leq k_{1} + k_{2} + k_{3} = |\pi_{e}(G)| - |\text{nse}(G)| \leq 29 \) has no solution. Hence this case is impossible.

Therefore, \( \pi(G) = \{2, 3, 7, 13\} \). We know that \( |P_{13}| = 13 \), we will show that \( 91 \notin \pi_{e}(G) \). Suppose that \( 91 \in \pi_{e}(G) \). We know that if \( P \) and \( Q \) are Sylow 13-subgroups of \( G \), then \( P \) and \( Q \) are conjugate, which implies that \( C_{G}(P) \) and \( C_{G}(Q) \) are conjugate in \( G \). Therefore, \( m_{91} = \phi(91) \cdot n_{13} \cdot k \), where \( k \) is the number of cyclic subgroups of order 7 in \( C_{G}(P_{13}) \). Since \( n_{13} = 378 \), we have \( 4536 \mid m_{91} \).

On the other hand, \( 91 \mid (1 + m_{13} + m_{7} + m_{91}) \), which is a contradiction. Hence \( 91 \notin \pi_{e}(G) \).

Since \( 91 \notin \pi_{e}(G) \), the group \( P_{7} \) acts fixed point freely on the set of elements of order 13, and so \( |P_{7}| \mid m_{13} = 4536 \), which implies that \( |P_{7}| = 7 \). Also we can prove that \( 26 \) and \( 21 \notin \pi_{e}(G) \). As \( 21 \notin \pi_{e}(G) \), the group \( P_{7} \) acts fixed point freely on the set of elements of order 7, and so \( |P_{7}| \mid m_{7} = 2106 \), which implies that \( |P_{7}| \mid 81 \). Since \( 26 \notin \pi_{e}(G) \), the group \( P_{7} \) acts fixed point freely on the set of elements of order 13, and so \( |P_{7}| \mid m_{13} = 4536 \), which implies that \( |P_{7}| \mid 8 \). Therefore, \( |G| = 2^{n} \cdot 3^{m} \cdot 7 \times 13 \), where \( n \leq 3 \) and \( m \leq 4 \).

We claim that \( G \) is a nonsolvable group. Suppose \( G \) is a solvable group. Since \( n_{13} = 378 \), we have \( 7 \equiv 1 \) (mod 13) by Lemma 2.1, which is a contradiction.

Hence \( G \) is a nonsolvable group. As \( G \) is a nonsolvable group and \( p \mid |G| \), where \( p \in \{7, 13\} \), \( G \) has a normal series \( 1 \triangleleft N \triangleleft H \triangleleft G \) such that \( N \) is a maximal solvable normal subgroup of \( G \) and \( H/N \) is a nonsolvable minimal normal subgroup of \( G/N \).

Then \( H/N \) is a non-abelian simple \( K_{3} \)-group or \( K_{4} \)-group.

Let \( H/N \) be a non-abelian simple \( K_{3} \)-group. By Lemma 2.2, \( H/N \cong \text{PSL}(2, 7) \) or \( \text{PSL}(2, 8) \). Let \( H/N \cong \text{PSL}(2, 7) \). Assume \( P_{7} \in \text{Syl}_{7}(G) \). Then \( P_{7}N/N \in \text{Syl}_{7}(H/N) \)
where \(0 \leq t \leq 7\). Since \(n_t(H/N) = n_t(PGL(2, 7)) = 8\), we have \(351 = 8t\), which is a contradiction.

Now let \(H/N \cong PGL(2, 8)\). Assume \(P \in Syl_7(G)\). Then \(P/N \in Syl_7(H/N)\). By Lemma 2.4, \(n_7(H/N) = n_7(G)\) for some positive integer \(t\) and \(7 \nmid t\). Since \(n_7(H/N) = n_7(PGL(2, 7)) = 36\), we have \(351 = 36t\), which is a contradiction.

Hence \(H/N\) is a non-abelian simple \(K_4\)-group. By Lemma 2.3, \(H/N \cong PGL(2, 13)\) or \(PGL(2, 27)\). Assume that \(H/N \cong PGL(2, 13)\) and let \(P \in Syl_7(G)\). Thus \(P/N \in Syl_7(H/N)\) and \(n_7(H/N) = n_7(G)\) for some positive integer \(t\) and \(7 \nmid t\). Since \(n_7(H/N) = 78\), we have \(351 = 78t\), which is a contradiction.

Let \(K/N = C_G(H/N)\). Then \(H/N \trianglelefteq G/K \trianglelefteq \text{Aut}(H/N)\), i.e., \(G/K\) is an almost simple group with socle \(H/N\). Thus \(G/K \cong PGL(2, 27)\), \(PGL(2, 27), \text{PGL}(2, 27)\) or \(\text{PGL}(2, 27)\) or \(\text{PGL}(2, 27)\).

Assume \(|G| = 4 \times 81 \times 7 \times 13\). As \(G\) does not contain any elements of order 16, 21, 26, 39, 49, 91, 169 and 243, we have \(\pi_e(G) \subseteq \{1, 2, 4\} \cup \{3\} \cup \{9, 9 \times 2, 9 \times 4\} \cup \{27, 27 \times 2, 27 \times 4\} \cup \{81, 81 \times 2, 81 \times 4\} \cup \{7, 2 \times 7, 4 \times 7, 8 \times 7\} \cup \{13\}\). Hence \(|\pi_e(G)| \leq 20\). Therefore, \(|G| = 8 \times 27 \times 7 \times 13 = 7722 + 728k_1 + 2106k_2 + 4536k_3\) where \(0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 15\). By an easy computer calculation we can get that this equation has no solution.

Let \(|G| = 8 \times 27 \times 7 \times 13\). Since \(\pi_e(G) \subseteq \{1, 2, 4, 8\} \cup \{3\} \cup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \cup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \cup \{7, 2 \times 7, 4 \times 7, 8 \times 7\} \cup \{13\}\), we have \(|\pi_e(G)| \leq 21\). Therefore, \(|G| = 8 \times 27 \times 7 \times 13 = 7722 + 728k_1 + 2106k_2 + 4536k_3\) where \(0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 16\). By an easy computer calculation we can get that this equation has no solution. Assume \(|G| = 8 \times 81 \times 7 \times 13\). Since \(\pi_e(G) \subseteq \{1, 2, 4, 8\} \cup \{3\} \cup \{9, 9 \times 2, 9 \times 4, 9 \times 8\} \cup \{27, 27 \times 2, 27 \times 4, 27 \times 8\} \cup \{81, 81 \times 2, 81 \times 4, 81 \times 8\} \cup \{7, 2 \times 7, 4 \times 7, 8 \times 7\} \cup \{13\}\), we have \(|\pi_e(G)| \leq 25\). Therefore, \(|G| = 8 \times 81 \times 7 \times 13 = 7722 + 728k_1 + 2106k_2 + 4536k_3\) where \(0 \leq k_1 + k_2 + k_3 = |\pi_e(G)| - |\text{nse}(G)| \leq 20\). By an easy computer calculation we can get that this equation has no solution. Therefore, \(|G| = 4 \times 27 \times 7 \times 13\). By [2], since \(PGL(2, 27)\) is a simple \(K_4\)-group, we can conclude that \(G \cong PGL(2, 27)\), and the proof is complete.

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make the proof of our main results substantially simplified.

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