On $\delta$-continuity in mixed fuzzy topological spaces

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ABSTRACT: In this paper we introduce and study the notion of $\delta$-continuity on mixed fuzzy topological spaces. We have investigated this notion in the light of the notion of $q$-neighbourhoods, $q$-coincidence, fuzzy $\delta$-closure, fuzzy $\delta$-interior. We have established the relationship between fuzzy continuity and fuzzy $\delta$-continuity in mixed fuzzy topological spaces.

Key Words: fuzzy closure; fuzzy $\delta$-closure; fuzzy continuity; fuzzy $\delta$-open sets; fuzzy $q$-neighbourhoods.

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1. Introduction

The notion of topological space has been expanded in different directions in the recent past. The notion of bitopological space and mixed topological space has been introduced and their different properties have been investigated. Bitopological spaces has been studied by Ganguly and Singha [9], Tripathy and Sarma [25,26] and others. The notion of mixed topology lies in the theory of strict topology of the spaces of continuous functions on locally compact spaces. The concept of mixed topology is pity old. Mixed topology is a technique of mixing two topologies on a set in order to get a third topology. The works on mixed topology is due to Cooper [7], Buck [5], Das and Baishya [8], Tripathy and Ray [27,28], Wiweger [30] and others.

In 1965 L.A. Zadeh introduced the concept of fuzzy sets. Since then the notion of fuzziness has been applied for the study in all the branches of science and technology. It has been applied in mathematical analysis for introducing and investigating different classes of sequence spaces of fuzzy numbers by Tripathy and Baruah [15,16,17], Tripathy and Borgohain [18,19], Tripathy and Dutta [21,22], Tripathy and Sarma [23,24] and many others in the recent past. The notion of fuzziness has been applied in topology and the notion of fuzzy topological spaces

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has introduced and investigated by many researches on topological spaces. Different properties of fuzzy topological spaces have been investigated by Arya and Singal [2,3], Chang [6], Azad [4], Das and Baishya [8], Ganster et al. [11], Ganguly and Singha [9], Ghanim et al. [12], Katsaras and Liu [13], Warren [29], Wong [31,32], Tripathy and Debnath [20] and many others. Recently mixed fuzzy topological spaces have been investigated from different aspect by Das and Baishya [8], Tripathy and Ray [27,28] and others.

The notion of mixed topology has been investigated by Alexiewicz and Semadeni [1]. They have introduced this notion via two normed spaces. Some stronger forms of fuzzy continuity and $\delta$-continuity in fuzzy settings have been studied by Ganguly and Saha [10], in fuzzy topological spaces. In this paper we introduce the concepts of $\delta$-continuity in mixed fuzzy topological spaces. Mixed fuzzy topological spaces have been investigated recently from different aspects by Das and Baishya [8], Rashid and Ali [14] and Tripathy and Ray [27].

2. Preliminaries and Definitions

Let $X$ be a non-empty set and $I$, the unit interval $[0,1]$. A fuzzy set $A$ on $X$ is characterized by a function $\mu_A : X \to I$, where $\mu_A$ is called the membership function of $A$. $\mu_A(x)$ representing the membership grade of $x$ in $A$. The empty fuzzy set is defined by $\mu_0(x) = 0$ for all $x \in X$. Also $X$ can be regarded as a fuzzy set in itself defined by $\mu_X(x) = 1$ for all $x \in X$. Further, an ordinary subset $A$ of $X$ can be regarded as a fuzzy set in $X$ if its membership function is taken as usual characteristic function of $A$ that is $\mu_A(x) = 1$ for all $x \in X$ and $\mu_A(x) = 0$ for all $x \in X - A$. Two fuzzy set $A$ and $B$ are said to be equal if $\mu_A(x) = \mu_B(x)$. A fuzzy set $A$ is said to be contained in a fuzzy set $B$, written as $A \subseteq B$, if $\mu_A \leq \mu_B$. Complement of a fuzzy set $A$ in $X$ is defined by $\mu_{\complement A} = 1 - \mu_A$. We write $A^c = \complement A$. Union and intersection of a collection $\{A_i : i \in N\}$ of fuzzy sets in $X$, are written as $\bigcup_{i \in N} A_i$ and $\bigcap_{i \in N} A_i$ respectively. Their membership functions are defined as follows.

$$
\mu_{\bigcup_{i \in N} A_i}(x) = \sup\{\mu_{A_i}(x) : i \in N\}, \quad \text{for all } x \in X.
$$

$$
\mu_{\bigcap_{i \in N} A_i}(x) = \sup\{\mu_{A_i}(x) : i \in N\}, \quad \text{for all } x \in X.
$$

**Definition 2.1.** A fuzzy topology $\tau$ on $X$ is a collection of fuzzy sets in $X$ such that $\emptyset, X \in \tau$; if $A_i \in \tau$, $i \in N$, then $\bigcup_{i} A_i \in \tau$; if $A, B \in \tau$ then $A \cap B \in \tau$. The pair $(X, \tau)$ is called a fuzzy topological space (fts). Members of $\tau$ are called open fuzzy sets and the complement of an open fuzzy set is called a closed fuzzy set.

**Definition 2.2.** If $(X, \tau)$ is a fuzzy topological space, then the closure and interior of a fuzzy set $A$ in $X$, denoted by $\text{cl} A$ and $\text{int} A$ respectively, are defined by $\text{cl} A = \bigcap\{B : B \text{ is a closed fuzzy set in } X \text{ and } A \subseteq B\}$ and $\text{int} A = \bigcap\{V : V \text{ is an open fuzzy set in } X \text{ and } V \subseteq A\}$. Clearly, $\text{cl} A$ (respectively $\text{int} A$) is the smallest (respectively largest) closed (respectively open) fuzzy set in $X$ containing (respectively contained in) $A$. If there are more than one topologies on $X$, then the
Definition 2.3. A collection \( B \) of open fuzzy sets in a fts \( X \) is said to be an open base for \( X \) if every open fuzzy set in \( X \) is a union of members of \( B \).

Definition 2.4. If \( A \) is a fuzzy set in \( X \) and \( B \) is a fuzzy set in \( Y \) then, \( A \times B \) is a fuzzy set on \( X \times Y \) defined by \( \mu_{A \times B}(x, y) = \min\{\mu_A(x), \mu_B(y)\} \) for all \( x \in X \) and for all \( y \in Y \). Let \( f \) be a function from \( X \) into \( Y \). Then, for each fuzzy set \( B \) in \( Y \), the inverse image of \( B \) under \( f \), written as \( f^{-1}[B] \), is a fuzzy set on \( X \) defined by \( \mu_{f^{-1}[B]}(x) = \mu_B(f(x)) \) for all \( x \in X \).

Definition 2.5. A fuzzy set \( A \) in a fuzzy topological space \((X, \tau)\) is called a neighborhood of a point \( x \in X \) if and only if there exists \( B \in \tau \) such that \( B \subseteq A \) and \( A(x) = B(x) > 0 \).

Definition 2.6. A fuzzy point \( x_\alpha \) is said to be quasi-coincident with \( A \), denoted by \( x_\alpha qA \), if and only if \( \alpha + A(x) > 1 \) or \( \alpha > (A(x))^c \).

Definition 2.7. A fuzzy set \( A \) is said to be quasi-coincident with \( B \) and is denoted by \( AqB \), if and only if there exists a \( x \in X \) such that \( A(x) + B(x) > 1 \).

Remark 2.8. It is clear that if \( A \) and \( B \) are quasi-coincident at \( x \) both \( A(x) \) and \( B(x) \) are not zero at \( x \) and hence \( A \) and \( B \) intersect at \( x \).

Definition 2.9. A fuzzy set \( A \) in a fts \((X, \tau)\) is called a quasi-neighborhood of \( x_\lambda \) if and only if \( A_1 \in \tau \) such that \( A_1 \subseteq A \) and \( x_\lambda qA_1 \). The family of all \( Q \)-neighborhoods of \( x_\lambda \) is called the system of \( Q \)-neighborhood of \( x_\lambda \). Intersection of two quasi-neighborhoods of \( x_\lambda \) is a quasi-neighborhood of \( x_\lambda \). Let \((X, \tau_1)\) and \((X, \tau_2)\) be two fuzzy topological spaces and let \( \tau_1(\tau_2) \) be defined as follows:

\[
\tau_1(\tau_2) = \{ A \in I^X : \text{for every fuzzy set } B \text{ in } X \text{ with } AqB, \text{ there exists a } \tau_2 \text{-open set } A_\alpha, \text{ such that } A_\alpha qB \text{ and } \tau_1 \text{-closure, } \overline{A_\alpha} \subseteq B \}\.
\]

Then \( \tau_1(\tau_2) \) is a topology on \( X \) and this is called mixed fuzzy topology and the space \((X, \tau_1(\tau_2))\) is called mixed fuzzy topological space is introduced and studied by Tripathy and Ray [27].

Definition 2.10. A fuzzy point \( x_\lambda \) is called a \( \delta \)-cluster point of a fuzzy set \( S \) in a fuzzy topological spaces \((X, \tau)\) if and only if every regular open \( Q \)-neighborhood of \( x_\lambda \) is quasi-coincident with \( S \).

The set of \( \delta \)-cluster points of \( S \) is denoted by \( [S]_\delta \) or \( \delta - clS \). If \( S = [S]_\delta \) then \( S \) is said to be \( \delta \)-closed. The complement of a \( \delta \)-closed set is said to \( \delta \)-open.

Now, we introduce the notion of \( \delta \)-cluster point in a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) as follows:
Definition 2.11. A fuzzy point $x_\lambda$ in a mixed fuzzy topological space is said to be $\delta$-cluster point of a fuzzy set $A$ if and only if, every regularly open $Q$-neighborhood of $x_\lambda$ is quasi-coincident with $A$.

The set of all $\delta$-cluster points of $A$ is denoted by $[A]_\delta$ and $[A] = A \cup A^\delta$, where $A^\delta$ denote the set of all points of $[A]_\delta$ which are not in $A$.

3. Main Results

Definition 3.1. A fuzzy set $A$ in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$ is said to be regularly open set in $X$ if and only if $\text{int}(\text{cl}A) = A$ (closure with respect to $\tau_2$ and interior with respect to $\tau_1$).

Example 3.2. Let $X = \{x, y\}$ be any non-empty set. Let $A = \{(x, 0.5), (y, 0.6)\}$ and $B = \{(x, 0.6), (y, 0.5)\}$ be two fuzzy sets in $X$. Then $\tau_1 = \{\overline{\mathbb{T}}, B\}$ and $\tau_2 = \{\overline{\mathbb{T}}, A\}$ are two fuzzy topologies on $X$.

Now, we construct a mixed topology from these two fuzzy topologies. Since the mixed topology $\tau_1(\tau_2)$ is coarser than $\tau_2$, and it is obvious that $\overline{\mathbb{T}}, A$ belongs to the mixed topology $\tau_1(\tau_2)$, so we have to verify whether $A \in \tau_1(\tau_2)$ or $A \notin \tau_1(\tau_2)$. If $A \in \tau_1(\tau_2)$ and the fuzzy set $B$ in $X$ be such that $A_q B$, so by definition there exists a $\tau_2$-open set such that $B_q \mathbb{T}$ and $\tau_1$-closure $\text{cl}(\mathbb{T}) \subseteq A$.

But, $\tau_1$-closure $\text{cl}(\mathbb{T}) = \mathbb{T}$ and $A \supseteq \mathbb{T}$.

Therefore $A \notin \tau_1(\tau_2)$ and consequently $\tau_1(\tau_2) = \{\overline{\mathbb{T}}, \mathbb{T}\}$. In this mixed fuzzy topology the only regularly open set is $\mathbb{T}$, because $\text{int}(\text{cl}(\mathbb{T})) = \mathbb{T}$ and $\text{int}(\text{cl}(\overline{\mathbb{T}})) = \mathbb{T}$.

**Lemma 3.3.** (Das and Baishya [8]) Let $\tau_1$ and $\tau_2$ be two topologies on a set $X$. If every $\tau_1$-quasi neighbourhood of $x_\lambda$ is $\tau_2$-quasi neighbourhood of $x_\lambda$, for all fuzzy points $x_\lambda$, then $\tau_1$ is coarser than $\tau_2$.

**Theorem 3.4.** Every $\delta$-closed set is closed in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$.

**Proof:** Let $A$ be any $\delta$-closed set in $X$.

Then $A = [A]_\delta$. We show that $A$ is closed in $X$ i.e. $A = \overline{\mathbb{A}}$, for this it is sufficient to show that $\overline{\mathbb{A}} \subseteq [A]_\delta$.

Suppose, $\overline{\mathbb{A}} \notin [A]_\delta$.

Then there exists $x \in X$, such that $\overline{\mathbb{A}}(x) > [A]_\delta(x)$.

$\Rightarrow \overline{A}(x) = \min\{F(x) : F \text{ is fuzzy closed in } X \text{ and } A \subseteq F\} > \max\{A(x), A^\delta(x)\}$. 
Let $F_1$ be the smallest such fuzzy closed set in $X$ containing $A$, then $F_1(x) > A(x)$ and $F_1(x) > A^\delta(x)$. Which contradicts the fact that $A^\delta \supseteq F_1$.

Thus, we must have $\overline{\overline{A}} \subseteq [A]_\delta$, and so $A \subseteq A \subseteq [A]_\delta = A \Rightarrow A = \overline{A}$.

Hence $A$ is a closed fuzzy set in mixed fuzzy topological space $X$. \hfill \Box

**Theorem 3.5.** Every $\delta$-open set is open in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$.

**Proof:** Let $S$ be any $\delta$-open set in $X$.

Since $S$ is $\delta$-open so by definition, its complement, $S^c = 1 - S$ is $\delta$-closed. But we know that every $\delta$-closed set is closed in mixed fuzzy topological spaces by Theorem 3.4.

Therefore $S^c$ is closed in $X$ and consequently $S$ is open in $X$. \hfill \Box

**Theorem 3.6.** For any two fuzzy topologies $\tau_1$ and $\tau_2$ on a set $X$, the mixed fuzzy topology $\tau_1(\tau_2)$ is coarser than $\tau_2$.

**Proof:** Let $A$ be any $\tau_1(\tau_2)$ quasi-neighborhood of a fuzzy point $x_\lambda$. Then there exists a $\tau_1(\tau_2)$-open set $B$ such that $x_\lambda qB$ and $B \subseteq A$.

A fuzzy point can be considered as a fuzzy singleton set.

Therefore $x_\lambda qB$ and $B \in \tau_1(\tau_2)$ implies that there exists $\tau_2$-open set $S$ such that $S qx_\lambda$ and $\tau_1$-closure of $S$, such that $S \subseteq \overline{S} \subseteq B \subseteq A$.

$\Rightarrow A$ is $\tau_2$-quasi neighborhood of the fuzzy point $x_\lambda$.

Hence $\tau_1(\tau_2)$ is coarser than $\tau_2$. \hfill \Box

**Theorem 3.7.** Every fuzzy regularly open set in a mixed fuzzy topological space is fuzzy $\delta$-open.

**Proof:** Let $A$ be any fuzzy regularly open set in a mixed fuzzy topological space $(X, \tau_1(\tau_2))$. We show that $A$ is fuzzy $\delta$-open. It is sufficient to show that $A^c$ is $\delta$-closed i.e. $[A^c]_\delta = A^c$.

Clearly, $A^c \subset [A^c]_\delta \Rightarrow (A^c)^c \supset [A^c]_\delta^c$

$\Rightarrow [A^c]_\delta^c \subset A$. \hfill (3.1)
Now, we show that \( [A^c]_\delta^c \supset A \).

Suppose, \( A \supseteq [A^c]_\delta^c \). Then there exists an element \( x \in X \), such that \( A(x) > [A^c]_\delta^c(x) \). If \( A(x) = \alpha \), then \( 0 < \alpha < 1 \) and \( A(x) > [A^c]_\delta^c(x) \Rightarrow x_\alpha \notin [A^c]_\delta^c \).

Now, \( \alpha > [A^c]_\delta^c(x) \).

\[ \Rightarrow [A^c]_\delta^c(x) > 1 - \alpha. \]  

(3.2)

Also, we have \( A(x) - \alpha + 1 > 1 \).

\[ \Rightarrow A(x) + (1 - \alpha) > 1. \]

(3.3)

\[ \Rightarrow x_1 - \alpha \notin [A^c]_\delta^c. \]

Hence, from (3.2) and (3.3) we get \( Aq[A^c]_\delta^c \).

Since, \( A \) is regularly open set, therefore, \( Aq[A^c]_\delta^c \Rightarrow AqA^c \). Which leads to a contradiction.

Therefore,

\[ A \subseteq [A^c]_\delta^c. \]  

(3.4)

Hence from (3.1) and (3.4) we can conclude that \( [A^c]_\delta^c = A^c \) i.e. \( A^c \) is \( \delta \)-closed i.e. \( A \) is \( \delta \)-open. \( \square \)

**Theorem 3.8.** In a mixed fuzzy topological space a \( \delta \)-open set need not be regularly open.

**Proof:** The results follows from the following example. \( \square \)

**Example 3.9.** Let us consider a non-empty set \( X = \{ x, y \} \) and consider the following fuzzy sets in \( X \).

\[ A = \{(x, .7), (y, .3)\} \quad \text{and} \quad B = \{(x, .3), (y, .7)\}. \]  

Then the collection of fuzzy sets \( \tau_1 = \{0, T, B\} \) and \( \tau_2 = \{0, T, A\} \) are two fuzzy topologies on \( X \).

Now, we construct a mixed fuzzy topology on \( X \) from these two fuzzy topologies \( \tau_1 \) and \( \tau_2 \).
Since the mixed fuzzy topology is coarser than $\tau_2$, so we need only to verify whether the fuzzy set $A$ is in $\tau_1(\tau_2)$ or not?

Let us consider a fuzzy set $S$ in $X$ such that $A \subseteq S$.

Now, the only $\tau_2$-open sets are $\emptyset$ and $A$ such that $A \subseteq S$ and $\tau_2$. Again, $\tau_1$-closure of $A = \{F : F$ is $\tau_1$-closed and $A \subseteq F\}$

$= \emptyset \land A = A \subseteq A$.

Hence, $A \in \tau_1(\tau_2)$ and so $\tau_1(\tau_2) = \{\emptyset, \tau, A\}$.

In this mixed fuzzy topology, the fuzzy regularly open sets are $\emptyset, \tau$, but $A$ is not regularly open because $\tau_1 - \text{Int}(\tau_2 - \text{cl}(A)) = \tau$.

Now, we shall show that $A$ is $\delta$-open. It is sufficient to prove that $A^c = F$ is $\delta$-closed.

Consider the fuzzy point $x_3$ of $B$, Then there exists a regularly open $q$-neighborhood $U$ of a fuzzy point $y_3$ of $Y$, such that $x_3$ is a $\delta$-cluster point of $B$. Similarly we can show that the fuzzy point $y_7$ in $B$ is also $\delta$-cluster point of $B$. Therefore, $B$ is $\delta$-closed and so $A$ is $\delta$-open.

Thus we conclude that a $\delta$-open set need not be regularly open.

4. $\delta$-continuity and continuity

In this section we introduce the notion of fuzzy $\delta$-continuity and continuity between two mixed fuzzy topological spaces and investigate some properties of fuzzy $\delta$-continuity.

**Definition 4.1.** A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_1(\tau_2))$ is said to be fuzzy $\delta$-continuous mapping if for any regularly open $q$-neighborhood $U$ of a fuzzy point $y_\lambda$ of $Y$, there exists a regularly open $q$-neighborhood $V$ of $x_\lambda$ of $X$ such that $f(V) \subseteq U$, where $f(x) = y$.

**Definition 4.2.** A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_1(\tau_2))$ is said to be fuzzy continuous mapping if for any open $q$-neighbourhood $U$ of a fuzzy point $y_\lambda$ of $Y$, there exists an open $q$-neighbourhood $V$ of $x_\lambda$ of $X$ such that $f(V) \subseteq U$, where $f(x) = y$.

The following is an alternative for the above definition.

**Definition 4.3.** A mapping $f : (X, \tau_1(\tau_2)) \to (Y, \tau_1(\tau_2))$ is said to be fuzzy continuous mapping if for any open fuzzy set $U$ in $X$, there exists an open fuzzy set $V$ in $X$ such that $f(V) \subseteq U$, where $f(x) = y$.

**Theorem 4.4.** If $f : X \to Y$ and $g : Y \to Z$ be two fuzzy $\delta$-continuous mapping from a mixed fuzzy topological spaces $X$ into another mixed fuzzy topological space $Y$ and from a mixed fuzzy topological spaces $Y$ into another mixed fuzzy topological space $Z$ respectively, then the mapping $g \circ f : X \to Z$ is also fuzzy $\delta$-continuous.
Proof: Let $U$ be any fuzzy regularly open $q$-neighbourhood of a fuzzy point $z_\lambda$ of $Z$. Since $g$ is fuzzy $\delta$-continuous mapping, so by definition of $\delta$-continuity, there exists a fuzzy regularly open $q$-neighbourhood $V$ of a fuzzy point $y_\lambda$ of $Y$ such that $g(V) \subset U$, where $g(y) = z$.

Since $f$ is fuzzy $\delta$-continuous, so for the fuzzy regularly open $q$-neighbourhood $V$ of a fuzzy point $y_\lambda$ of $Y$, there exists a fuzzy regularly open $q$-neighbourhood $W$ of $x_\lambda$ of $X$ such that $f(W) \subset V$, where $f(x) = y$.

Thus, for any fuzzy regularly open $q$-neighbourhood $U$ of a fuzzy point of $Z$, there exists a fuzzy regularly open $q$-neighbourhood $W$ of the fuzzy point $x_\lambda$ of $X$ such that $f(W) \subset V \Rightarrow g(f(W)) \subset g(V)$ \Rightarrow (g \circ f)(W) \subset U$, where $(g \circ f)(x) = z$.

Hence, $(g \circ f)$ is a fuzzy $\delta$-continuous mapping. \square

Theorem 4.5. Let $f : (X, \tau_1(\tau_2)) \rightarrow (Y, \tau_3(\tau_4))$ be two mappings, then the following conditions are equivalent:

(i) $f$ is $\delta$-continuous.

(ii) $f([A]_\delta) \subset (f[A]_\delta)$ for any fuzzy set $A$ in $X$.

(iii) For every fuzzy $\delta$-closed set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-closed in $X$.

(iv) For every fuzzy $\delta$-open set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-open in $X$.

(v) For every fuzzy regularly open set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-open in $X$.

(vi) For every fuzzy regularly closed set $A$ in $Y$, $f^{-1}(A)$ is fuzzy $\delta$-closed in $X$.

Proof: (i) $\Rightarrow$ (ii) Suppose $f : X \rightarrow Y$ is fuzzy $\delta$-continuous. Consider a fuzzy point $y_\alpha$ in $f([A]_\delta)$, then $f(x) = y \Rightarrow x_\alpha \in [A]_\delta$. Let $U$ be a fuzzy regularly open $q$-neighbourhood of the fuzzy point $y_\alpha$. Since $f$ is fuzzy $\delta$-continuous so there exists a fuzzy regularly open $q$-neighbourhood $V$ of $x_\alpha$ such that $f(V) \subset U$, where $f(x) = y$.

Now, $x_\alpha \in [A]_\delta$ and $V$ is a neighbourhood of $x_\alpha$, therefore $V q A$. 

Next we show that \( f(V)qf(A) \), which will imply that \( Uqf(A) \) and \( U \) is a fuzzy regularly open \( q \)-neighbourhood of \( y_\alpha \), so we conclude that \( y_\alpha \in [f(A)]_\delta \).

Suppose, \( f(V) \) is not quasi-coincident with \( f(A) \)

\[ \Rightarrow \text{For all } y \in Y \text{ such that } f(V)(y) + f(A)(y) \leq 1. \]

\[ \Rightarrow \sup_{z \in f^{-1}(y)} V(z) + \sup_{z \in f^{-1}(y)} A(z) \leq 1. \]

\[ \Rightarrow V(z) + A(z) \leq 1, \text{ for all } z \in f^{-1}(y). \]

In particular, \( V(x) + A(x) \leq 1 \), which implies \( V \) is not quasi-coincident with \( A \). Which leads to a contradiction. Hence, we must have \( f(V)qf(A) \Rightarrow Uqf(A) \).

Therefore, \( y_\alpha \in [f(A)]_\delta \), and so \( [f(A)]_\delta \subset (f[A])_\delta \).

\( (ii) \Rightarrow (iii) \) Let \( A \) be any fuzzy \( \delta \)-closed set in \( Y \).

Then \([A]_\delta = A\). Next we show that \([f^{-1}(A)]_\delta = f^{-1}(A)\).

For any fuzzy set \( A \) in \( Y \), \( f^{-1}(A) \) is a fuzzy set in \( X \).

Hence, by the given condition, \( f([f^{-1}(A)]_\delta \subset [f(f^{-1}[A])_\delta] \).

\[ \Rightarrow f([f^{-1}(A)]_\delta \subset [A]_\delta. \]

\[ \Rightarrow f([f^{-1}(A)]_\delta \subset A \text{ (since } [A]_\delta = A). \]

\[ \Rightarrow [f^{-1}(A)]_\delta \subset f^{-1}(A). \]

Further from definition of \( \delta \)-closure we have \([f^{-1}(A)]_\delta \supset f^{-1}(A)\). Therefore, \( f^{-1}(A) \) is fuzzy \( \delta \)-closed set in \( X \).

\( (iii) \Rightarrow (iv) \) Easy, so omitted.

\( (iv) \Rightarrow (v) \). This implication follows directly from the fact that every regularly open set in a mixed fuzzy topological space is fuzzy \( \delta \)-open (see Theorem 3.7).

\( (v) \Rightarrow (vi) \) Easy, so omitted.

\[ \square \]

**Theorem 4.6.** Every fuzzy \( \delta \)-continuous function is fuzzy continuous between two mixed fuzzy topological spaces.

**Proof:** Let \( f : (X, \tau_1(\tau_2)) \rightarrow (Y, \tau_3(\tau_4)) \) be a fuzzy \( \delta \)-continuous function from a mixed fuzzy topological space \((X, \tau_1(\tau_2))\) into another mixed fuzzy topological
space \((Y, \tau_3(\tau_4))\). Then for every fuzzy \(\delta\)-closed set \(A\) in \(Y\), \(f^{-1}(A)\) is fuzzy \(\delta\)-closed in \(X\).

We know that every fuzzy \(\delta\)-closed set is fuzzy closed set in mixed fuzzy topological space (see Theorem 3.4). Since \(A\) is fuzzy closed set in \(Y\) such that \(f^{-1}(A)\) is fuzzy closed set in \(X\), so by the definition of fuzzy continuity between two mixed fuzzy topological spaces \(f\) is fuzzy continuous function.

But the converse is not necessarily true in general. We give here one example that ensures that fuzzy continuity does not implies \(\delta\)-continuity. \(\square\)

**Example 4.7.** Consider the identity function \(E : (X, \tau_1(\tau_2)) \to (X, \tau_3(\tau_4))\) from a mixed fuzzy topological space \(X\) into another mixed fuzzy topological space \(X\).

Where the mixed fuzzy topological space \((X, \tau_1(\tau_2))\) is defined as follows:

Let \(X = x, y\) and the fuzzy sets are defined by

\[
A_1 = \{(x, 0.2), (y, 0.8)\}, A_2 = \{(x, 0.2), (y, 0.2)\},
A_3 = \{(x, 0.8), (y, 0.2)\}, A_4 = \{(x, 0.8), (y, 0.8)\}.
\]

Then the collection \(\tau_1 = \{\bar{\emptyset}, \emptyset, A_1, A_2, A_3, A_4\}\) will form a fuzzy topology on \(X\).

Consider the following fuzzy sets in \(X\)

\[
B_1 = \{(x, 0.3), (y, 0.7)\}, B_2 = \{(x, 0.7), (y, 0.3)\},
B_3 = \{(x, 0.3), (y, 0.3)\}, B_4 = \{(x, 0.7), (y, 0.7)\}.
\]

Then the collection of fuzzy sets \(\tau_2 = \{\bar{\emptyset}, \emptyset, B_1, B_2, B_3, B_4\}\) will form a fuzzy topology on \(X\).

We construct a mixed fuzzy topology \(\tau_1(\tau_2)\) on \(X\) from these two fuzzy topologies \(\tau_1\) and \(\tau_2\). Since the mixed fuzzy topology is coarser than \(\tau_2\), so we need only to verify whether the fuzzy sets \(B_1, B_2, B_3, B_4\) are in \(\tau_1(\tau_2)\) or not.

Consider any fuzzy set \(U = \{(x, 0.6), (y, 0.5)\}\) in \(X\). Then \(B_1 \not\in U\).

Now, \(B_1 \tau_1(\tau_2)\) if there exists \(\tau_2\)-open set \(B_2\) such that \(B_2 \bar{\tau} U\) and \(\tau_1\)-closure \(\bar{B_2} = \wedge\{F : F\) is \(\tau_1\) - closed and \(B_2 \subseteq F\) = \(\bar{F} \wedge A_4\) . But, \(A_4 \subseteq B_1\) and so \(B_1 \not\in \tau_1(\tau_2)\).

Similarly we can show that \(B_2, B_3, B_4 \not\in \tau_1(\tau_2)\).
Hence $\tau_1(\tau_2) = \{\overline{U}, T\}$.

Another topology on $X$ is defined as in Example 3.8. i.e. $\tau_3(\tau_4) = \{\overline{U}, T, A\}$.

Then the identity function is fuzzy continuous because inverse images of open set is open. But this function is not $\delta$-continuous because the inverse image of $\delta$-open set $A$ is not $\delta$-open.

**Conclusion:** In this paper we introduce the notions of fuzzy $\delta$-open set, fuzzy $\delta$-closed set and fuzzy $\delta$-continuity in mixed fuzzy topological spaces. We have investigated some of their properties. These notions can be applied for further investigations from different aspects.

**References**


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