Generalized Locally-$\tau^*g$-closed sets

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ABSTRACT: In this paper, we define and study a new class of generally locally closed sets called $I$-locally-$\tau^*g$-closed sets in ideal topological spaces. We also discuss various characterizations of $I$-locally-$\tau^*g$-closed sets in terms of $g$-closed sets and $\beta^*_g$-closed sets.

Key Words: Ideal topological space, $g$-open set, $g$-closed set, $g$-local function, $(\cdot)^*g$-operator, $\tau^*_g$-open and $\tau^*_g$-closed.

Contents

1 Introduction 171
2 Preliminaries 171
3 $I$-locally-$\tau^*_g$-closed sets 172

1. Introduction

According to Bourbaki [3], a locally closed set is an intersection of an open set and a closed set. In [6], Levine defined a new class of generalized open and closed sets, and discussed their characterizations in detail. In [1], Balachandran, Sundaram and Maki defined and studied generalized locally closed sets using generalized closed sets and generalized open sets. In this paper, we introduce and study a new class of $I$-locally-$\tau^*_g$-closed sets using $g$-local functions defined in [2] with respect to the family of generalized open sets and ideal. Also we discuss various properties of this operator in detail.

2. Preliminaries

An ideal $\mathcal{I}$ [5] on $X$ is a nonempty collection of subsets of $X$ satisfying the following: (i) If $A \in \mathcal{I}$ and $B \subset A$, then $B \in \mathcal{I}$, and (ii) if $A \in \mathcal{I}$ and $B \in \mathcal{I}$, then $A \cup B \in \mathcal{I}$. A topological space $(X, \tau)$ together with an ideal $\mathcal{I}$ is called an ideal topological space and is denoted by $(X, \tau, \mathcal{I})$. For each subset $A$ of $X$, $A^*(\mathcal{I}, \tau) = \{x \in X \mid U \cap A \notin \mathcal{I} \text{ for every open set } U \text{ containing } x\}$ is called the local function of $A$ [5] with respect to $\mathcal{I}$ and $\tau$. We simply write $A^*$ instead of $A^*(\mathcal{I}, \tau)$ in case there is no chance for confusion. We often use the properties of the local function stated in Theorem 2.3 of [4] without mentioning it. Moreover, $cl^*(A) = A \cup A^*$ [8] defines a Kuratowski closure operator for a topology $\tau^*$, on $X$ which is finer than $\tau$. A subset $A$ of a topological space $(X, \tau)$ is said to be $g$-closed [6], if $cl(A) \subset U$ whenever $A \subset U$ and $U$ is open. The complement of a $g$-closed
set is called a g-open set [6]. The collection of all g-open sets in a topological space \((X, \tau)\) is denoted by \(\tau_g\). The g-closure of \(A\) denoted by \(cl_g(A)\) [2], defined as the intersection of all \(g\)-closed sets containing \(A\) and the \(g\)-interior of \(A\) denoted by \(int_g(A)\), defined as the union of all \(g\)-open sets contained in \(A\). For every \(A \in \mathcal{V}(X)\), \(A^*(\mathcal{I}, \tau_g) = \{x \in X \mid U \cap A \not\subseteq \mathcal{I} \text{ for every } g\text{-open set } U \text{ containing } x\}\) is called the \(g\)-local function of \(A\) [2] with respect to \(\mathcal{I}\) and \(\tau_g\) and is denoted by \(A^*_g\). Also, \(cl^*_g(A) = A \cup A^*_g\) [2] is a Kurotowski closure operator for a topology \(\tau^*_g = \{X - A \mid cl^*_g(A) = A\}\) [2] on \(X\) which is finer than \(\tau_g\). A subset \(A\) of a topological space \((X, \tau)\) is said to be \(\tau^*_g\)-closed [1], if \(A = U \cap V\) where \(U\) is open and \(V\) is \(g\)-closed. A subset \(A\) of an ideal space \((X, \tau, \mathcal{I})\) is said to be \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed [7], if \(A = U \cap V\) where \(U\) is open and \(V\) is \(g\)-closed.

3. \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed sets

**Definition 3.1.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. A subset \(A\) of an ideal topological space \((X, \tau, \mathcal{I})\) is said to be an \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed set if there exists a \(\tau^*_g\)-open set \(U\) and a \(\tau^*_g\)-closed set \(V\) such that \(A = U \cap V\).

The following Theorem 3.2 gives a characterization of \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed sets in terms of \(\tau^*_g\)-open sets.

**Theorem 3.2.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \(A \subset X\). Then the following are equivalent.

(a) \(A\) is an \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed set.

(b) \(A = U \cap cl^*_g(A)\) for some \(\tau^*_g\)-open set \(U\).

(c) \(A^*_g - A\) is a \(\tau^*_g\)-closed set.

(d) \((X - A^*_g) \cup A = A \cup (X - cl^*_g(A))\) is a \(\tau^*_g\)-open set.

(e) \(A \subset int^*_g(A) \cup (X - A^*_g)\).

**Proof:** (a) \(\Rightarrow\) (b). If \(A\) is an \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed set, then there exists a \(\tau^*_g\)-open set \(U\) and a \(\tau^*_g\)-closed set \(F\) such that \(A = U \cap F\). Clearly, \(A \subset U \cap cl^*_g(A)\). Since \(F\) is \(\tau^*_g\)-closed, \(cl^*_g(A) \subset cl^*_g(F) = F\) and so \(U \cap cl^*_g(A) \subset U \cap F = A\). Therefore, \(A = U \cap cl^*_g(A)\) for some \(\tau^*_g\)-open set \(U\).

(b) \(\Rightarrow\) (c). Now \(A^*_g - A = A^*_g \cap (X - A) = A^*_g \cap (X - (U \cap cl^*_g(A))) = A^*_g \cap (X - U)\).

Therefore, \(A^*_g - A\) is \(\tau^*_g\)-closed.

(c) \(\Rightarrow\) (d). Since \(X - (A^*_g - A) = (X - A^*_g) \cup A\), \((X - A^*_g) \cup A\) is \(\tau^*_g\)-open. Clearly, \((X - A^*_g) \cup A = A \cup (X - cl^*_g(A))\).

(d) \(\Rightarrow\) (e). The proof is clear.

(e) \(\Rightarrow\) (a). Since \(A^*_g\) is a \(g\)-closed set, \(X - A^*_g = int^*_g(X - A^*_g) \subset int^*_g(A \cup (X - A^*_g))\).

Then by hypothesis, \(A \cup (X - A^*_g) \subset int^*_g(A \cup (X - A^*_g))\) and so \(A \cup (X - A^*_g)\) is \(\tau^*_g\)-open. Since \(A = (A \cup (X - A^*_g)) \cap cl^*_g(A)\), \(A\) is an \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed set. □

Clearly, every open subset of an ideal topological space \((X, \tau, \mathcal{I})\) is always an \(\mathcal{I}\)-locally-\(\tau^*_g\)-closed, since every open set is a \(\tau^*_g\)-open set and \(X\) is \(\tau^*_g\)-closed. The
following Example 3.3 shows that the converse is not true in general. Also, every \( \tau^*_g \)-closed set is a 3-locally-\( \tau^*_g \)-closed set, since \( X \) is \( \tau^*_g \)-open. Example 3.4 below shows that the converse is not true in general.

**Example 3.3.** Let \((X, \tau)\) be a non-discrete topology. If \( \mathcal{I} = \emptyset(X) \), then every subset of \( X \) is \( * \)-closed and so every subset of \( X \) is \( \tau^*_g \)-closed and hence \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed. So there exists \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed sets which are not open.

**Example 3.4.** Let \( X = \{a, b, c, d\} \), \( \tau = \{\emptyset, X, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\} \) and \( \mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\} \). If \( A = \{b\} \), then \( A^*_g = \{b, c, d\} \notin A \) and so \( A \) is not a \( \tau^*_g \)-closed set. Besides, since \( A \) is \( \tau^*_g \)-open, \( A \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

**Theorem 3.5.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A \subset X \). If \( A \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set and \( A^*_g = X \), then \( A \) is a \( \tau^*_g \)-open set.

**Proof:** If \( A \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set, then by Theorem 3.2(e), \( A \subset \text{int}^*_g(A \cup (X - A^*_g)) \). Since \( A^*_g = X \) and so \( A \subset \text{int}^*_g(A) \) which implies that \( A \) is \( \tau^*_g \)-open.

**Corollary 3.6.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A^*_g = X \), where \( A \subset X \). Then \( A \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set if and only if \( A^*_g \) is a \( \tau^*_g \)-open set.

**Theorem 3.7.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space and \( A \) be an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed subset of \( X \). Then, the following hold.

(a) If \( B \) is a \( \tau^*_g \)-closed set, then \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

(b) If \( B \) is a \( \tau^*_g \)-open set, then \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

(c) If \( B \) is either a \( g \)-open or a \( g \)-closed set, then \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

(d) If \( B \) is either an open or a closed set, then \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

**Proof:** Since \( A \) is \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed, there exists a \( \tau^*_g \)-open set \( U \) and a \( \tau^*_g \)-closed set \( F \) such that \( A = U \cap F \).

(a) Let \( B \) be \( \tau^*_g \)-closed. Then \( A \cap B = (U \cap F) \cap B = U \cap (F \cap B) \), where \( F \cap B \) is \( \tau^*_g \)-closed. Hence, \( A \cap B \) is \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed.

(b) If \( B \) is a \( \tau^*_g \)-open set, then \( A \cap B = (U \cap F) \cap B = (U \cap B) \cap F \), where \( U \cap B \) is \( \tau^*_g \)-open. Therefore, \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

(c) If \( B \) is either a \( g \)-open or a \( g \)-closed set, then \( B \) is either \( \tau^*_g \)-open or \( \tau^*_g \)-closed. Therefore, by (a) and (b), \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

(d) Since every open and closed set is \( \tau^*_g \)-open and \( \tau^*_g \)-closed respectively, the proof is clear.

**Theorem 3.8.** Let \((X, \tau, \mathcal{I})\) be an ideal topological space. Then the intersection of two \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed sets is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.

**Proof:** Let \( A \) and \( B \) be \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed subsets of \( (X, \tau, \mathcal{I}) \). Then \( A = U_1 \cap V_1 \) and \( B = U_2 \cap V_2 \) for some \( \tau^*_g \)-open sets \( U_1 \) and \( U_2 \) and \( \tau^*_g \)-closed sets \( V_1 \) and \( V_2 \). Now \( A \cap B = (U_1 \cap V_1) \cap (U_2 \cap V_2) = (U_1 \cap U_2) \cap (V_1 \cap V_2) \), where \( U_1 \cap U_2 \) is \( \tau^*_g \)-open and \( V_1 \cap V_2 \) is \( \tau^*_g \)-closed. This implies that \( A \cap B \) is an \( \mathcal{I} \)-locally-\( \tau^*_g \)-closed set.
Corollary 3.9. The family of all $\tau^*_g$-closed sets in any ideal topological space $(X, \tau, \mathcal{I})$ is closed under arbitrary intersection.

Theorem 3.10. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A$ is a $g$-locally-closed set, then $A$ is an $\mathcal{I}$-locally-$\tau^*_g$-closed set.

Proof: If $A$ is $g$-locally-closed, then there exists a $g$-open set $U$ and a $g$-closed set $V$ such that $A = U \cap V$. Since every $g$-closed set is $\tau^*_g$-closed and every $g$-open set is $\tau^*_g$-open, $A$ is $\mathcal{I}$-locally-$\tau^*_g$-closed.

Theorem 3.11. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$ where $\mathcal{I} = \{\emptyset\}$. Then $A$ is a $g$-locally closed set if and only if $A$ is an $\mathcal{I}$-locally-$\tau^*_g$-closed set.

Proof: By Theorem 3.10, every $g$-locally closed set is an $\mathcal{I}$-locally-$\tau^*_g$-closed set. Conversely, since $\mathcal{I} = \{\emptyset\}$, $A^*_g = \text{cl}_g(A)$ which implies that $\tau^*_g$-closed sets coincide with $g$-closed sets. Therefore, $\mathcal{I}$-locally-$\tau^*_g$-closed sets coincide with $g$-locally-closed sets when $\mathcal{I} = \{\emptyset\}$.

Clearly, every $*$-closed set is an $\mathcal{I}$-locally-$\tau^*_g$-closed set. The following Example 3.12 shows that the converse is not true in general.

Example 3.12. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$ and $\mathcal{I} = \{\emptyset, \{a\}, \{c\}, \{a, c\}\}$. If $A = \{b, c, d\}$, $\text{cl}^*(A) = X$ and so $A$ is not $*$-closed. But $A^*_g = \{b, c, d\} = A$ which implies that $A$ is $\tau^*_g$-closed and so $\mathcal{I}$-locally-$\tau^*_g$-closed set.

In ideal topological spaces, locally closed sets are $\mathcal{I}$-locally-$\tau^*_g$-closed sets, since closed sets are $\tau^*_g$-closed sets. The following Example 3.13 shows that the converse is not true in general.

Example 3.13. Let $X = \{a, b, c, d\}$, $\tau = \{\emptyset, X, \{b\}, \{a, b\}, \{a, b, c\}, \{a, b, d\}\}$. If $\mathcal{I} = \varphi(X)$, then every subset of $X$ is $\tau^*_g$-closed. If $A = \{b, c, d\}$, then $A$ is $\mathcal{I}$-locally-$\tau^*_g$-closed. Since $X$ is the only open set containing $A$ and $A$ is not closed, $A$ is not a locally closed set.

Theorem 3.14. Let $(X, \tau, \mathcal{I})$ be an ideal topological space and $A \subset X$. If $A$ is $\mathcal{I}$-locally-$*$-closed, then $A$ is $\mathcal{I}$-locally-$\tau^*_g$-closed.

Proof: If $A$ is $\mathcal{I}$-locally-$*$-closed, then there exists an open set $U$ and a $*$-closed set $V$ such that $A = U \cap V$. Since every $*$-closed set is $\tau^*_g$-closed, $A$ is $\mathcal{I}$-locally-$\tau^*_g$-closed.

The following Example 3.15 shows that the converse of Theorem 3.13 is not true in general.

Example 3.15. Consider Example 3.12, if $A = \{a\}$, $\text{cl}^*(A) = \{a, d\}$, then $A$ is not $*$-closed, but $\text{cl}^*_g(A) = A$ implies that $A$ is $\tau^*_g$-closed. Hence $A$ is $\mathcal{I}$-locally-$\tau^*_g$-closed. Since $X$ is the only open set containing $A$ and $A$ is not $\mathcal{I}$-locally-$*$-closed.
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References


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