Some remarks on statistical summability of order $\tilde{\alpha}$ defined by generalized De la Vallée-Poussin Mean

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ABSTRACT: In this article we define $(\lambda, \mu)-$statistical summability and $(V, \lambda, \mu)-$summability of order $\tilde{\alpha}$ for double sequences and obtain some relations between these summability methods. We demonstrate examples which shows our method of summability is more general for double sequences.

Key Words: Statistical Convergence, $\lambda-$statistical convergence, Double sequences.

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1. Introduction

Fast [6] introduced the notion of statistical convergence as a generalized summability method in order to assign limits to those sequences which are not convergent in usual sense. He used the concept of natural density of subsets of $\mathbb{N}$, the set of positive integers. The natural density of a set $K \subset \mathbb{N}$, is denoted by $\delta(K)$ and is defined by

$$\delta(K) = \lim_{n} \frac{1}{n} \sum_{k=1}^{n} \chi_{K}(k)$$

provided the limit exists, where $\chi_{K}$ denotes the characteristic function of $K$. As the sum on the right side of the above expression denotes the cardinality of the set $\{k \leq n : k \in K\}$ so Fast [6] defined statistical convergence as follows.

Definition 1.1. [6] A sequence $x = (x_k)$ of numbers is said to be statistically convergent to a number $L$ provided that, for every $\epsilon > 0$,

$$\delta(\{k \leq n : |x_k - L| > \epsilon\}) = 0.$$

In this case, we write $S = \lim_{k \to \infty} x_k = L$.

Let $S(x)$ denotes the set of all statistically convergent sequences.

Although, statistical convergence was introduced in the mid of last century but a rapid development on statistical convergence starts with the papers of Šalát [20].
Fridy [7] and Connor [5]. For more details and related concepts, we refer to [12], [19], [21,22,23,24] and [28].

In [13], Mursaleen presented an interesting extension of statistical convergence namely $\lambda-$statistical convergence and show how it is related with $(V, \lambda)-$summability.

Let $\lambda = (\lambda_n)$ be a non-decreasing sequence of positive numbers tending to $\infty$ with $\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1$. The generalized de la Vallée-Poussin mean is defined by

$$t_n(x) = \frac{1}{\lambda_n} \sum_{k \in I_n} x_k,$$

where $I_n = [n - \lambda_n + 1, n]$.

A sequence $x = (x_k)$ of numbers is said to be $(V, \lambda)-$summable to a number $L$ (see [11]) if $t_n(x) \rightarrow L$ as $n \rightarrow \infty$.

**Definition 1.2.** [13] A sequence $x = (x_k)$ of numbers is said to be $\lambda-$statistically convergent to a number $L$ provided that for every $\epsilon > 0$,

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \left| \{ n - \lambda_n + 1 \leq k \leq n : |x_k - L| \geq \epsilon \} \right| = 0.$$

In this case, the number $L$ is called $\lambda-$statistical limit of the sequence $x = (x_k)$ and we write $S_{\lambda} - \lim_{k \rightarrow \infty} x_k = L$. We denote the set of all $\lambda-$statistically convergent sequences by $S_{\lambda}(x)$.

Further, an interesting generalization of statistical convergence was introduced by Çolak [2] under the name of "$\text{statistical convergence of order } \alpha$" for some $\alpha \in (0,1]$. This new idea was further investigated by Çolak and Bektaş in [4] via $(V, \lambda)-$summability and obtained some interesting results. Before we go further we quote the following definition.

**Definition 1.3.** [3] Let $\lambda = (\lambda_n)$ be a sequence of real numbers as defined above and $0 < \alpha \leq 1$ be given. The sequence $x = (x_k)$ is said to be $\lambda-$statistically convergent of order $\alpha$ if there is a number $L$ such that

$$\lim_{n \rightarrow \infty} \frac{1}{\lambda_n} \left| \{ k \in I_n : |x_k - L| \geq \epsilon \} \right| = 0.$$

In this case, we write $S_{\lambda}^{\alpha} - \lim_{k \rightarrow \infty} x_k = L$. The set of all $\lambda-$statistically convergent sequences of order $\alpha$ is denoted by $S_{\lambda}^{\alpha}(x)$.

We next give some ideas and developments on double sequences which have been frequently appeared in literature.

A double sequence $x = (x_{ij})$ of real numbers is said to be convergent in Priegshein’s sense or $P-$convergent (See [18]) if for every $\epsilon > 0$ there exists $n \in \mathbb{N}$ such that $|x_{ij} - L| < \epsilon$ whenever $i, j \geq n$. The number $L$ is called Priegshein limit of $x = (x_{ij})$ and we write $P-\lim x = L$. 

Double sequences were initially discussed by Bromwich [1] and Hardy [8]. Later, many authors including Móricz [16], Patterson [17], Tripathy and Sarma [25, 26, 27], Kumar [9] and Kumar and Mursaleen [10] etc. have shown their interest to study double sequences and related convergence problems. Mursaleen and Edely [15] and Mursaleen et al. [14] respectively extended Definition 1.1 and Definition 1.2 on double sequences and obtained some analogous results. However, Çolak and Altin [4] introduced statistical convergence of order \( \alpha \) for these kind of sequences.

**Definition 1.4.** [15] A double sequence \( x = (x_{ij}) \) of real numbers is said to be statistically convergent to \( L \) if for every \( \epsilon > 0 \)

\[
P - \lim_{n,m \to \infty} \frac{1}{nm} |\{(i,j) \in \mathbb{N} \times \mathbb{N}, i \leq n, j \leq m : |x_{ij} - L| \geq \epsilon\}| = 0.
\]

In this case, we write \( S_2 - \lim_{i,j \to \infty} x_{ij} = L \) and \( S_2(x) \) denotes the set of all statistically convergent double sequences.

Let \( \lambda = (\lambda_n) \) and \( \mu = (\mu_m) \) be two non-decreasing sequences of positive real numbers tending to \( \infty \) with \( \lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1 \) and \( \mu_{m+1} \leq \mu_m + 1, \mu_1 = 1 \). The generalized de la Vallée-Poussin mean of \( x = (x_{ij}) \) is defined by

\[
t_{mn}(x) = \frac{1}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} x_{ij},
\]

where \( I_n = [n - \lambda_n + 1, n] \) and \( I_m = [m - \mu_m + 1, m] \). Moreover, a double sequence \( x = (x_{ij}) \) is said to be \( (V, \lambda, \mu) \)-summable to a number \( L \) provided that \( t_{mn}(x) \to L \) as \( m, n \to \infty \).

**Definition 1.5.** [14] A double sequence \( x = (x_{ij}) \) of numbers is said to be \( (\lambda, \mu) \)-statistically convergent to a number \( L \) provided for every \( \epsilon > 0 \),

\[
P - \lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0.
\]

In this case, the number \( L \) is called \( (\lambda, \mu) \)-statistical limit of the sequence \( x = (x_{ij}) \) and we write \( S_{(\lambda, \mu)} - \lim_{i,j \to \infty} x_{ij} = L \).

Let, \( S_{(\lambda, \mu)}(x) \) denotes the set of all \( (\lambda, \mu) \)-statistically convergent double sequences of numbers.

In this article, we aim to define \((\lambda, \mu)\)-statistical convergence and \((V, \lambda, \mu)\)-summability of order \( \alpha \) and obtain some relevant connections. Throughout we take \( a, b, c, d \in (0, 1] \) as otherwise indicated. We will write \( \tilde{\alpha} \) as an alternative of \((a, b)\) and \( \tilde{\beta} \) as an alternative of \((c, d)\). Also we define: \( \tilde{\alpha} \leq \tilde{\beta} \iff a \leq c \) and \( b \leq d \); \( \tilde{\alpha} < \tilde{\beta} \iff a < c \) and \( b < d \); \( \tilde{\alpha} \equiv \tilde{\beta} \iff a = c \) and \( b = d \); \( \tilde{\alpha} \in (0, 1] \iff a \in (0, 1] \) and \( \tilde{\beta} \in (0, 1] \iff c, d \in (0, 1] \); \( \tilde{\alpha} \geq 1 \) in case \( a = b = 1 \); \( \tilde{\beta} \geq 1 \) in case \( c = d = 1 \) and \( \tilde{\alpha} > 1 \) in case \( a > 1, b > 1 \).
2. Main Results

In this section, we present our main results. We begin with the following definition:

**Definition 2.1.** Let $\lambda = (\lambda_n)$ and $\mu = (\mu_m)$ be two non-decreasing sequences of positive real numbers tending to $\infty$ with

$$\lambda_{n+1} \leq \lambda_n + 1, \lambda_1 = 1; \mu_{m+1} \leq \mu_m + 1, \mu_1 = 1$$

and $\tilde{\alpha} \in (0, 1]$ be given.

A double sequence $x = (x_{ij})$ of numbers is said to be $(\lambda, \mu)$–statistically convergent of order $\tilde{\alpha}$ if there exists a number $L$ such that for every $\epsilon > 0$

$$\lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| = 0,$$

where $\lambda^a = (\lambda^a_n) = (\lambda_1^a, \lambda_2^a, \lambda_3^a, \ldots)$; $\mu^b = (\mu^b_m) = (\mu_1^b, \mu_2^b, \mu_3^b, \ldots)$ and $\lambda^a_n \mu^b_m$ denotes the usual multiplication of the corresponding entries of the sequences $\lambda^a$ and $\mu^b$. In this case, the number $L$ is called $(\lambda, \mu)$–statistical limit of the sequence $x = (x_{ij})$ of order $\tilde{\alpha}$ and we write $S^{\tilde{\alpha}}_{(\lambda, \mu)} = \lim_{i,j} x_{ij} = L$.

Let $S^{\tilde{\alpha}}_{(\lambda, \mu)}(x)$ denotes the set of all $(\lambda, \mu)$–statistically convergent double sequences of order $\tilde{\alpha}$.

For $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.1 coincides with $(\lambda, \mu)$–statistical convergence of double sequences of [14]. For the choice $\lambda = (n)$ and $\mu = (m)$, Definition 2.1 coincides with statistical convergence of double sequences of order $\tilde{\alpha}$ of [3]. Moreover, if we take $\lambda = (n); \mu = (m)$ and $\tilde{\alpha} = (a, b) = (1, 1)$, Definition 2.1 coincides with statistical convergence of double sequences of [15].

**Theorem 2.2.** For $\tilde{\alpha} \in (0, 1]$, if $S^{\tilde{\alpha}}_{(\lambda, \mu)} = \lim_{i,j} x_{ij} = x_0$, then $x_0$ is unique.

**Proof:** Easy, so omitted. $\Box$

We next provide an example to show that the Definition 2.1 is well defined for $\tilde{\alpha} \in (0, 1]$ but not for $\tilde{\alpha} > 1$ in general.

**Example 2.3.** Let $x = (x_{ij})$ be defined as follows:

$$x_{ij} = \begin{cases} 1 & \text{if } i + j \text{ even} \\ 0 & \text{if } i + j \text{ odd} \end{cases}$$

Then for $\tilde{\alpha} > 1$,

$$\lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) \in I_n \times I_m : |x_{ij} - 1| \geq \epsilon\}| \leq \lim_{n,m \to \infty} \frac{[\lambda_n \mu_m] + 1}{2 \lambda_n \mu_m} = 0$$

and

$$\lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m} |\{(i, j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}| \leq \lim_{n,m \to \infty} \frac{[\lambda_n \mu_m] + 1}{2 \lambda_n \mu_m} = 0.$$

This shows that $S^{\tilde{\alpha}}_{(\lambda, \mu)} = \lim_{i,j} x_{ij} = 0$ and $S^{\tilde{\alpha}}_{(\lambda, \mu)} = \lim_{i,j} x_{ij} = 1$ which leads to a contradiction to Theorem 2.2.
We state the following result without proof.

**Theorem 2.4.** Let \( x = (x_{ij}) \) and \( y = (y_{ij}) \) be two double sequences of complex numbers and \( \alpha \in (0,1] \).

(i) If \( S_{(\lambda,\mu)}^\alpha - \lim x_{ij} = L \) and \( c \in \mathbb{C} \), then \( S_{(\lambda,\mu)}^\alpha - \lim (c x_{ij}) = c L \).

(ii) If \( S_{(\lambda,\mu)}^\alpha - \lim x_{ij} = L \) and \( S_{(\lambda,\mu)}^\alpha - \lim y_{ij} = M \), then \( S_{(\lambda,\mu)}^\alpha - \lim (x_{ij} + y_{ij}) = L + M \).

**Definition 2.5.** Let \( \alpha \) be any real number such that \( \alpha \in (0,1] \) and \( p \) be a positive real number. A double sequence \( x = (x_{ij}) \) is said to be strongly \((V,\lambda,\mu)\)-summable of order \( \alpha \) to a number \( L \) provided that

\[
\lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p = 0,
\]

where \( I_n = [n - \lambda_n + 1, n] \) and \( I_m = [m - \mu_m + 1, m] \). In this case, the number \( L \) is called strong \((V,\lambda,\mu)\)-statistical limit of the sequence \( x = (x_{ij}) \) of order \( \alpha \).

Let \( |w_{ij}|_\alpha(x) \) denote the set of all strongly \((V,\lambda,\mu)\)-summable double sequences of order \( \alpha \).

For \( \alpha = (a,b) = (1,1) \), Definition 2.5 coincides with strong \((V,\lambda,\mu)\)-summability of double sequences of [14]. For \( \lambda = (n) \) and \( \mu = (m) \), Definition 2.5 coincides with strong \( p \)-Cesàro summability of double sequences of order \( \alpha \) of [3]. However, if we take \( \lambda = (n) \); \( \mu = (m) \) and \( \alpha = (a,b) = (1,1) \), Definition 2.5 coincides with strong \( p \)-Cesàro summability of double sequences of [15].

**Theorem 2.6.** Let \( \tilde{\alpha}, \tilde{\beta} \in (0,1] \) such that \( \alpha \preceq \tilde{\beta} \). Then \( S_{(\lambda,\mu)}^\alpha(x) \subseteq S_{(\lambda,\mu)}^{\tilde{\beta}}(x) \) and the inclusion is strict for some \( \tilde{\alpha} \) and \( \tilde{\beta} \) such that \( \alpha \prec \tilde{\beta} \).

**Proof:** Let \( x = (x_{ij}) \in S_{(\lambda,\mu)}^\alpha(x) \). Since, \( \alpha \preceq \tilde{\beta} \) so \( a \leq c \) and \( b \leq d \); which for any \( \epsilon > 0 \) gives the inequality

\[
\frac{1}{\lambda_n \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}| \leq \frac{1}{\lambda_n \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon\}|;
\]

and therefore the result follows immediately from the fact that \( x = (x_{ij}) \in S_{(\lambda,\mu)}^\alpha(x) \). For rest part of the Theorem we consider the following example. Define \( x = (x_{ij}) \) by

\[
x_{ij} = \begin{cases} 
  ij, & \text{if } n - \lfloor \sqrt{n} \rfloor + 1 \leq i \leq n \text{ and } m - \lfloor \sqrt{m} \rfloor + 1 \leq j \leq m \\
  0, & \text{otherwise}
\end{cases}
\;
\text{then}
\]

\[
\frac{1}{\lambda_n \mu_m^d} |\{(i,j) \in I_n \times I_m : |x_{ij} - 0| \geq \epsilon\}|
\]

\[
= \frac{1}{\lambda_n \mu_m^d} \left| \left\{(i,j) \in I_n \times I_m : n - \lfloor \sqrt{n} \rfloor + 1 \leq i \leq n \text{ and } m - \lfloor \sqrt{m} \rfloor + 1 \leq j \leq m \right\} \right| \leq \frac{\sqrt{\lambda_n \mu_m^d}}{\lambda_n \mu_m^d}.
\]
It follows, for $\tilde{\beta} \in (\frac{1}{2}, 1]$ (i.e. for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$), we have
\[
\lim_{n,m \to \infty} \frac{1}{\lambda_n \mu_m^d} \left| \{(i, j) \in I_n \times I_m : |x_{ij} - 0| \geq \varepsilon \} \right| \leq \lim_{n,m \to \infty} \frac{|\sqrt{\lambda_n \mu_m^d}|}{\lambda_n \mu_m^d} = 0.
\]
This shows that $x = (x_{ij}) \in S_{(\lambda, \mu)}(x)$, but one can easily verify that $x \notin S_{(\lambda, \mu)}(x)$ for $\tilde{\alpha} \in (0, \frac{1}{2}]$ (i.e. for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$).

Corollary 2.7. Let $\tilde{\alpha}, \tilde{\beta} \in (0, 1]$.

(i) If $\tilde{\beta} \equiv 1$, then $S_{(\lambda, \mu)}(x) \subseteq S_{(\lambda, \mu)} = S(\lambda, \mu)$ and the inclusion is strict.

(ii) $S_{(\lambda, \mu)}(x) = S_{(\lambda, \mu)}(x) \iff \tilde{\alpha} \equiv \tilde{\beta}$.

(iii) $S_{(\lambda, \mu)}(x) = S_{(\lambda, \mu)}(x) \iff \tilde{\alpha} \equiv 1$.

Theorem 2.8. Let $\lambda = (\lambda_n), \mu = (\mu_m)$ be two sequences as defined above and $\tilde{\alpha} \in (0, 1]$, then

(i) $S_{(\lambda, \mu)}(x) \subseteq S_2(x)$ for all $\lambda, \mu$ and $\tilde{\alpha} \in (0, 1]$. 

(ii) $S_2(x) \subseteq S_{(\lambda, \mu)}(x)$, if and only if, $\liminf_{n \to \infty} \frac{\lambda_n}{m} > 0$ and $\liminf_{m \to \infty} \frac{\mu_m}{m} > 0$.

Proof: (i) By the nature of the sequences $(\lambda_n)$, $(\mu_m)$ and from the expression $\frac{\lambda_n \mu_m}{n m} \leq 1$, the result follows.

(ii) Let, $\liminf_{n \to \infty} \frac{\lambda_n}{m} > 0$, $\liminf_{m \to \infty} \frac{\mu_m}{m} > 0$ and $x = (x_{ij}) \in S_2(x)$. For given $\varepsilon > 0$, we have,
\[
\{(i, j), i \leq n and j \leq m : |x_{ij} - L| \geq \varepsilon \} \supset \{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \varepsilon \},
\]

it follows that,
\[
\frac{1}{nm} \left| \{(i, j), i \leq n and j \leq m : |x_{ij} - L| \geq \varepsilon \} \right| \geq \frac{1}{nm} \left| \{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \varepsilon \} \right| = \left( \frac{\lambda_n}{n} \right) \left( \frac{\mu_m^d}{m} \right) \frac{1}{\lambda_n \mu_m^d} \left| \{(i, j) \in I_n \times I_m : |x_{ij} - L| \geq \varepsilon \} \right|.
\]

Taking limit as $n, m \to \infty$ we have, $S_2(x) \subseteq S_{(\lambda, \mu)}(x)$.

Conversely, suppose that either $\liminf_{n \to \infty} \frac{\lambda_n}{m}$ or $\liminf_{m \to \infty} \frac{\mu_m}{m}$ or both are zero. Then we can choose two subsequences $(n_p)$ and $(m_q)$ such that $\lambda_n^{\frac{1}{p}} < \frac{1}{p}$ and $\frac{\mu_m}{q} < \frac{1}{q}$. Define double sequence $x = (x_{ij})$ as follows:
\[
x_{ij} = \begin{cases} 1 & \text{if } i \in I_{n_p} \text{ and } j \in I_{m_q} \quad (p, q = 1, 2, 3, \ldots) \\ 0 & \text{otherwise,} \end{cases}
\]

Then clearly $x \in S_2(x)$, but $x \notin S_{(\lambda, \mu)}(x)$. From Corollary 2.7, since $S_{(\lambda, \mu)}(x) \subseteq S_{(\lambda, \mu)}(x)$, we have $x \notin S_{(\lambda, \mu)}(x)$. Hence, $\liminf_{n \to \infty} \frac{\lambda_n}{m} > 0$ and $\liminf_{m \to \infty} \frac{\mu_m}{m} > 0$. 

**Theorem 2.9.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$ and $p$ be a positive real number. Then $[w^2_p]_\alpha(x) \subseteq [w^2_p]_\beta(x)$ and the inclusion is strict for some $\alpha$ and $\beta$ such that $\alpha < \beta$.

**Proof:** Let $x = (x_{ij}) \in [w^2_p]_\alpha(x)$, then for $\alpha \in (0, 1]$ and a positive real number $p$

$$\lim_{n,m \to \infty} \frac{1}{x_{n,m}^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p = 0.$$  

Also for given $\alpha$ and $\beta$ such that $\alpha \leq \beta$, one can write

$$\lim_{n,m \to \infty} \frac{1}{x_{n,m}^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p \leq \lim_{n,m \to \infty} \frac{1}{x_{n,m}^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p = 0$$

which implies $x = (x_{ij}) \in [w^2_p]_\beta(x)$. Hence, $[w^2_p]_\alpha(x) \subseteq [w^2_p]_\beta(x)$. The following example will show that the inclusion is strict. Define the sequence $x = (x_{ij})$ by

$$x_{ij} = \begin{cases} 1, & \text{if } n - \sqrt{\lambda} + 1 \leq i \leq n \text{ and } m - \sqrt{\mu} + 1 \leq j \leq m \\ 0, & \text{otherwise} \end{cases}$$

Then for $\beta \in (\frac{1}{2}, 1]$ (that is for $\frac{1}{2} < c \leq 1$ and $\frac{1}{2} < d \leq 1$),

$$\frac{1}{x_{n,m}^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - 0|^p \leq \frac{\sqrt{\lambda_n} \sqrt{\mu_n}}{\sqrt{\lambda_n^p} \mu_n^p} = \frac{1}{\sqrt{\lambda_n} \sqrt{\mu_n}}.$$  

Since $\frac{1}{\sqrt{\lambda_n} \sqrt{\mu_n}} \rightarrow 0$ as $n, m \to \infty$, therefore $x = (x_{ij}) \in [w^2_p]_\beta(x)$, but for $\alpha \in (0, \frac{1}{2})$ (that is for $0 < a \leq \frac{1}{2}$ and $0 < b \leq \frac{1}{2}$)

$$\frac{\sqrt{\lambda_n} - 1}{\sqrt{\mu_n}} \leq \frac{1}{x_{n,m}^p} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - 0|^p$$  

and $\frac{\sqrt{\lambda_n} - 1}{\sqrt{\mu_n}} \rightarrow \infty$ as $n, m \to \infty$, which implies $x = (x_{ij}) \notin [w^2_p]_\alpha(x)$.

Hence the inclusion is strict. \qed

**Corollary 2.10.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$ and $p$ be a positive real number.

Then

(i) $[w^2_p]_\alpha(x) = [w^2_p]_\beta(x) \iff \alpha \equiv \beta$.

(ii) $[w^2_p]_\alpha(x) \subseteq w^2_p$ for each $\alpha \in (0, 1]$ and $0 < p < \infty$.

**Theorem 2.11.** Let $\alpha, \beta \in (0, 1]$ such that $\alpha \leq \beta$ and $p$ be a positive real number.

If a sequence $x = (x_{ij})$ is strongly $(V, \alpha, \beta)$-summable to $L$ of order $\alpha$, then it is $(\alpha, \beta)$-statistically convergent to $L$ of order $\beta$, i.e., $[w^2_p]_\alpha(x) \subset S_{(\alpha, \beta)}(x)$.
Proof: For any sequence \( x = (x_{ij}) \) and \( \epsilon > 0 \)

\[
\sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p \geq \sum_{(i,j) \in I_n \times I_m, \ |x_{ij} - L| \geq \epsilon} |x_{ij} - L|^p + \sum_{(i,j) \in I_n \times I_m, \ |x_{ij} - L| < \epsilon} |x_{ij} - L|^p
\]

which implies

\[
\frac{1}{\lambda_n \mu_m} \sum_{(i,j) \in I_n \times I_m} |x_{ij} - L|^p \geq \frac{1}{\lambda_n \mu_m} \left| \left\{ (i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon \right\} \right| \cdot \epsilon^p
\]

\[
\geq \frac{1}{\lambda_n \mu_m} \epsilon^{\tilde{\alpha}} \left| \left\{ (i,j) \in I_n \times I_m : |x_{ij} - L| \geq \epsilon \right\} \right| \cdot \epsilon^p.
\]

It follows that if \( x = (x_{ij}) \) is strong \((V, \lambda, \mu)\)-summable to \( L \) of order \( \tilde{\alpha} \), then it is \((\lambda, \mu)\)-statistically convergent to \( L \) of order \( \tilde{\beta} \).

For particular choice of \( \tilde{\alpha} \equiv \tilde{\beta} \) in above Theorem we have the following result.

Corollary 2.12. Let \( \tilde{\alpha}, \tilde{\beta} \in (0, 1] \) such that \( \tilde{\alpha} \preceq \tilde{\beta} \);

(i) If \( \tilde{\alpha} \equiv \tilde{\beta} \) then \( \text{[w}_2^\alpha \tilde{\alpha}(x) \subset S^{\tilde{\alpha}}(\lambda, \mu)(x) \).

(ii) For \( \tilde{\beta} \equiv 1 \), \( \text{[w}_2^{\tilde{\alpha}} \tilde{\alpha}(x) \subset S(\lambda, \mu)(x) \).

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