Statistical convergence of double sequences on probabilistic normed spaces defined by $[V, \lambda, \mu]$-summability

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ABSTRACT: In this paper, we aim to generalize the notion of statistical convergence for double sequences on probabilistic normed spaces with the help of two nondecreasing sequences of positive real numbers $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ such that each tending to $\infty$, also $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$, and $\mu_{n+1} \leq \mu_n + 1$, $\mu_1 = 1$. We also define generalized statistically Cauchy double sequences on PN space and establish the Cauchy convergence criteria in these spaces.

Key Words: Statistical convergence; $\lambda$-statistical convergence; Probabilistic normed spaces.

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1. Introduction

Before we go into the motivation for this paper and present main results, we move through the background of the topic. Menger [12] provoked a crucial generalization of a metric space and called it a probabilistic metric space. This concept was further developed by various authors [2,3,4], [6], [11] and [23,24]. Probabilistic normed space, which is an important family of probabilistic metric spaces, were firstly defined by Šterneš [25]. Alsina et al. [1] gave a new definition of probabilistic normed space making Šterneš definition a special case. As a result, a productive theory agreeable with ordinary normed spaces and probabilistic metric spaces originated.

The notion of statistical convergence of sequence of numbers was introduced by Fast [5] and Schoenberg [22] independently in 1951 and discussed by [7], [13,14], [16,17,18,19,20,21], [26,27], [29] and [31]. During last few years, statistical convergence has been applied in various fields like fourier analysis, ergodic theory and number theory. Mursaleen [15] generalized the notion of statistical convergence with the help of a non-decreasing sequence $\lambda = (\lambda_n)$ of positive numbers tending to $\infty$ with $\lambda_{n+1} \leq \lambda_n + 1$, $\lambda_1 = 1$ and called respectively $\lambda$-statistical convergence. Karakus extended the concept of the statistical convergence for single and double
sequences on probabilistic normed spaces in [8] and [9]. Tripathy et al. [28] discussed the double sequence spaces with the help of Orlicz function and in [30], they extended the concept to double sequence spaces of fuzzy numbers. Recently, Kumar and Mursaleen [10] defined $(\lambda, \mu)$-statistical convergence of double sequences on intuitionistic fuzzy normed spaces. Following Kumar and Mursaleen [10], in this paper, we aim to define strongly $(\lambda, \mu)$-statistical convergence of double sequences on probabilistic normed spaces.

2. Background and preliminaries

First, we recall some notations and basic definitions those will be used in this paper. By a distribution function we mean a function $F : \mathbb{R} \cup \{-\infty, +\infty\} \to [0, 1]$ that is left-continuous and non-decreasing on $\mathbb{R}$ with $F(-\infty) = 0$ and $F(\infty) = 1$. We normalize all distribution functions to be left continuous on unextended real line $\mathbb{R} = (-\infty, +\infty)$. Moreover, for any $a \geq 0$, $\varepsilon_a$ is the distribution function defined by

$$\varepsilon_a(x) = \begin{cases} 0, & x \leq a \\ 1, & x > a \end{cases}.$$ 

Let $\Delta$ denotes the set of all the distribution functions, $\Delta^+ = \{F : F \in \Delta \text{ with } F(0) = 0\}$ and $D^+ \subseteq \Delta^+$ is the set $D^+ = \{F \in \Delta^+ : \lim_{t \to -\infty} F(t) = 1\}$ where $\lim_{t \to -\infty} F(t) = \lim_{x \to -\infty} F(t(x))$. For $F, G \in \Delta^+$, $F \leq G$ if $F(x) \leq G(x)$ for all $x \in \mathbb{R}$ and $(\Delta^+, \leq)$ is a partially ordered set. The maximal element for $\Delta^+$ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0 \end{cases}.$$ 

**Definition 2.1.** A triangle function is a mapping $\tau$ from $\Delta^+ \times \Delta^+$ into $\Delta^+$ such that, for all $F, G, H, K$ in $\Delta^+$,

(i) $\tau(F, \varepsilon_0) = F$;

(ii) $\tau(F, G) = \tau(G, F)$;

(iii) $\tau(F, G) \leq \tau(H, K)$ whenever $F \leq H, G \leq K$;

(iv) $\tau(\tau(F, G), H) = \tau(F, \tau(G, H))$.

Particular and relevant triangle functions are the functions $\tau_T, \tau_{T^*}$ and those of the form $\Pi_T$ which, for any continuous $t$-norm $T$, and any $x > 0$, are given by

$$\tau_T(F, G)(x) = \sup\{T(F(s), G(t)) : s + t = x\}$$

$$\tau_{T^*}(F, G)(x) = \inf\{T^*(F(s), G(t)) : s + t = x\}$$

and

$$\Pi_T(F, G)(x) = T(F(x), G(x)).$$

In 1993, using triangle functions, Alsina et al. [1] defined probabilistic normed spaces as follows:
Definition 2.2. [1] A probabilistic normed space, briefly PN-space, is a quadruple \((V, v, \tau, \tau^*)\) where \(V\) is a real linear space, \(\tau\) and \(\tau^*\) are continuous triangle functions with \(\tau \leq \tau^*\) and \(v\), the probabilistic norm, is a mapping from \(V\) into the space of distribution function \(\Delta^+\) such that writing \(v_p\) for \(v(p)\) for all \(p, q \in V\), the following conditions hold:

(i) \(v_p = \varnothing\) if and only if \(p = \theta\), the null vector in \(V\),
(ii) \(v_{-p} = v_p\),
(iii) \(v_{p+q} \geq \tau(v_p, v_q)\),
(iv) \(v_p \leq \tau^*(v_{1-p}, v_{1-(1-\alpha)p})\) for every \(\alpha \in [0, 1]\).

If, instead of (i), we only have \(v_p = \varepsilon_0\), then we shall speak of a probabilistic pseudo normed space, briefly a PPN-space. If the inequality (iv) is replaced by the equality \(v_p = \tau_{\alpha p}(v_{1-p}, v_{1-(1-\alpha)p})\), then the PN-space is called a Šerstnev space, in this case, a condition stronger than (ii) holds, namely
\[v_{\lambda p} = v_p(\frac{\lambda}{\alpha}), \forall \lambda \neq 0, \forall p \in V,\]

here \(j\) is the identity map on \(\mathbb{R}\). A Šerstnev space is denoted by \((V, v, \tau)\).

There is a natural topology in PN-space \((V, v, \tau, \tau^*)\), called the strong topology. It is defined, for \(t > 0\), by the neighbourhoods
\[N_p(t) = \{q \in V : d_S(v_{q-p}, \varepsilon_0) < t\} = \{q \in V : v_{q-p}(t) > 1 - t\}\]

The strong neighbourhood system for \(V\) is the union \(\bigcup_{p \in V} N_p \lambda\) where \(N_p = \{N_p \lambda : \lambda > 0\}\). The strong neighbourhood system for \(V\) determines a Hausdroff topology for \(V\).

Definition 2.3. Let \((V, v, \tau, \tau^*)\) be a PN-space. A sequence \((p_n)_n\) in \(V\) is said to be strongly convergent to \(p\) in \(V\) if for each \(\lambda > 0\), there exists a positive integer \(N\) such that \(p_n \in N\lambda(\lambda), \forall n \geq N\).

Definition 2.4. Let \((V, v, \tau, \tau^*)\) be a PN-space. A sequence \((p_n)_n\) in \(V\) is called strongly Cauchy sequence if, for every \(\lambda > 0\), there is a positive integer \(N\) such that \(v_{p_n - p_m}(\lambda) > 1 - \lambda\), whenever \(m, n > N\).

Definition 2.5. A PN-space \((V, v, \tau, \tau^*)\) is said to be strongly complete in the strong topology if and only if every strongly Cauchy sequence in \(V\) is strongly convergent to a point in \(V\).

Lemma 2.6. If \(|\alpha| \leq |\beta|\), then \(v_{\beta p} \leq v_{\alpha p}\) for every \(p \in V\).

Definition 2.7. The natural density of a set \(K\) of positive integers is defined by
\[\delta(K) = \lim_{n \to \infty} \frac{1}{n} \{k \in K : k \leq n\}\] Where \(|\{k \in K : k \leq n\}|\) denotes the number of elements of \(K\) not exceeding \(n\).

Definition 2.8. Let \((V, v, \tau, \tau^*)\) be a PN-space. A sequence \((p_n)_n\) in \(V\) is said to be strongly statistical convergent to \(p\) in \(V\) if for each \(\lambda > 0\),
\[\delta\{n \in N : p_n \notin N\lambda(\lambda)\} = 0\]
The element $p$ is called the statistical limit of the sequence $(p_n)_n$ with respect to the probabilistic norm $v$ and we write $st_v \rightarrow \lim p_n = p$

**Definition 2.9.** Let $(V, v, \tau, \tau^*)$ be a PN-space. A sequence $(p_n)_n$ in $V$ is called strongly statistical Cauchy sequence if, for every $\lambda > 0$, there is a positive integer $N$ such that

$$\delta(\{n \in N : p_n \notin N_{p_n}(\lambda)\}) = 0.$$  
Namely, $(p_n)$ is strongly statistically Cauchy if and only if, for every $\lambda > 0$ there exists a number $N$ such that $d_{L}(v_{p_n - p_N}, \varepsilon_0) < \lambda$ for a.a.n.

3. **Strong $(\lambda, \mu)$-statistical convergence of double sequences on a PN-space**

In this section we define and study Strong $(\lambda, \mu)$-statistical convergence of double sequences on probabilistic normed spaces.

**Definition 3.1.** Let $\lambda = (\lambda_n)$ and $\mu = (\mu_n)$ be two nondecreasing sequences of positive real numbers such that each tending to $\infty$ and

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1,$$

$$\mu_{n+1} \leq \mu_n + 1, \mu_1 = 1.$$  
Let $I_n = [n - \lambda_n + 1, n]$ and $I_m = [m - \mu_m + 1, m]$. For any set $K \subseteq N \times N$, the number

$$\delta_{\lambda, \mu}(K) = \lim_{m,n \to \infty} \frac{1}{\lambda_n \mu_m}|\{(i, j) : i \in I_n, j \in I_m, (i,j) \in K\}|,$$

is called the $(\lambda, \mu)$-density of the set $K$ provided the limit exists.

A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)$-statistical convergent to a number $\xi$ provided that for each $\varepsilon > 0$,

$$\lim_{m,n \to \infty} \frac{1}{\lambda_n \mu_m}|\{(i, j) : i \in I_n, j \in I_m, |x_{ij} - \xi| \geq \varepsilon\}| = 0,$$

i.e., the set $K(\varepsilon) = \frac{1}{\lambda_n \mu_m}|\{(i, j) : i \in I_n, j \in I_m, |x_{ij} - \xi| \geq \varepsilon\}$ has $(\lambda, \mu)$-density zero. In this case the number $\xi$ is called the $(\lambda, \mu)$-statistical limit of the sequence $x = (x_{ij})$ and we write $St_{(\lambda, \mu)} - \lim_{i,j \to \infty} x_{ij} = \xi$.

Now we define the strong $(\lambda, \mu)$-statistical convergence of double sequences with respect to PN-space.

**Definition 3.2.** Let $(V, v, \tau, \tau^*)$ be a PN-space. A double sequence $x = (x_{ij})$ of elements in $V$ is said to be strongly $(\lambda, \mu)$-statistical convergent to $\xi$ in $V$ if for each $\lambda > 0$,

$$\delta_{\lambda, \mu}(\{(i, j) : i \in I_n, j \in I_m, x_{ij} \notin N_{\xi}(\lambda)\}) = 0.$$  
equivalently

$$\delta_{\lambda, \mu}(\{(i, j) : i \in I_n, j \in I_m, x_{ij} \in N_{\xi}(\lambda)\}) = 1.$$
In this case the element $\xi$ is called the strong $(\lambda, \mu)$-statistical limit of the sequence $x = x_{ij}$ with respect to the probabilistic norm $v$ and we write $s(l_{v, 1}^{(\lambda, \mu)}) \rightarrow \lim_{i,j \rightarrow \infty} x_{ij} = \xi$.

Let $S(l_{v, 1}^{(\lambda, \mu)})$ denotes the set of all strongly $(\lambda, \mu)$-statistical convergent double sequences with respect to the probabilistic norm $v$.

**Lemma 3.3.** Let $(V, v, \tau, \tau^*)$ be a PN-space and $x = x_{ij}$ be a double sequence of elements in $V$. Then for each $\lambda > 0$, the following statements are equivalent

(i) $s(l_{v, 1}^{(\lambda, \mu)}) \rightarrow \lim_{i,j \rightarrow \infty} x_{ij} = x$.
(ii) $\delta_{(\lambda, \mu)} \{ \{ i, j \} : i \in I_n, j \in I_m, x_{ij} \notin N_\epsilon(\lambda) \} = 0$.
(iii) $\delta_{(\lambda, \mu)} \{ \{ i, j \} : i \in I_n, j \in I_m, x_{ij} \notin N_\epsilon(\lambda) \} = 1$.
(iv) $s(l_{v, 1}^{(\lambda, \mu)}) \rightarrow \lim_{i,j \rightarrow \infty} v_{x_{ij} - \xi} = 1$.

**Theorem 3.4.** Let $(V, v, \tau, \tau^*)$ be a PN-space. If a double sequence $x = x_{ij}$ of elements in $V$ is strongly $(\lambda, \mu)$-statistical convergent with respect to probabilistic norm $v$, then its $s(l_{v, 1}^{(\lambda, \mu)})$-limit is unique.

**Proof:** The proof of the Theorem can be established using standard techniques, so we omit. □

**Theorem 3.5.** Let $(V, v, \tau, \tau^*)$ be a PN-space. If $x = x_{ij}$ be a double sequence of elements in $V$ such that $v - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$ then $s(l_{v, 1}^{(\lambda, \mu)}) - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$.

**Proof:** Let $v - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$. For each $\lambda > 0$, there exists a positive integer $m$ such that $v_{x_{ij} - \xi(\lambda)} > 1 - \lambda$ for every $i, j \geq m$. It follows that the set $\{ \{ i, j \} : i \in I_n, j \in I_m, x_{ij} \notin N_\epsilon(\lambda) \}$ has atmost finitely many terms. It follows that

\[
\delta_{(\lambda, \mu)} \{ \{ i, j \} : i \in I_n, j \in I_m, x_{ij} \notin N_\epsilon(\lambda) \} = 0
\]

This shows that $s(l_{v, 1}^{(\lambda, \mu)}) - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$. □

**Theorem 3.6.** Let $(V, v, \tau, \tau^*)$ be a PN space. The $s(l_{v, 1}^{(\lambda, \mu)}) - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$, if and only if, there exists a subset $K = \{ \{ i, j \} : i, j = 1, 2, 3, \ldots \}$ such that $\delta_{(\lambda, \mu)}(K) = 1$ and $v - \lim_{(i,j) \in K, i,j \rightarrow \infty} x_{ij} = \xi$.

**Proof:** **Necessity**— Suppose that $s(l_{v, 1}^{(\lambda, \mu)}) - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$. For $\lambda > 0$, consider the sets

\[
M_v(\lambda) = \{ \{ i, j \} : i \in I_n, j \in I_m, v_{x_{ij} - \xi(\lambda)} > 1 - \frac{1}{\lambda} \}
\]

\[
K_v(\lambda) = \{ \{ i, j \} : i \in I_n, j \in I_m, v_{x_{ij} - \xi(\lambda)} \leq 1 - \frac{1}{\lambda} \}
\]

Since $s(l_{v, 1}^{(\lambda, \mu)}) - \lim_{i,j \rightarrow \infty} x_{ij} = \xi$, it follows that $\delta_{(\lambda, \mu)}(K_v(\lambda)) = 0$. Furthermore, for $\lambda = 1, 2, 3, \ldots$, we observe $M_v(\lambda) \supset M_v(\lambda + 1)$ and

\[
\delta_{(\lambda, \mu)}(M_v(\lambda)) = 1. \tag{3.1}
\]
Now we have to show that for \((i, j) \in M_v(\lambda)\), \(v - \lim_{i,j \to \infty} x_{ij} = \xi\). Suppose, for \((i, j) \in M_v(\lambda)\), \((x_{ij})\) is not convergent to \(\xi\) with respect to the probabilistic norm \(v\). Then, there exists some \(\beta > 0\) such that
\[
\{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - \xi(\lambda) \leq 1 - \beta \}
\]
for infinitely many terms \((x_{ij})\).

Let \(M_v(\beta) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - \xi(\lambda) > 1 - \beta \}\) and \(\beta > \frac{1}{2}\) for \(\lambda = 1, 2, 3, ...\). Then, we have
\[
\delta_{(\lambda, \mu)}(M_v(\beta)) = 0. \tag{3.2}
\]

Also, \(M_\delta(\lambda) \subset M_v(\beta)\) implies that \(\delta_{(\lambda, \mu)}(M_\delta(\lambda)) = 0\). In this way, we obtained a contradiction to (3.1) as \(\delta_{(\lambda, \mu)}(M_v(\lambda)) = 1\). Hence \(v - \lim_{i,j \to \infty} x_{ij} = \xi\).

**Sufficiency** - Suppose that there exists a subset \(K = \{ (i, j) : i, j = 1, 2, 3, ... \}\) such that \(\delta_{(\lambda, \mu)}(K) = 1\) and \(v - \lim_{i,j \in K \to \infty} x_{ij} = \xi\). But then for \(\lambda > 0\), we can find out a positive integer \(m\) such that
\[
v_{x_{ij} - \xi(\lambda)} > 1 - \lambda
\]
for all \(i, j \geq m\). If we take,
\[
K_v(\lambda) = \{ (i, j) : i \in I_n, j \in I_m, x_{ij} \notin N_\xi(\lambda) \}
\]
Then, it is easy to see that
\[
K_v(\lambda) \subseteq N \times N - \{ (i, j) : i \in I_n, j \in I_m, x_{ij} \in N_\xi(\lambda) \}
\]
and consequently
\[
\delta_{(\lambda, \mu)} K_v(\lambda) \leq 1 - 1 = 0.
\]
Hence, \(st_v(\lambda, \mu) - \lim_{i,j \to \infty} x_{ij} = \xi\). \qed

Now we define strongly \((\lambda, \mu)\)-statistically Cauchy double sequences in PN-space and establish the Cauchy convergence criteria in these spaces.

**Definition 3.7.** Let \((V, v, \tau, \tau^*)\) be a PN-space. A double sequence \(x = (x_{ij})\) of elements in \(V\) is said to be strongly \((\lambda, \mu)\)-statistically Cauchy with respect to the probabilistic norm \(v\) if for each \(\lambda > 0\) there exists a positive integers \(n\) and \(m\) such that for all \(i, p \geq n\) and \(j, q \geq m\),
\[
\delta_{(\lambda, \mu)} \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij} - x_{pq}}(\lambda) \leq 1 - \lambda \} = 0.
\]
or equivalently
\[
\delta_{(\lambda, \mu)} \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij} - x_{pq}}(\lambda) > 1 - \lambda \} = 1.
\]

**Theorem 3.8.** Let \((V, v, \tau, \tau^*)\) be a PN-space. If a double sequence \(x = x_{ij}\) of elements in \(V\) is strongly \((\lambda, \mu)\)-statistical convergent, if and only if, it is strongly \((\lambda, \mu)\)-statistical Cauchy with respect to probabilistic norm \(v\).

**Proof:** First suppose that there exists \(\xi \in V\) such that \(st_v^{(\lambda, \mu)} - \lim_{i,j \to \infty} x_{ij} = \xi\). Let \(\lambda > 0\) be given. Choose \(\gamma > 0\) such that
\[
\tau(1 - \gamma, 1 - \gamma) > 1 - \lambda \tag{3.3}
\]
For \(\lambda > 0\), if we define
\[
A(\gamma) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij} - \xi(\lambda)}(\lambda) \leq 1 - \gamma \}
\]
then
\[ A^C(\gamma) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - \xi(\frac{1}{\lambda}) > 1 - \gamma \} \]

Since \( a_{x_{ij}}^{(\lambda, \mu)} \rightarrow_{i,j \rightarrow \infty} x_{ij} = \xi \), it follows that \( \delta_{(\lambda, \mu)}(A(\gamma)) = 0 \) and consequently \( \delta_{(\lambda, \mu)}(A^C(\gamma)) = 1 \). Let \((p, q) \in (A^C(\lambda))\). Then

\[ v_{x_{pq}} - \xi(\frac{1}{\lambda}) > 1 - \gamma. \]  \hspace{1cm} (3.4)

If we take

\[ B(\lambda) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - x_{pq}(\lambda) \leq 1 - \lambda \}, \]

then to prove the result it is sufficient to prove that \( B(\lambda) \subseteq A(\gamma) \). For \((m, n) \in B(\lambda), v_{x_{mn}} - x_{pq}(\lambda) \leq 1 - \lambda \)

If \( v_{x_{mn}} - x_{pq}(\lambda) \leq 1 - \lambda \), then we have \( v_{x_{mn}} - \xi(\frac{1}{\lambda}) \leq 1 - \gamma \) and therefore \((m, n) \in A(\gamma)\). As otherwise i.e., if \( v_{x_{mn}} - \xi(\lambda) > 1 - \lambda \), then by using (3.3) and (3.4) we have

\[ 1 - \lambda \geq v_{x_{ij}} - x_{pq}(\lambda) \geq \tau(v_{x_{mn}} - \xi(\frac{1}{\lambda}), v_{x_{pq}} - \xi(\frac{1}{\lambda})) > \tau(1 - \gamma, 1 - \gamma) > 1 - \lambda, \]

which is not possible. Hence \( B(\lambda) \subseteq A(\gamma) \).

Conversely- Suppose that \( x = (x_{ij}) \) is strongly \((\lambda, \mu)\)-statistical Cauchy but not strongly \((\lambda, \mu)\)-statistical convergent with respect to the probabilistic norm \( v \).

Then there exists positive integers \( p \) and \( q \) such that if we take

\[ A(\lambda) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - x_{pq}(\lambda) \leq 1 - \lambda \} \]

and

\[ B(\lambda) = \{ (i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - \xi(\frac{1}{\lambda}) > 1 - \lambda \}. \]

then \( \delta_{(\lambda, \mu)}(A(\lambda)) = \delta_{(\lambda, \mu)}(B(\lambda)) = 0 \) and consequently

\[ \delta_{(\lambda, \mu)}(A^C(\lambda)) = \delta_{(\lambda, \mu)}(B^C(\lambda)) = 1. \]  \hspace{1cm} (3.5)

Since

\[ v_{x_{ij}} - x_{pq}(\lambda) \geq 2v_{x_{ij}} - \xi(\frac{1}{\lambda}) > 1 - \lambda \]

If \( v_{x_{ij}} - \xi(\frac{1}{\lambda}) > \frac{1 - \lambda}{2} \).

It follows that

\[ \delta_{(\lambda, \mu)}\{(i, j) : i \in I_n, j \in I_m, v_{x_{ij}} - x_{pq}(\lambda) > 1 - \lambda \} = 0 \}

i.e., \( \delta_{(\lambda, \mu)}(A^C(\lambda)) = 0 \). But then we obtained a contradiction to (3.5) as \( \delta_{(\lambda, \mu)}(A^C(\lambda)) = 1 \). Hence, \((x_{ij})\) is strongly \((\lambda, \mu)\)-statistical convergent with respect to the probabilistic norm \( v \).

On combining Theorem 3.6 and Theorem 3.8, we obtain the following result.

**Theorem 3.9.** Let \((V, v, \tau, \tau^*)\) be a PN-space and \( x = x_{ij} \) be a double sequence of elements in \( V \). Then, the following conditions are equivalent:

(i) \( x \) is a strongly \((\lambda, \mu)\)-statistical convergent with respect to the probabilistic norm \( v \).

(ii) \( x \) is a strongly \((\lambda, \mu)\)-statistical Cauchy with respect to the probabilistic norm \( v \).

(iii) there exists a subset \( K = \{(i, j) : i, j = 1, 2, 3, \ldots \} \) such that \( \delta_{(\lambda, \mu)}(K) = 1 \) and \( v - \lim_{(i, j) \in K, i, j \rightarrow \infty} x_{ij} = \xi \).
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