Upper Bound of Second Hankel determinant for generalized Sakaguchi type spiral-like functions

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ABSTRACT: In this paper, the authors introduce a generalized Sakaguchi type spiral-like function class $S(\lambda, \beta, s, t)$ and obtain sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the function $f$ in the above class. Relevances of the main result are also briefly indicated.

Key Words: Analytic functions, Starlike functions, Sakaguchi type functions, $\lambda$-spiral-like functions, Second Hankel determinant, Toeplitz determinants

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1. Introduction and Motivation

Let

$$U := \{z : z \in \mathbb{C} \text{ and } |z| < 1\}$$

be the unit disk in the complex $z$-plane. Let $\mathcal{A}$ be the class of functions $f$ of the form:

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are analytic in $U$ and satisfy the following normalization condition:

$$f(0) = f'(0) - 1 = 0.$$ 

Further, by $S$ we shall denote the class of all functions $f$ in $\mathcal{A}$ which are univalent in $U$.

A function $f \in \mathcal{A}$ is said to be $\lambda$-spiral starlike function of order $\beta$, denoted by $SP(\lambda, \beta)$ if and only if the following inequality holds true:

$$\Re \left[ e^{i\lambda} \frac{zf'(z)}{f(z)} \right] > \beta \quad (0 \leq \beta < 1, \quad |\lambda| \leq \frac{\pi}{2}; \quad z \in U).$$

For $\beta = 0$, the class $SP(\lambda, 0)$ reduces to $S_p(\lambda)$ which has been studied by Spacek [22]. Observed that for $\lambda = 0$, $S_p(0) = S^*$, the familiar class of starlike functions

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in $\mathbb{U}$. Recently, Frasin [9] introduced and studied a generalized Sakaguchi type function class $S(\alpha, s, t)$ as follows. A function $f(z) \in A$ is said to be in the class $S(\alpha, s, t)$ if it satisfies
\[
\Re \left[ \frac{(s - t)zf'(z)}{f(sz) - f(tz)} \right] > \alpha
\]
for some $\alpha (0 \leq \alpha < 1)$, $s, t \in \mathbb{C}$, $s \neq t$ and for all $z \in \mathbb{U}$.

Motivated by work of Frasin [9], we introduce here a new subclass of $A$ as follows:

**Definition 1.1.** A function $f(z) \in A$ is said to be in the generalized Sakaguchi type spiral-like class $S(\lambda, \beta, s, t)$ if it satisfies
\[
\Re \left[ e^{i\lambda} \frac{(s - t)zf'(sz)}{f(sz) - f(tz)} \right] > \beta \cos \lambda \quad (z \in \mathbb{U}),
\]
for some $\beta (0 \leq \beta < 1)$, $s$ and $t$ are real parameters, $s > t$ and $\lambda$ is real with $|\lambda| < \frac{\pi}{2}$.

It may be noted that for $s = 1$, $\lambda = 0$, the class $S(0, \beta, 1, t) = S(\beta, t)$ has been studied by Owa et al. [23, 24], Goyal and Goswami [10] and Cho et al. [4]; while for $s = 1$, $\lambda = 0$, $\beta = 0$, $t = -1$, the class $S(0, 0, 1, -1) = S(0, -1)$ has introduced and studied by Sakaguchi [21]. Further, for $\lambda = t = 0$, $s = 1$, the above class reduces to the well-known subclass of $A$ consisting of univalent starlike functions (see [6]).

The $q$th Hankel determinant for $q \geq 1$ and $n \geq 1$ is stated by Noonan and Thomas [19] as
\[
H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.
\]

A good amount of literature is available about the importance of Hankel determinant. It is useful in the study of power series with integral coefficients (see [3]), meromorphic functions (see [25]) and also singularities (see [5]). Noor (see [20]) determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with a bounded boundary while Ehrenborg (see [7]) studied the Hankel determinant of exponential polynomials.

For $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to
\[
H_2(1) = \begin{vmatrix} 1 & a_2 \\ a_2 & a_3 \end{vmatrix} = a_3 - a_2^2
\]
and
\[
H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2.
\]
It is well-known [6] that for \( f \in S \) and given by (1.1), the sharp inequality \( |a_3 - a_2^3| \) holds. Fekete-Szegö (see [8]) then further generalized the estimate \( |a_3 - \mu a_2^2| \) with \( \mu \) real and \( f \in S \). For a given family \( \mathcal{F} \) of the functions in \( \mathcal{A} \), the sharp upper bound for the nonlinear functional \( |a_2a_4 - a_3^2| = |H_2(1)| \) is popularly known as the second Hankel determinant. Second Hankel determinant for various subclasses of analytic functions were obtained by various authors. For details, (see [1,2,11,12,13,14,15,18]).

Following the techniques devised by Libera and Zlotkiewicz (see [16,17]), in the present paper, the authors determine a sharp upper bound of the second Hankel determinant \( |H_2(1)| \) for the function \( f \) belonging to the class \( S(\lambda, \beta, s, t) \).

2. Preliminaries

Let \( \mathcal{P} \) denote the class of functions normalized by
\[
p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n,
\]
which are regular in \( U \) and satisfying \( \Re\{p(z)\} > 0 \) for every \( z \in U \). Here \( p(z) \) is called caratheodory function (see [6]).

To investigate the main result, we need the following lemmas.

Lemma 2.1. (see [6]) If \( p \in \mathcal{P} \), then \( |c_n| \leq 2 \), for each \( n \geq 1 \) and the inequality is sharp for the function \( 1 + z \).

Lemma 2.2. ([16], also see [17, p. 254]) Let the function \( p \in \mathcal{P} \) be given by the power series (2.1). Then
\[
2c_2 = c_1^3 + x(4 - c_1^2),
\]
and
\[
4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - (4 - c_1^2)c_1x^2 + 2(4 - c_1^2)(1 - |x|^2)y
\]
for some complex numbers \( x, y \) satisfying \( |x| \leq 1 \) and \( |y| \leq 1 \).

3. Main Result

Theorem 3.1. Let the function \( f \) given by (1.1) be in the class \( S(\lambda, \beta, s, t) \). Then
\[
|a_2a_4 - a_3^2| \leq \frac{4(1 - \beta)^2 \cos^2 \lambda}{(2s^2 - st - t^2)^2}.
\]
The estimate in (3.1) is sharp.

Proof: Let the function \( f(z) \) given by (1.1) be in the class \( S(\lambda, \beta, s, t) \). Then from the Definition 1.1, there exists an analytic function \( p \in \mathcal{P} \) in the unit disk \( U \) with \( p(0) = 1 \) and \( \Re(p(z)) > 0 \) such that
\[
e^{i\lambda(s-t)}z f'(sz) = (1 - \beta)p(z) + \beta |\cos \lambda + isin \lambda|
\]
\[
\implies e^{i\lambda(s-t)}z f'(sz) - e^{i\lambda}(f(sz) - f(tz))
\]
\[
= (f(sz) - f(tz))((1 - \beta)p(z) + \beta - 1)|\cos \lambda|.
\]
Replacing \( f(tz) \), \( f(sz) \), \( f'(sz) \) and \( p(z) \) by their equivalent series in (3.2), after simplification, we obtain

\[
e^{i\lambda}[a_2(s-t)z + a_3(2s^2-st-t^2)z^2 + a_4(3s^3-s^2t-st^2-t^3)z^3 + \cdots] = [c_1z + \{a_2(s+t)c_1 + c_2\}z^2 + \{a_3(s^2+st+t^2)c_1 + a_2(s+t)c_2 + c_3\}z^3 + \cdots](1-\beta)\cos \lambda.
\]

(3.3)

Equating the coefficients of \( z \), \( z^2 \) and \( z^3 \) in (3.3), we get

\[
a_2 = \frac{e^{-i\lambda}(1-\beta)\cos \lambda}{s-t} c_1,
\]

\[
a_3 = \frac{e^{-i\lambda}(1-\beta)\cos \lambda}{2s^2-st-t^2} \left[ e^{-i\lambda}(1-\beta)(s+t)\cos \lambda c_1^2 + (s-t)c_2 \right],
\]

\[
a_4 = \frac{e^{-i\lambda}(1-\beta)\cos \lambda}{5s^3-3st^2-2st} \left[ (3s^3-2t^3+2st^2-2st)(s-t)\cos \lambda c_1^2 + 2e^{-i\lambda}(1-\beta)\cos \lambda c_1^2 c_2 + c_1 c_3 + e^{-i\lambda}(1-\beta)^2\cos \lambda c_2^2 \right] \left[ (s-t)^2(2s^2-st-t^2)^2 \right],
\]

(3.4)

Substituting the values of \( a_2 \), \( a_3 \) and \( a_4 \) from (3.4) in the second Hankel functional \( |a_2a_4-a_3^2| \) for the function \( f \in S(\lambda, \beta, s, t) \), we obtain

\[
|a_2a_4-a_3^2| = \left| \frac{e^{-2i\lambda}(1-\beta)^2\cos \lambda}{(s-t)(3s^3-st^2-st^2-t^3)} \right|
\]

\[
+ \frac{3s^3-2t^3+s^2t-2st^2}{(s-t)(2s^2-st-t^2)} e^{-i\lambda}(1-\beta)\cos \lambda c_1^2 c_2
\]

\[
+ e^{-2i\lambda}(1-\beta)^2\cos \lambda \left( s^3 + 2s^2t + 2st^2 + t^3 \right) c_4
\]

\[
+ c_1 c_3 - \frac{e^{-2i\lambda}(1-\beta)^2\cos \lambda}{(s-t)^2(2s^2-st-t^2)^2} \left[ (s-t)^2 c_2^2 \right]
\]

\[
+ e^{-2i\lambda}(1-\beta)^2\cos \lambda \left( s^3 + 2s^2t + 2st^2 + t^3 \right) c_4
\]

\[
+ c_1 c_3 - \frac{e^{-2i\lambda}(1-\beta)^2\cos \lambda}{(s-t)^2(2s^2-st-t^2)^2} \left[ (s-t)^2 c_2^2 \right]
\]

(3.5)

Making use of the result \( |xa+yb| \leq |x||a| + |y||b| \), where \( x, y, a \) and \( b \) are real numbers and \( |e^{-in\lambda}| = 1 \), where \( n \) is a real number, after simplification, we obtain

\[
|a_2a_4-a_3^2| \leq \frac{(1-\beta)^2\cos \lambda}{(s-t)^2(3s^3-st^2-st^2-t^3)(2s^2-st-t^2)^2} \left| d_1 c_1 c_3 + d_2 \cos \lambda c_1^2 c_2 + d_3 c_2^2 + d_4 \cos \lambda c_1^2 \right|,
\]

(3.6)
where

\[ d_1 = (s - t)(2s^2 - st - t^2)^2, \]
\[ d_2 = (s^4t - 3s^2t^3 + 2st^4)(1 - \beta), \]
\[ d_3 = -(s - t)^2(3s^3 - st^2 - s^2t - t^3), \]
\[ d_4 = -(s^5 + 2s^4t - s^3t^2 - 2s^2t^3)(1 - \beta)^2. \] (3.7)

Substituting the values of \( c_2 \) and \( c_3 \) from Lemma 2.2 in the right hand side of (3.6), we have

\[
|d_1c_1c_3 + d_2c_2 + d_3c_2 + d_4c_4| = \left| \frac{d_1c_1}{4} [c_1^3 + 2c_1(4 - c_1^2)x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |z|^2)z + \frac{d_2c_2}{2}c_1^2 + x(4 - c_1^2)] + \frac{d_3}{4} (c_1^2 + x(4 - c_1^2))^2 + d_4c_4 \right|. \] (3.8)

Making use of well-known fact that \(|z| < 1\) in (3.8), upon simplification gives

\[
4|d_1c_1c_3 + d_2c_2 + d_3c_2 + d_4c_4| = \left| (d_1 + 2d_2c_2 + d_3 + 4d_4c_4) c_1^4 + 2d_2c_1(4 - c_1^2) + 2(d_1 + d_2c_2 + d_3)c_1^2(4 - c_1^2)|x| - ((d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3)(4 - c_1^2)|x|^2 \right|. \] (3.9)

Using the values of \( d_1, d_2, d_3 \) and \( d_4 \) given in (3.7), after simplification, we obtain

\[ d_1 + 2d_2c_2 + d_3 + 4d_4c_4 \lambda = (s - t)(s^4 - 3s^2t^2 + 2st^3) + 2(1 - \beta) \]
\[ (s^4t - 3s^2t^3 + 2st^4) \cos \lambda - 4(s^5 + 2s^4t - s^3t^2 - 2s^2t^3)(1 - \beta)^2 \cos^2 \lambda, \] (3.10)

\[ 2(d_1 + d_2c_2 + d_3) = 2[(s - t)(s^4 - 3s^2t^2 + 2st^3) + (s^4t - 3s^2t^3 + 2st^4)(1 - \beta)c_2 \lambda], \] (3.11)

and

\[ (d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3 = [(s - t)(2s^2 - st - t^2)^2 - (s - t)^2(3s^3 - st^2 - s^2t - t^3)]c_1^2 + 2(s - t)(2s^2 - st - t^2)^2c_1 + 4(s - t)^2(3s^3 - st^2 - s^2t - t^3). \] (3.12)
Consider

\[
\left[(s-t)(2s^2 - st - t^2)^2 - (s-t)^2(3s^3 - st^2 - s^2t - t^3)\right]c_1^2
+ 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3)
= (s-t)(s^4 - 3s^2t^2 + 2st^3)
\]

\[
\left[c_1^2 + \frac{2(2s^2 - st - t^2)^2}{s^4 - 3s^2t^2 + 2st^3} + 4\frac{(s-t)(3s^3 - st^2 - s^2t - t^3)}{s^4 - 3s^2t^2 + 2st^3}\right]
= (s-t)(s^4 - 3s^2t^2 + 2st^3)
\]

\[
= \left\{c_1 + \frac{(2s^2 - st - t^2)^2}{s^4 - 3s^2t^2 + 2st^3}\right\}^2
\]

\[
= \left\{\sqrt{4s^8 + 28s^6t^2 - 16s^4t + 29s^4t^4 - 32s^4t^3 + 10s^2t^6 - 20s^3t^5 - 4st^7 + t^8}\right\}
\]

\[
= (s-t)(s^4 - 3s^2t^2 + 2st^3)
\]

\[
\left[c_1 + \left\{\frac{2s^2 - st - t^2}{s^4 - 3s^2t^2 + 2st^3}\right\} + \sqrt{4s^8 + 28s^6t^2 - 16s^4t + 29s^4t^4 - 32s^4t^3 + 10s^2t^6 - 20s^3t^5 - 4st^7 + t^8}\right] \times
\left[c_1 + \left\{\frac{2s^2 - st - t^2}{s^4 - 3s^2t^2 + 2st^3}\right\} - \sqrt{4s^8 + 28s^6t^2 - 16s^4t + 29s^4t^4 - 32s^4t^3 + 10s^2t^6 - 20s^3t^5 - 4st^7 + t^8}\right]
\]

Since \(c_1 \in [0, 2]\), using the result \((c_1 + a)(c_2 + b) \geq (c_1 - a)(c_1 - b)\), where \(a, b \geq 0\) in the right-hand side of (3.13), upon simplification, we obtain

\[
\left[(s-t)(2s^2 - st - t^2)^2 - (s-t)^2(3s^3 - st^2 - s^2t - t^3)\right]c_1^2
+ 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3)
\geq \left[(s-t)(2s^2 - st - t^2)^2 - (s-t)^2(3s^3 - st^2 - s^2t - t^3)\right]c_1^2
- 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3)
\]

From (3.12) and (3.14), it follows that
−[(d_1 + d_3)c_1^2 + 2d_1c_1 - 4d_3] \leq -\left\{ (s-t)(2s^2 - st - t^2)^2 - (s-t)^2(3s^3 - st^2 - s^2t - t^3) \right\} c_1^2 \\
-2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3)\right]\right).

(3.15)

Substituting the values from the relation (3.10), (3.11) and (3.15) in the right hand side of (3.9), we get

\[ 4|d_1c_1 + d_2c_2 + d_3c^2 + d_4c^2λc_1 | \leq |(s-t)(s^4 - 3s^2t^2 + 2st^3) + 2(1-β)\]
\[ (s^3 - 3st^2 + 2st^3)\cos λ - 4(s^3 + 2s^2 t - s^2t^2 - 2s^2t^3)(1-β)^2\cos λ c_1^2 ]
\[ +2(s-t)(2s^2 - st - t^2)^2c_1 (4c_1 - c_1^2) + 2(s-t)(2s^2 - st - t^2)^2c_1^2(1-β)c_1^2 \]
\[ +2(s-t)(3s^2t + 2st^3)(1-β)c_1^2 \cos λ - 2(s-t)^2(3s^3 - st^2 - st^2 - t^3) | c_1^2 - 
\]
\[ 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3) | (4c_1 - c_1^2) | x |^2 \]

(3.16)

Choosing \( c_1 = c \in [0, 2] \), applying triangle inequality and replacing \( |x| \) by \( μ \) on the right hand sides of (3.16), we obtain

\[ 4|d_1c_1 + d_2c_2 + d_3c^2 + d_4c^2λc_1 | \leq |4\cos^2 λ(1-β)^2(s^5 + 2s^4 t - s^3 t^2 - 2s^2 t^3) 
\[ -(s-t)(s^4 - 3s^2t^2 + 2st^3) - 2(1-β)(s^3 - 3st^2 + 2st^3)\cos λ c_1^2 ]
\[ +2(s-t)(2s^2 - st - t^2)^2c_1^2(4c_1 - c_1^2) + 2(s-t)(2s^2 - st - t^2)^2c_1^2(1-β)c_1^2 \]
\[ +2(s-t)(3s^2t + 2st^3)(1-β)c_1^2 \cos λ - 2(s-t)^2(3s^3 - st^2 - st^2 - t^3) | c_1^2 - 
\]
\[ 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3) | (4c_1 - c_1^2) μ^2 \]
\[ = H(c, μ) \] (say) \( 0 \leq μ = |x| \leq 1, \ 0 \leq c \leq 2 \),

(3.17)

where

\[ H(c, μ) = 4\cos^2 λ(1-β)^2(s^5 + 2s^4 t - s^3 t^2 - 2s^2 t^3) 
\[ -(s-t)(s^4 - 3s^2t^2 + 2st^3) - 2(1-β)(s^3 - 3st^2 + 2st^3)\cos λ c_1^2 ]
\[ +2(s-t)(2s^2 - st - t^2)^2c_1^2(4c_1 - c_1^2) + 2(s-t)(2s^2 - st - t^2)^2c_1^2(1-β)c_1^2 \]
\[ +2(s-t)(3s^2t + 2st^3)(1-β)c_1^2 \cos λ - 2(s-t)^2(3s^3 - st^2 - st^2 - t^3) | c_1^2 - 
\]
\[ 2(s-t)(2s^2 - st - t^2)^2c_1 + 4(s-t)^2(3s^3 - st^2 - s^2t - t^3) | (4c_1 - c_1^2) μ^2 \].

(3.18)

Now we maximize the function \( H(c, μ) \) on the closed region \([0, 2] \times [0, 1]\). Differen-
Upon simplification, it follows from (3.18) partially with respect to \( \mu \), we get

\[
\frac{\partial H}{\partial \mu} = [2(s - t)(2s^2 - st - t^2)^2 + 2(s^4t - 3s^2t^3 + 2st^4)(1 - \beta)\cos \lambda
- 2(s - t)^2(3s^3 - st^2 - s^2t - t^3)]c^2(4 - c^2) + 2[[(s - t)(2s^2 - st - t^2)^2
- (s - t)^2(3s^3 - st^2 - s^2t - t^3)]c^2 - 2(s - t)(2s^2 - st - t^3)2c
+ 4(s - t)^2(3s^3 - st^2 - s^2t - t^3)](4 - c^2)\mu \tag{3.19}
\]

For \( 0 < \mu < 1 \), for fixed \( c \), \( 0 < c < 2 \), we observe from (3.19) that \( \frac{\partial H}{\partial \mu} > 0 \). Therefore, \( H(c, \mu) \) is an increasing function of \( \mu \) and hence cannot have a maximum value at any point in the interior of the closed square \([0, 2] \times [0, 1] \). Moreover, for a fixed \( c \in [0, 2] \), we have

\[
\max_{0 \leq \mu \leq 1} H(c, \mu) = H(c, 1) = T(c) \tag{3.20}
\]

Upon simplification, it follows from (3.18) and (3.20) that

\[
T(c) = 4[4\cos^2 \lambda(1 - \beta)^2(s^5 + 2s^4t - s^3t^2 - 2st^3)] - (s - t)(s^4 - 3s^2t^2 + 2st^3)
2(1 - \beta)(s^4t - 3s^2t^2 + 2st^4)\cos \lambda)c^4 + 2(s^4t - 3s^2t^2 + 2st^4)(1 - \beta)\cos \lambda + 3(s - t)
(s^4 - 3s^2t^2 + 2st^3))c^2(4 - c^2) + 4(s - t)^2(3s^3 - st^2 - s^2t - t^3)(4 - c^2).
\]

Therefore,

\[
T'(c) = 4[4\cos^2 \lambda(1 - \beta)^2(s^5 + 2s^4t - s^3t^2 - 2st^3)] - (s - t)(s^4 - 3s^2t^2 + 2st^3)
2(1 - \beta)(s^4t - 3s^2t^2 + 2st^4)\cos \lambda)c^3 + 2(s^4t - 3s^2t^2 + 2st^4)(1 - \beta)\cos \lambda + 3(s - t)
(s^4 - 3s^2t^2 + 2st^3))(8c - 4c^3) - 8(s - t)^2(3s^3 - st^2 - s^2t - t^3)c.
\]

and

\[
T''(c) = 12[4\cos^2 \lambda(1 - \beta)^2(s^5 + 2s^4t - s^3t^2 - 2st^3)] - (s - t)(s^4 - 3s^2t^2 + 2st^3)
2(1 - \beta)(s^4t - 3s^2t^2 + 2st^4)\cos \lambda)c^2 + 2(s^4t - 3s^2t^2 + 2st^4)(1 - \beta)\cos \lambda + 3(s - t)
(s^4 - 3s^2t^2 + 2st^3))(8 - 12c^2) - 8(s - t)^2(3s^3 - st^2 - s^2t - t^3).
\]

For extreme values of \( T(c) \), consider \( T'(c) = 0 \). From (3.22) we have \( c = 0 \). Putting the values of \( c = 0 \) in (3.23) and simplify, we get

\[
T''(c) = -8(s - t)[(9s^2t^2 - 6st^3 - 4s^4t + t^4) - 2(s^3t - 2st^3 + s^2t^2)(1 - \beta)\cos \lambda]
\leq 0 \quad (0 \leq \beta < 1, |\lambda| < \frac{\pi}{2}). \tag{3.24}
\]
By second derivative test, $T(c)$ has maximum values at $c = 0$ and for a fixed value of $\lambda$ ($|\lambda| < \frac{\pi}{2}$), we obtain

$$
\max_{0 \leq c \leq 2} T(c) = T(0) = 16(s - t)^2(3s^3 - st^2 - s^2t - t^3).
$$

(3.25)

Consider the maximum value of $T(c)$ only at $c = 0$, simplifying the relation (3.17) and (3.25), we obtain

$$
|d_1c_1c_3 + d_2\cos\lambda c_1^2c_2 + d_3c_2^2 + d_4\cos^2\lambda c_1^4| \leq 4(s - t)^2(3s^3 - st^2 - s^2t - t^3).
$$

(3.26)

From (3.6) and (3.26), after simplifying, we get

$$
|a_2a_4 - a_3^2| \leq \frac{4(1 - \beta)^2\cos^2\lambda}{(2s^2 - st - t^2)^2}.
$$

(3.27)

By choosing $c_1 = c = 0$ and selecting $x = -1$ in (2.2) and (2.3), we find that $c_2 = -2$ and $c_3 = 0$. Under such case it follows from (3.4) that $a_2 = 0$, $a_3 = -\frac{2\sin(1 - \beta)\cos\lambda}{2s^2 - st - t^2}$ and $a_4 = 0$. Substituting these values in the functional $|a_2a_4 - a_3^2|$, we observed that the equality is attained which shows our result is sharp. This completes the proof of Theorem 3.1.

Concluding Remark: In this paper, we have determined the sharp upper bound for the functional $|a_2a_4 - a_3^2|$ for the functions $f \in \mathcal{A}$ belonging to the class $S(\lambda, \beta, s, t)$. We conclude this paper by remarking that the above theorem include several previously established results for particular values of the parameters $\lambda$, $\beta$, $s$, $t$. For example, taking $s = 1$, $t = 0$ and $\beta = 0$ in Theorem 3.1 we get the result due to Krishna and Reddy (see [15]). Further, by letting $s = 1$, $t = 0$, $\beta = 0$ and $\lambda = 0$ in Theorem 3.1, we obtain the result $|a_2a_4 - a_3^2| \leq 1$. This result is sharp and coincides with that of Janteng et al. (see [13]). Now we are working on to find the sharp upper bound for the above function class using third Hankel determinant.

References


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