On semiderivations of $*$-prime rings

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Abstract: Let $R$ be a $*$-prime ring with involution $*$ and center $Z(R)$. An additive mapping $F : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that (i) $F(xy) = F(x)g(y) + g(x)F(y) = F(x)y + g(x)F(y)$ and (ii) $F(g(x)) = g(F(x))$ hold for all $x, y \in R$. In the present paper, some well known results concerning derivations of prime rings are extended to semiderivations of $*$-prime rings.

Key Words: $*$-prime rings, derivations, semiderivations.

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1. Introduction

Let $R$ will be an associative ring with center $Z$. For any $x, y \in R$ the symbol $[x, y]$ represents commutator $xy - yx$. Recall that a ring $R$ is prime if $xRy = 0$ implies $x = 0$ or $y = 0$. An additive mapping $*: R \to R$ is called an involution if $(xy)^* = y^*x^*$ and $(x^*)^* = x$ for all $x, y \in R$. A ring equipped with an involution is called a ring with involution or $*$-ring. A ring with an involution is said to be $*$-prime if $xRy = xRy^* = 0$ or $xRy = x^*Ry = 0$ implies that $x = 0$ or $y = 0$. Every prime ring with an involution is $*$-prime but the converse need not hold general. An example due to Oukhtite [7] justifies the above statement that is, $R$ be a prime ring, $S = R \times R^o$ where $R^o$ is the opposite ring of $R$. Define involution $*$ on $S$ as $*(x, y) = (y, x)$. $S$ is $*$-prime, but not prime. This example shows that every prime ring can be injected in a $*$-prime ring and from this point of view $*$-prime rings constitute a more general class of prime rings. In all that follows the symbol $S_a(R)$, first introduced by Oukhtite, will denote the set of symmetric and skew symmetric elements of $R$, i.e. $S_a(R) = \{x \in R \mid x^* = \pm x\}$.

An additive mapping $d : R \to R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. For a fixed $a \in R$, the mapping $I_a : R \to R$ given by $I_a(x) = [a, x]$ is a derivation which is said to be an inner derivation. The study of derivations in prime rings was initiated by E. C. Posner in [11]. Recently, Bresar defined the following notation in [1]: An additive mapping $F : R \to R$ is called a generalized derivation if there exists a derivation $d : R \to R$ such that

$$F(xy) = F(x)y + xd(y), \quad \text{for all} \ x, y \in R.$$
Basic examples are derivations and generalized inner derivations (i.e., maps of type $x \to ax + xb$ for some $a, b \in R$). Several authors consider the structure of a prime ring in the case that the derivation $d$ is replaced by a generalized derivation. Generalized derivations have been primarily studied on operator algebras.

In [2] J. Bergen has introduced the notion of semiderivations of a ring $R$ which extends the notion of derivations of a ring $R$. An additive mapping $F : R \to R$ is called a semiderivation if there exists a function $g : R \to R$ such that (i) $F(xy) = F(x)g(y) + xF(y) = F(x)y + g(x)F(y)$ and (ii) $F(g(x)) = g(F(x))$ hold for all $x, y \in R$. In case $g$ is an identity map of $R$, then all semiderivations associated with $g$ are merely ordinary derivations. On the other hand, if $g$ is a homomorphism of $R$ such that $g \neq 1$, then $f = g - 1$ is a semiderivation which is not a derivation. In case $R$ is prime and $F \neq 0$, it has been shown by Chang [3] that $g$ must necessarily be a ring endomorphism.

Let $S$ be a nonempty subset of $R$. A mapping $F$ from $R$ to $R$ is called centralizing on $S$ if $[F(x), x] \in Z$ for all $x \in S$ and is called commuting on $S$ if $[F(x), x] = 0$ for all $x \in S$. The study of such mappings was initiated by E. C. Posner in [11]. A famous result due to Herstein [5] states that if $R$ is a prime ring of characteristic not 2 which admits a nonzero derivation $d$ such that $[d(x), a] = 0$ for all $x \in R$, then $a \in Z$. Also, Herstein showed that if $d(R) \subset Z$, then $R$ must be commutative. On the other hand, in [4], Daif and Bell proved that if a semiprime ring $R$ has a derivation $d$ satisfying the following condition, then $I$ is a central ideal;

there exists a nonzero ideal $I$ of $R$ such that

either $d([x, y]) = [x, y]$ for all $x, y \in I$, or $d([x, y]) = -[x, y]$ for all $x, y \in I$.

Many authors have studied commutativity of prime and semiprime rings admitting derivations, generalized derivations and semiderivations which satisfy appropriate algebraic conditions on suitable subsets of the rings. Recently, some well-known results concerning prime rings have been proved for $\ast$--prime ring by Oukhtite et al. (see, [6-10], where further references can be found). In the present paper our objective is to generalize above results for semiderivations of a $\ast$--prime ring.

Throughout the paper, $R$ will be a $\ast$--prime ring and $F$ be a semiderivation of $R$ associated with a surjective function $g$ of $R$ such that $\ast F = F \ast$. Also, we will make some extensive use of the basic commutator identities:

\[
[x, yz] = y[x, z] + [x, y]z \\
[xy, z] = [x, z]y + x[y, z].
\]

2. Results

**Lemma 2.1.** Let $R$ be a $\ast$--prime ring and $a \in R$. If $R$ admits a semiderivation $F$ of $R$ such that $aF(x) = 0$ (or $F(x)a = 0$) for all $x \in R$, then $a = 0$ or $F = 0$.

**Proof:** For all $x, y \in R$, we get $aF(xy) = 0$, and hence

\[
aF(x)g(y) + axF(y) = 0,
\]
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and so
\[ aRF(y) = 0, \] for all $y \in R$.
Replacing $y$ by $y^*$ in this equation and using $\ast F = F\ast$, we find that
\[ aRF(y)^* = 0, \] for all $y \in R$.
Since $R$ is a $\ast$--prime ring, we have $a = 0$ or $F = 0$. Similarly holds case $F(x)a = 0$.

The following theorem is be obtained using the same methods in [3, Theorem 1].

**Theorem 2.2.** Let $R$ be a $\ast$--prime ring, $F$ a nonzero semiderivation of $R$ associated with a function $g$ (not necessarily surjective). Then $g$ is a homomorphism of $R$.

**Proof:** For any $x, y, z \in R$, we get
\[
F(z(x + y)) = F(z)g(x + y) + zF(x + y) = F(z)g(x + y) + zF(x) + zF(y).
\]
On the other hand,
\[
F(z(x + y)) = F(zx + zy) = F(z)g(x) + zF(x) + F(z)g(y) + zF(y).
\]
Comparing these two equations, we arrive at $F(z)(g(x + y) - g(x) - g(y)) = 0$, for all $x, y, z \in R$. Using Lemma 2.1 and $F \neq 0$, we obtain that $g(x + y) = g(x) + g(y)$, for all $x, y \in R$.

Now, let $x, y, z \in R$. Then
\[
F((xy)z) = g(xy)F(z) + F(xy)z
= g(xy)F(z) + g(x)F(y)z + F(xy)yz.
\]
On the other hand,
\[
F((xy)z) = F(x(yz)) = g(x)F(yz) + F(x)yz = g(x)g(y)F(z) + g(x)F(y)z + F(x)yz.
\]
Hence we get $(g(xy) - g(x)g(y))F(z) = 0$, for all $x, y, z \in R$. Again using Lemma 2.1 and $F \neq 0$, we have
\[ g(xy) = g(x)g(y), \] for all $x, y \in R$. 
\[ \Box \]
Theorem 2.3. Let $R$ be a $*$-prime ring, $F$ a semiderivation of $R$ such that $F(R) \subseteq Z$, then $F = 0$ or $R$ is commutative.

Proof: By the hypothesis, we have
\[ F(xy) \in Z, \text{ for all } x, y \in R. \]
That is
\[ F(x)g(y) +xF(y) \in Z, \text{ for all } x, y \in R. \]
Commuting this term with $x$ and using the hypothesis, we get
\[
0 = [F(x)g(y)+xF(y),x] = F(x)[g(y),x]
\]
Since $F(x) \in Z$ and $g$ is surjective function of $R$, we arrive at
\[ F(x)R[y,x] = 0, \text{ for all } x, y \in R. \]
Using $*F = F*$, for any $x \in S_{a_*}(R)$, we have
\[ F(x^*)R[y,x] = 0, \text{ for all } x \in S_{a_*}(R), y \in R. \]
Since $R$ is a $*$-prime ring, we arrive at
\[ F(x) = 0 \text{ or } [y,x] = 0, \text{ for all } x \in S_{a_*}(R), y \in R. \]
Using the fact that $x + x^* \in S_{a_*}(R)$, $x - x^* \in S_{a_*}(R)$ for all $x \in R$, we easily deduce $F(x \pm x^*) = 0$ or $[y, x \pm x^*] = 0$. Hence we obtain $R$ is union of its two additive subgroups such that
\[ K = \{x \in R \mid F(x) = 0\} \]
and
\[ L = \{x \in R \mid x \in Z\}. \]
Clearly each of $K$ and $L$ is additive subgroup of $R$. Moreover, $R$ is the set-theoretic union of $K$ and $L$. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, we have $F = 0$ and the second case, $R$ is commutative. \hfill \Box

Theorem 2.4. Let $R$ be a 2-torsion free $*$-prime ring, $F$ a semiderivation of $R$ such that $F^2(x) = 0$, for all $x \in R$, then $F = 0$.

Proof: Assume that
\[ F^2(x) = 0, \text{ for all } x \in R. \]
Replacing $x$ by $xy$ in this equation, we get
\[ 0 = F^2(xy) = F(F(x)g(y) + xF(y)) \]
\[ = F^2(x)g^2(y) + F(x)F(g(y)) + F(x)g(F(y)) + xF^2(y) \]
and so
\[ 2F(x)F(g(y)) = 0, \text{ for all } x, y \in R. \]

Using $R$ is a 2–torsion free and $g$ is surjective function of $R$, we have
\[ F(x)F(y) = 0, \text{ for all } x, y \in R. \]

By Lemma 2.1, we complete the proof. $\square$

**Theorem 2.5.** Let $R$ be a 2–torsion free $\ast-$prime ring and $a \in R$. If $R$ admits a semiderivation $F$ such that $[F(x), a] = 0$, for all $x \in R$, then $F = 0$ or $a \in Z$.

**Proof:** Replacing $x$ by $xy$ and using the hypothesis, we have
\[ 0 = [a, F(xy)] = [a, F(x)y + g(x)F(y)] \]
\[ = F(x)[a, y] + [a, g(x)]F(y) \] (2.1)
Writing $g$ for $F(y)$ in this equation and again using the hypothesis, we obtain that
\[ [a, g(x)]F^2(y) = 0, \text{ for all } x, y \in R. \]

Since $g$ is surjective function of $R$, we have
\[ [a, x]F^2(y) = 0, \text{ for all } x, y \in R. \]

Substituting $xz$ for $x$ in this equation, we get
\[ [a, x]RF^2(y) = 0, \text{ for all } x, y \in R. \]

Since $\ast F = F \ast$, it reduces
\[ [a, x]RF^2(y)^\ast = 0, \text{ for all } x, y \in R. \]

By the $\ast-$primeness of $R$, we find that
\[ a \in Z \text{ or } F^2(y) = 0, \text{ for all } y \in R. \]

If $F^2(y) = 0$, for all $y \in R$, then $F = 0$ by Theorem 2.4. $\square$

**Theorem 2.6.** Let $R$ be a 2–torsion free $\ast-$prime ring and $F$ a semiderivation of $R$ such that $[F(R), F(R)] = 0$, then $F = 0$ or $R$ is commutative.

**Proof:** By Theorem 2.5, we have $F = 0$ or $F(R) \subseteq Z$. If $F(R) \subseteq Z$, then $F = 0$ or $R$ is commutative by Theorem 2.3. $\square$
Theorem 2.7. Let $R$ be a $\ast$−prime ring, $F$ a semiderivation of $R$ such that $[F(x), x] = 0$, for all $x \in R$, then $F = 0$ or $R$ is commutative.

Proof: Linearizing the hypothesis, we have

$$[F(x), y] + [F(y), x] = 0,$$

for all $x, y \in R$. Replacing $y$ by $yx$ in this equation and using the hypothesis, we get

$$0 = [F(x), yx] + [F(yx), x] = [F(x), y]x + [F(y)x + g(y)F(x), x],$$

and so

$$[g(y), x]F(x) = 0,$$

for all $x, y \in R$.

Since $g$ is surjective function of $R$, we have

$$[y, x]F(x) = 0,$$

for all $x, y \in R$.

Writing $yz$ for $y$ and using this equation, we obtain that

$$[y, x]RF(x) = 0,$$

for all $x, y \in R$.

Using the same arguments as we used in the last part of proof of the Theorem 2.3, we get the required result.

Theorem 2.8. Let $R$ be a $\ast$−prime ring, $F$ a nonzero semiderivation of $R$ such that $F([x, y]) = 0$, for all $x, y \in R$, then $R$ is commutative.

Proof: Replacing $y$ by $xy$ in the hypothesis, we get

$$0 = F([x, y]) = F(x)g([x, y]) + xF([x, y]) = F(x)g([x, y]).$$

We know that $g$ is homomorphism of $R$ by Theorem 1. Hence we have

$$F(x)[g(x), g(y)] = 0,$$

for all $x, y \in R$.

Since $g$ is surjective function of $R$, we get

$$F(x)[g(x), y] = 0,$$

for all $x, y \in R$.

Writing $yz$ for $y$ and using this equation, we obtain that

$$F(x)R[g(x), z] = 0,$$

for all $x, z \in R$.

Using $\ast F = F\ast$, for any $x \in S_a(R)$, we have

$$F(x)^\ast R[g(x), z] = 0,$$

for all $x \in S_a(R), z \in R$. 

\[\square\]
Since $R$ is a $*$-prime ring, we arrive at
\[ F(x) = 0 \text{ or } [g(x), y] = 0, \text{ for all } x \in S_a(R), y \in R. \]

Using the fact that $x + x^* \in S_a(R), x - x^* \in S_a(R)$ for all $x \in R$, we easily deduce
\[ F(x \pm x^*) = 0 \text{ or } [g(x \pm x^*), y] = 0, \]
Hence we obtain that $R$ is union of its two additive subgroups such that
\[ K = \{ x \in R \mid F(x) = 0 \} \]
and
\[ L = \{ x \in R \mid [g(x), y] = 0, \text{ for all } y \in R \}. \]
Clearly each of $K$ and $L$ is additive subgroup of $R$. Moreover, $R$ is the set-theoretic union of $K$ and $L$. But a group can not be the set-theoretic union of two proper subgroups, hence $K = R$ or $L = R$. In the former case, we have $F = 0$, a contradiction. So, we must have $L = R$. Hence $R$ is commutative.

**Theorem 2.9.** Let $R$ be a $*$-prime ring, $F$ a nonzero semiderivation of $R$ such that $F([x, y]) = \pm [x, y]$, for all $x, y \in R$, then $R$ is commutative.

**Proof:** Replacing $y$ by $xy$ in the hypothesis, we get
\[ F(x[x, y]) = \pm x[x, y] \]
\[ F(x)g([x, y]) + xF([x, y]) = \pm x[x, y], \]
and so
\[ F(x)g([x, y]) = 0. \]
Using the same arguments as we used in the last part of proof of the Theorem 2.8, we get the required result. □

**References**


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