Fixed Point Theorem in Fuzzy Metric Space

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ABSTRACT: In this paper we prove a fixed point theorem on a fuzzy set defining a new class of fuzzy metric space as *structure fuzzy metric space*

Key Words: Fixed point, fuzzy metric, continuous t-norm.

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1. Introduction


In this paper we will study a fixed point theorem from view point of a new class of fuzzy metric defined on a fuzzy set. This concept came to exist when the author was investigating properties in a generalized closed set of bitopological space using topological ideal. Often topological ideal is simply stated as ideal.

A non-empty collection $I$ of subsets of a set $X$ is said to be an ideal if it follows following two conditions

1. If $A \in I$ and $B \subseteq A$ then $B \in I$.
2. If $A \in I$, $B \in I$, then $A \cup B \in I$.

Fixed point theorems in any areas are most useful. Mathematical economists, physicists, computer scientists etc are using fixed point theorems in their respective research areas. Now a days fuzzy fixed point theorems are also playing crucial role in mathematical economics, social choices, auction theory.

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application of convex topological fixed point theory can be found in the 1994’s Nobel laureate John Fr. Nash’s classic seminal paper of equilibrium point of "Non-cooperative games" [14]. His proof is based on Kakutani’s fixed point theorem [15], which is the generalization of Brouwer’s fixed point theorem.

2. Preliminary definitions

In this section we discuss some existing definitions.

**Definition 2.1.** ([6]) A binary operation $*: [0, 1] \times [0, 1] \to [0, 1]$ is a continuous $t$-norm if $*$ satisfies the following conditions

(a) $*$ is commutative and associative;

(b) $*$ is continuous;

(c) $a * 1 = a \forall a \in [0, 1]$;

(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$ and $a, b, c, d \in [0, 1]$.

**Definition 2.2.** ([12]) Let $X$ be a non-empty set, $*$ be a continuous $t$-norm and $M: X^2 \times [0, \infty) \to [0, 1]$ be a fuzzy set. Consider the following conditions for all $x, y, z \in X$ and $t, s \in [0, \infty)$:

(M1) $M(x, y, 0) = 0$

(M2) $M(x, x, t) = 1$

(M3) $M(x, y, t) = 1 \Rightarrow x = y$

(M4) $M(x, y, t) = M(y, x, t)$

(M5) $M(x, y, t + s) \geq M(x, z, t) * M(z, y, s)$

(M6) $M(x, y, .) : [0, \infty) \to [0, 1]$ is left continuous

Then $(X, M, *)$ is said to be a fuzzy metric space.

3. Main result

This section contains some new definitions, terminologies and they are used to prove one theorem in fuzzy fixed point theory.
**Definition 3.1.** $(X, M, \ast)$ is said to be a structure fuzzy metric space (SFMS) if it satisfies conditions $(M1), (M3), (M4), (M5)$ and $(M6)$ of Definition 2.2.

**Example 3.2.** If $X = R$, define $a \ast b = ab$ and $M(x, y, t) = \frac{1}{2} \frac{1}{t} |x - y| + |x| + |y| t$ then $(X, M, \ast)$ is a SFMS.

**Definition 3.3.** A sequence $< x_n >$ in a SFMS is said to be structure convergent if there exists $x \in X$ such that $\lim_{n \to \infty} M(x_n, x, t) = 1 \forall t > 0$. Then $x$ is said to be structure limit of $< x_n >$ and denoted by $\lim_{n \to \infty} x_n = x$.

**Definition 3.4.** A sequence $< x_n >$ in a SFMS $(X, M, \ast)$ is said to be structure Cauchy sequence if for each $t > 0$ and $r \in N$ such that $\lim_{n \to \infty} M(x_n + r, x_n, t) = 1$.

$(X, M, \ast)$ is said to be structure complete if every structure Cauchy sequence in it is structure convergent.

**Definition 3.5.** Let $(X, M, \ast)$ be a SFMS, $f$ and $h$ are self maps on $X$. Then $f$ and $h$ are said to be normalized at $x$ if and only if $M(fhx, hfx, t) = 1 \forall t \in [0, \infty)$.

The functions $f$ and $h$ are said to be normalized on $X$ if $f$ and $h$ are normalized at all points $x$ of $X$.

**Definition 3.6.** The functions $f$ and $h$ are said to be common domain normalized (CDN) if they are normalized at the coincidence point of $f$ and $h$.

**Remark 3.7.** A SFMS has a unique limit point.

**Proof:** Proof is easy, so omitted.

Now we discuss the main theorem of this section.

**Theorem 3.8.** Let $(X, M, \ast)$ be a SFMS and let $f, h : X \to X$ be two mappings with the following conditions,

(a) $f(X) \subset h(X)$

(b) Either $f(X)$ or $h(X)$ is structure complete

(c) $M(fx, fy, kt) \geq M(hx, hy, t)$ for all $x, y \in X$ and $0 < k < 1, t \in [0, \infty)$

(d) $\lim_{t \to \infty} M(x, y, t) = 1$

Then $f$ and $h$ have a coincidence point; moreover if $f$ and $h$ are CDN then $f$ and $h$ have a unique fixed point.
Proof:  
By condition (a), for some \( x_0 \in X \); we have \( x_1 \in X \) such that \( fx_0 = hx_1 = y_1 \) (say). Thus by using mathematical induction, we have \( fx_n = hx_{n+1} = y_{n+1} \) where \( n \in N \) and \( y_0 = hx_0 \).

For \( 0 < k < 1 \) and \( t \in [0, \infty) \) we have \( M(y_1, y_2, kt) = M(fx_0, fx_1, kt) \geq M(hx_0, hx_1, t) = M(y_0, y_1, t) \)

\[
M(y_1, y_2, kt) = M(fx_1, fx_2, kt) \geq M(hx_1, hx_2, t) = M(y_1, y_2, t) \geq M(y_0, y_1, \frac{t}{k}).
\]

Thus \( M(y_2, y_3, t) \geq M(y_0, y_1, \frac{t}{k^2}) \).

Proceeding by mathematical induction we have \( M(y_n, y_{n+1}, t) \geq M(y_0, y_1, \frac{t}{k^n}) \)

Thus for \( r \in N, t \in [0, \infty) \) we have \( M(y_n, y_{n+r}, t) \geq M(y_0, y_{n+1}, \frac{t}{k^r}) \)

\[
M(y_{n+1}, y_{n+r+1}, \frac{t}{k^r}) \geq M(y_{n+2}, y_{n+r+2}, \frac{t}{k^r}) \geq \ldots \geq M(y_n, y_1, \frac{t}{k^n}) \geq M(y_0, y_1, \frac{t}{k^n}) = \lim_{n \to \infty} M(y_n, y_{n+r}, t) = 1.
\]

Thus \( y_n \) is a structure Cauchy sequence. Let \( h(X) \) is structure complete; then there exists \( u \in h(X) \) such that \( \lim_{n \to \infty} y_{n+1} = \lim_{n \to \infty} hx_{n+1} = u = \lim_{n \to \infty} fx_n \). Let \( hp = u \) for some \( p \in X \)

Thus \( M(fp, hp, kt) = \lim_{n \to \infty} M(fp, fx_n, kt) \geq \lim_{n \to \infty} M(hp, hx_n, t) \geq \lim_{n \to \infty} M(u, hx_n, t) = 1 \). So, \( fp = hp \) and it proves that \( f \) and \( h \) have a coincidence point.

Now let \( f \) and \( h \) are normalized at some coincidence point \( \theta \). Thus from definition, we have \( M(fh\theta, hf\theta, t) = 1 \forall t \geq 0 \). This condition \( (M3) \) implies \( fh\theta = hf\theta \).

Let \( f\theta = h\theta = v \) then \( M(fv, v, kt) = M(fv, f\theta, kt) \geq M(hv, h\theta, t) = M(hfv, f\theta, t) \geq M(hv, h\theta, \frac{t}{k^n}) \geq \ldots \geq M(hv, h\theta, \frac{1}{k^n}) \). If \( n \to \infty \) then we must have \( fv = v \). In similar manner we can show that \( hv = v \). Thus \( v \) is common fixed point of \( f \) and \( h \). Proceeding in the same way we may work for \( h(X) \).

Uniqueness of fixed point: Let \( \lambda \) be another common fixed point of \( f \) and \( h \); then

\[
M(v, \lambda, kt) = M(fv, f\lambda, kt) \geq M(hv, h\lambda, t) = M(v, \lambda, t) = M(fv, f\lambda, t) \geq M(hv, h\lambda, \frac{t}{k^n}) = M(v, \lambda, \frac{t}{k^n}) \geq \ldots \geq M(v, \lambda, \frac{1}{k^n}).\]

Thus if \( n \to \infty \) then \( v = \lambda \). Hence the proof.

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