Remarks on Heron's cubic root iteration formula

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Abstract
The existence as well as the computation of roots appears in number theory, algebra, numerical analysis and other areas. The present study illustrate the contributions of several authors towards the extraction of different order roots of real number. Different methods with several approaches are studied to extract the roots of real number. Some of the methods described earlier are equivalent as observed in the present study. Heron developed a general iteration formula to determine the cube root of a real number \( N \) i.e.
\[
\sqrt[3]{N} = a + \frac{bd}{bd + aD}(b - a),
\]
where \( a^3 < N < b^3 \), \( d = N - a^3 \) and \( D = b^3 - N \). Although the direct proof of the above method is not available in literature, some authors have proved the same with the help of conjectures. In the present investigation, the proof of Heron’s method is explained and is generalized for any odd order roots. Thereafter it is observed that Heron’s method is a particular case of the generalized method.

Keyword: Cube root; Higher order roots; Heron’s method. Mathematics Subject Classifications 2010: 01A30; 01A35; 11A07.

1 Introduction
Recently, attempts have been made by many to find the cube and higher order roots of a real number in various methods with different approaches. Heron’s iteration formula to determine the cube root of a number \( N \) was \( \sqrt[3]{N} = a + \frac{bd}{bd + aD}(b - a) \), where \( a^3 < N < b^3 \), \( d = N - a^3 \) and \( D = b^3 - N \) as stated by Deslauriers and Dubuc [13]. According to Heath [16], a conjecture on Heron’s cubic root iteration formula was made by Wertheim [37] taking \( b = a + 1 \). Assuming some elementary considerations, Eneström [14] proved the Wertheim’s conjecture. Again Taisbak [34] made a conjecture about Heron’s method and provided possible proofs for it with the help of difference operators. Many researchers like Hess [17], Taisbak [35], Crisman and Veatch [11] are also continuing their research on Heron’s work. Recently, Gadzia et al. [15] extended the Wertheim [37] conjecture to find the odd order roots of a number and suggested possible proofs for even order roots. Again they showed that Al-Samawal’s and Lagrange’s method are equivalent for \( n \)th root extraction of a real number.

A direct proof of Heron’s general cubic root iteration formula is provided and it is extended for any odd order roots in the present work. Further, many counterexamples are discussed in support of the work. In addition to above some contributions made by different authors for extraction of roots are described.

2 Historical Background
In the middle of the tenth century, the book written by al-Udli̇dī̇sī (Book of chapters on Hindu arithmetic, kitāb al-fusil fī al-hīsāb al-hindī [2]) described the earliest work in Arabic that treats the computation of cube roots. The oldest description of the extraction of cube roots was also found in China, in the
classical work of the Nine chapters on the Mathematical Art which was translated by Chemla and Guo [8] in French and Kangshen et al. [21] in English. Later, Jia Xian used five rows in computation of cube root of a real number in the early eleventh century. Jia’s algorithm differs from the Nine chapters in surface structure. Again Jia gave two methods for cube root extraction and one method for fourth root extraction as quoted by Yang Hui in 1261.

The work presented in Principles of Hindu Reckoning (kitāb al-fusūl fi al-hisāb al-hindī) written c.1000 by the Persian Kushyar ibn Labban, The Sufficient on Hindu Calculation (al-Muqni fi al-hisāb al-hindī) by Alī ibn Ahmad al-Nasawi (written before 1030 [3]) and The Completion of Arithmetic (al-Takmilā fī al-hisāb) by Abd al-Qahir ibn Tahir al-Baghdādī were roughly contemporary with Jia Xian. Kushyār’s work was translated into English by Levey and Petruck [23]. Al-Nasawi’s text on the cube root was translated into German by Paul [28] and the work by Ibn Tahir was edited by Saidan [3]. The generalization of the algorithm for higher order roots was known in the twelfth century by al-Samawal al-Maghribi (Rashed[30]) and also by Nasîr al-Dīn al-Tūsî in the thirteenth century [1]. Fifth and fourth order root of a number was formulated by Al-Samawal (1172) and Nasir al-Din al-Tusi (1265), respectively using sexagesimal system. The algorithm described by Al-Tusi was very close to the earlier proposed algorithm of Kushyār. Their procedures coincide completely with the methods given by Jia Xian and Al-Samawal. This approach was later followed by Nizam al-Din al- Nisâbûri, who described the extraction of cube and higher roots elaborately in “The Epistle on Arithmetic”, around fourteenth century [24].

The earliest works on arithmetic from the Maghreb and Muslim Spain came from the twelfth and thirteenth centuries. During this period Al-Hassrūr, Ibn al-Yasamin and Ibn Mun’im all included cube extraction in some of their treatises. With the help of complete binomial expansions Ibn Mun’im extracted the fifth and seventh roots of a number. The fifth root of a 13 digits number was extracted by a Mathematician in Kairouan, Tunisia, before 1241 as described by Rashed [31] using binomial expansion. According to Lamrabet [22], Ibn al-Yasamin proposed an algorithm for the extraction of cube root on the development of \((a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3\). In 1427, Jamshid al-Kashi extracted the fifth root of a decimal number and sixth root of a sexagesimal number. The same algorithm was described by al-Qatrawnî [22] in fifteenth century.

Later, many researchers have studied the cube and other roots of a real number in different ways (Ruffini [32], Horner [18, 19]). For the first time Paul [28] studied, the extraction of higher roots in Arabic (Islamic) mathematics, which was focused on the work of Jamshid al-Kashi. Burr [6] developed several iteration methods for computing cube roots, when a fast square root was available. Some methods are superior to the conventional Newton’s method in particular situations was observed by him. Padro and Saez [26] generalized the algorithms established by Shanks [33] and Peralta [29] for computing square roots modulo of a prime to algorithms for computing cube roots, which played an important role in cryptosystems. Ahmadi et al. [4] calculated the number of nonzero coefficients (Hamming weight) in the polynomial representation of \(x^3 \in \mathbb{F}_3[x]/(f)\), where \(f \in \mathbb{F}_3[x]\) is an irreducible trinomial. Cho et al. [10] found that the shifted polynomial basis and variation of polynomial basis reduces Hamming weight of \(x^3 \) and \(x^2\) and suggested a suitable shifted polynomial basis which was eliminating modular reduction process in cube roots computation. Tapia [36] presented an algorithm which extracts the principal Q-th root of any real number for any given number system, which was an extension of the square root algorithm. Newton’s method is a first way to compute n-th root of a number and has quadratic convergence. Chen and Hsieh [9] proposed a new class of iterative methods based on Pade approximation to Taylor’s series of the differentiable function for computing the roots which was faster than Newton’s method. The fourth degree algorithm in the work of Chen and Hsieh [9] converges two times faster than Newton’s method. Duben [12] analyzed a double iteration process to find n-th root of a positive real number which was equivalent to the Newton’s method. Higher order methods were also mentioned for finding n-th roots. The study of Johansson [20] supported the hypotheses formulated by Paul [28] and Chemla [7] on an early scientific connection between China and Persia. Parakh [27] observed that, the methods adopted today for teaching in schools are the extension of Aryabhatta’s root extraction methods.

Recently, Padhan et al. [25] gave a general method to determine the cube and higher order roots of any real number, which is very simple and similar to the method proposed by Black [5]. This method has better performance in area, power consumption than the method adopted by Black [5] while
implementing on FPGA.

3 Iteration formulae for odd order roots

In this section, the Heron’s cubic root iteration formula is proved and extended this for any odd order roots.

3.1 Heron’s root iteration formula

Let us state the Heron’s cubic root iteration formula as Theorem 3.1.

**Theorem 3.1** If \( a^3 < N < b^3 \), then cube root of \( N \) is defined as

\[
\sqrt[3]{N} = a + \frac{bd}{bd + aD}(b - a),
\]

where \( d = N - a^3 \) and \( D = b^3 - N \).

**Proof:** Let \( x \) be the cube root of \( N \). Assume that \((x - a)^3 = \delta_1\) and \((b - x)^3 = \delta_2\).

\[
\begin{align*}
(x - a)^3 &= \delta_1 \\
\Rightarrow x^3 - 3x^2a + 3xa^2 - a^3 &= \delta_1 \\
\Rightarrow 3x^2a - 3xa^2 &= x^3 - a^3 - \delta_1 \\
\Rightarrow 3xa(x - a) &= d - \delta_1, \quad \text{where} \ d = x^3 - a^3. 
\end{align*}
\]

Similarly, taking \( D = b^3 - x^3 \), we get

\[
3bx(b - x) = D - \delta_2. \tag{3.2}
\]

Dividing eqn. (3.2) by eqn. (3.1), we have

\[
\frac{D - \delta_2}{d - \delta_1} = \frac{3bx(b - x)}{3xa(x - a)} = \frac{b(b - x)}{a(x - a)}. \tag{3.3}
\]

As the value of \( \delta_1 \) and \( \delta_2 \) are very small, from eqn. (3.6), we get

\[
\begin{align*}
\frac{D}{d} &= \frac{b(b - x)}{a(x - a)} \\
\Rightarrow \frac{aD}{bd} &= \frac{b - a}{x - a} - 1 \\
\Rightarrow x - a &= \frac{bd}{bd + aD}(b - a) \\
\Rightarrow x &= a + \frac{bd}{bd + aD}(b - a) \\
\Rightarrow \sqrt[3]{N} &= a + \frac{bd}{bd + aD}(b - a).
\end{align*}
\]

\( \square \)
3.2 Generalization of Heron’s root iteration formula

In the Theorem 3.2, the general formula for any odd order root of a number $N$ is discussed.

**Theorem 3.2** If $a^n < N < b^n$, $n = 2m + 1$, then approximate $n$th root of $N$ is defined as

$$\sqrt[n]{N} = a + \frac{b^n d}{b^m d + a^m D} (b - a),$$

where $d = N - a^n$ and $D = b^n - N$.

**Proof:** Let $x$ be the $n$th root of $N$. Assume that $(x - a)^{2m+1} = \delta_1$ and $(b - x)^{2m+1} = \delta_2$.

$$\Rightarrow x^{2m+1} - (2m + 1)c_1 x^{2m} a + (2m + 1)c_2 x^{2m-1} a^2 - ... + (2m + 1)c_{2m} x^2 a^{2m} - a^{2m+1} = \delta_1$$

$$\Rightarrow (2m + 1)c_1 x^{2m} a - (2m + 1)c_2 x^{2m-1} a^2 + ... - (2m + 1)c_{2m} x^2 a^{2m} = x^{2m+1} - a^{2m+1} - \delta_1$$

$$\Rightarrow (2m + 1)c_1 x a(x^{2m-1} - mx^{2m-2}a + ... - a^{2m-1}) = d - \delta_1,$$

where $d = x^{2m+1} - a^{2m+1}$.

Similarly, taking $b^{2m+1} - x^{2m+1} = D$, we have

$$(2m + 1)c_1 b x\{b^{2m-1} - mb^{2m-2}x + ... - x^{2m-1}\} = D - \delta_2.$$  \hspace{1cm} (3.5)

Dividing eqn. (3.5) by eqn. (3.4), we get

$$\frac{D - \delta_2}{d - \delta_1} = \frac{(2m + 1)c_1 b x\{b^{2m-1} - mb^{2m-2}x + ... - x^{2m-1}\}}{(2m + 1)c_1 x a(x^{2m-1} - mx^{2m-2}a + ... - a^{2m-1})}$$

$$= \frac{b^{2m-1} - mb^{2m-2}x + ... - x^{2m-1}}{a(x^{2m-1} - mx^{2m-2}a + ... - a^{2m-1})}$$

(neglecting the very small terms $(x - a)^3$, $(x - a)^5$, ..., $(x - a)^{2m-1}$ and simplifying)

$$= \frac{b^m}{a^m} \left( \frac{b - a}{x - a} - 1 \right).$$  \hspace{1cm} (3.6)

As the values of $\delta_1$ and $\delta_2$ are very small, from eqn. (3.6), we get

$$\frac{D}{d} = \frac{b^m}{a^m} \left( \frac{b - a}{x - a} - 1 \right)$$

$$\Rightarrow \frac{b - a}{x - a} = \frac{a^m D}{b^m d} + 1$$

$$\Rightarrow x - a = \frac{b^m d + a^m D}{b^m d + a^m D} (b - a)$$

$$\Rightarrow x = a + \frac{b^m d}{b^m d + a^m D} (b - a)$$

$$\Rightarrow \sqrt[n]{N} = a + \frac{b^m d}{b^m d + a^m D} (b - a).$$

\[\square\]

**Remark 3.1** When $m = 1$, Theorem 3.2 reduces to Heron’s cubic root iteration formula.

**Example 3.1** Evaluation of 5th root of 100.

It is clear that $2.5^5 < 100 < 2.6^5$ that is $97.65625 < 100 < 118.81376$. According to Theorem 3.2,
\[ a = 2.5, \quad b = 2.6, \quad d = 100 - 97.65625 = 2.34375, \quad D = 118.81376 - 100 = 18.81376. \]

Therefore,
\[
\sqrt[5]{100} = 2.5 + \frac{2.6^2 \times 2.34375}{2.6^2 \times 2.34375 + 2.5^2 \times 18.81376} (2.6 - 2.5)
\]
\[ = 2.5118742259504. \]

It can be easily verified that \((2.5118742259504)^5 = 99.997572463296\) and the error is very minimum that is 0.002429536704.

**Example 3.2** Evaluation of 7th root of 100.

It is clear that \(1.9^7 < 100 < 2^7\) that is \(89.3871739 < 100 < 128\). According to Theorem 3.2, \(a = 1.9, \quad b = 2, \quad d = 100 - 89.3871739 = 10.6128261, \quad D = 128 - 100 = 28.\)

Therefore,
\[
\sqrt[7]{100} = 1.9 + \frac{2^3 \times 10.6128261}{2^3 \times 10.6128261 + 1.9^3 \times 28} (2 - 1.9)
\]
\[ = 1.9306557847757. \]

It can be easily checked that \((1.9306557847757)^7 = 99.984793600089\) and the error is 0.015206399911.

**Remark 3.2** The error can be made as minimum as required by taking the values of \(a\) and \(b\) closed enough to the root.

### 4 Conclusion

A direct proof of Heron’s general cubic root iteration formula is described and extended for any odd order roots. It is observed that the Heron’s general cubic root iteration formula is a particular case of the present study. Counterexamples are discussed in support of the present investigation.

### References


