A Takeuchi-Yamada type equation with variable exponents

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ABSTRACT: We prove continuity of the flows and upper semicontinuity of global attractors for a Takeuchi-Yamada type equation with variable exponents.

Key Words: Variable exponents, parabolic problems, global attractors, upper semicontinuity.

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1. Introduction

The study of the continuity with respect to initial conditions and parameters is important to verify the stability of a PDE model. Currently, some researchers investigated in which way the parameter $p(x)$ affects the dynamic of problems involving the $p(x)$-Laplacian, analyzing the continuity properties of the flows and of the global attractors with respect to the parameter $p(x)$. B. Amaziane, L. Pankratov and V. Prytula studied homogenization of $p_\epsilon(x)$-Laplacian elliptic equations (see [2]) and B. Amaziane, L. Pankratov and A. Piatnitski studied nonlinear flow through double porosity media in variable exponent Sobolev spaces (see [1]) where the authors considered the following initial boundary value problem

$$
\begin{align*}
\omega^\epsilon(x) \frac{\partial u^\epsilon}{\partial t}(t) - \text{div}(k^\epsilon(x)\nabla u^\epsilon |\nabla u^\epsilon|^{p^\epsilon(x)-2}) = g(t, x) & \quad \text{in} \quad Q \\
u^\epsilon = 0 & \quad \text{on} \quad [0, t] \times \partial \Omega, \\
u^\epsilon(0, x) = u_0(x) & \quad \text{in} \quad \Omega,
\end{align*}
$$

where $\Omega \subset \mathbb{R}^n$ ($n = 2, 3$) is a bounded domain, $Q$ denotes the cylinder $[0, T] \times \Omega$, $T > 0$ is given, $g \in C([0, T]; L^2(\Omega))$ and $u_0 \in H^2(\Omega)$ are given functions. They...
studied the minimization problem for functionals in the limit of small $\epsilon$ and obtained the homogenized functional. We considered in [11] the following nonlinear PDE problem

\[
\begin{align*}
    \frac{\partial u}{\partial t}(t) - \frac{\partial}{\partial x} \left( \frac{\partial u}{\partial x}(t) \right)^{p_s(x)-2} \frac{\partial u}{\partial x}(t) &= B(u_s(t)), \quad t > 0 \\
    u_s(0) &= u_{0s},
\end{align*}
\]

under Dirichlet homogeneous boundary conditions, where $u_{0s} \in H := L^2(I)$, $I := (c,d)$, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $p_s(x) \in C^1(I)$, $p^-_s := \text{ess inf } p_s > 2$ for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(I)$ ($p > 2$ constant) as $s \to \infty$. We proved continuity of the flows and upper semicontinuity of the family of global attractors $\{A_s\}_{s \in \mathbb{N}}$ as $s$ goes to infinity.

In this work we consider the nonlinear perturbation $|u|^{p_s(x)-2}u$ of the $p(x)$--Laplacian, i.e., we consider the following nonlinear PDE problem

\[
\begin{align*}
    \frac{\partial u}{\partial t}(t) - \text{div}(|\nabla u_s(t)|^{p_s(x)-2}\nabla u_s(t)) + |u_s(t)|^{p_s(x)-2}u_s(t) &= B(u_s(t)), \quad t > 0 \\
    u_s(0) &= u_{0s},
\end{align*}
\]

under homogeneous Neumann Boundary conditions, where $u_{0s} \in H := L^2(\Omega)$, $\Omega \subset \mathbb{R}^n (n \geq 1)$ is a smooth bounded domain, $B : H \to H$ is a globally Lipschitz map with Lipschitz constant $L \geq 0$, $p_s(\cdot) \in C^1(\Omega)$, $p^-_s := \text{ess inf } p_s \geq p$, $p^+_s := \text{ess sup } p_s \leq a$, for all $s \in \mathbb{N}$, and $p_s(\cdot) \to p$ in $L^\infty(\Omega)$ as $s \to \infty$ ($p > 2$ and $a$ are constants). We prove continuity of the flows and upper semicontinuity of the family of global attractors $\{A_s\}_{s \in \mathbb{N}}$ as $s$ goes to infinity for the problem (1.1).

In [5], Chafee and Infante considered the equation

\[(P1) \ u_t = \lambda u_{xx} + u - u^3,
\]

and Takeuchi and Yamada considered in [14] the following more general equation involving the $p$-Laplacian operator

\[(P2) \ u_t = \lambda (|u|^p-2u)_x + |u|^{q-2}u(1 - |u|^r),
\]

where $p > 2$, $q \geq 2$, $r > 0$ and $\lambda > 0$ are constants. Note that taking $p = q = r = 2$, problem (P2) becomes problem (P1). The authors in [4] proved the continuity of the flows and upper semicontinuity of a family of global attractors for the problem (P2) when $p = q$ and $p \to 2$.

Considering the problem

\[u_t = \lambda \text{div}(|\nabla u|^{p(x)-2}\nabla u) + |u|^{q-2}u(1 - |u|^r(x)),
\]

with $q \equiv 2$ and $r(x) := p(x) - 2 > 0$, we obtain

\[(P3) \ u_t = \lambda \text{div}(|\nabla u|^{p(x)-2}\nabla u) + u(1 - |u|^{p(x)-2}).
\]

Note that the problem

\[
\begin{align*}
    \left\{ \begin{array}{ll}
    u_t &= \lambda \text{div}(|\nabla u|^{p(x)-2}\nabla u) + u(1 - |u|^{p(x)-2}), & t > 0 \\
    u(0) &= u_0,
    \end{array} \right.
\]

\]}
can be seen as
\[
\begin{cases}
u_t - \lambda \text{div}( |\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u = u, & t > 0, \\
u(0) = u_0, &
\end{cases}
\] (1.2)
and \(\tilde{B}(u) := u\) is a globally Lipschitz map. So, all the results developed in this paper for an abstract globally Lipschitz external forcing term can be applied to the Takeuchi-Yamada type equation (1.2). The bifurcation studies of solutions to problem (1.2) with respect to the parameter \(\lambda\) remains an open problem.

The study of continuity properties with respect to initial conditions and exponent parameters for the problem \(u_t = \lambda(|u|^p - 2u)x + u\) were already contemplated in the papers \([10,11]\).

The paper is organized as follows. In Section 2 we present properties on the operator and we guarantee existence of global solution and global attractor for problem (1.1). In Section 3 we obtain uniform estimates for solutions of (1.1). In Section 4 we prove that the solutions \(\{u_s\}\) of (1.1) go to the solution \(v\) of the limit problem (4.1) and, after that, we obtain the upper semicontinuity of the global attractors for the problem (1.1).

2. Properties on the operator

The authors in [13] proved that the operator
\[
Au := -\text{div}( |\nabla u|^{p(x)-2} \nabla u) + |u|^{p(x)-2} u
\]
where \(p(\cdot)\) is continuous in \(\Omega\) and \(p^- > 2\), is the realization of the operator \(A_1 : X \to X^*, X := W^{1,p(x)}(\Omega),\)
\[
A_1 u(v) := \int_{\Omega} (|\nabla u(x)|^{p(x)-2} \nabla u(x) \cdot \nabla v(x) dx + \int_{\Omega} |u(x)|^{p(x)-2} u(x) v(x) dx,
\]
i.e., \(A(u) = A_1 u, \text{ if } u \in D(A) := \{u \in X; A_1 u \in H\}\) and it is a maximal monotone operator in \(H\). Besides, \(A\) is the subdifferential of the proper, convex and lower semicontinuous function \(\varphi_{p(x)} : H \to \mathbb{R} \cup \{+\infty\}\) defined by
\[
\varphi_{p(x)}(u) := \left\{ \begin{array}{ll}
\int_{\Omega} \frac{1}{p(x)} |\nabla u|^{p(x)} dx + \int_{\Omega} \frac{1}{p(x)} |u|^{p(x)} dx, & \text{if } u \in X, \\
+\infty, & \text{otherwise}.
\end{array} \right.
\] (2.1)
Moreover, we have the following properties on the operator

Lemma 2.1. [13]

\[
\langle Au, u \rangle_{X^*,X} \geq \left\{ \begin{array}{ll}
\frac{1}{2p^+ + 1} ||u||_{p(x)}^{p^+ x}, & \text{if } ||u||_{p(x)} \leq 1 \text{ and } ||\nabla u||_{p(x)} \leq 1, \\
\frac{1}{2p^- + 1} ||u||_{p^-(x)}^{p^- x}, & \text{if } ||u||_{p(x)} \geq 1 \text{ and } ||\nabla u||_{p(x)} \geq 1, \\
||\nabla u||_{p(x)}^{p^+} + ||u||_{p(x)}^{p^+}, & \text{if } ||u||_{p(x)} \leq 1 \text{ and } ||\nabla u||_{p(x)} \geq 1, \\
||\nabla u||_{p(x)}^{p^+} + ||u||_{p(x)}^{p^+}, & \text{if } ||u||_{p(x)} \geq 1 \text{ and } ||\nabla u||_{p(x)} \leq 1.
\end{array} \right.
\]
By Consequence 3 in [13], it follows that the equation (1.1) determines a
continuous semigroup of nonlinear operators \( \{ T_s(t) : H \to H, t \geq 0 \} \), where for each
\( u_0 \in H, t \mapsto T_s(t)u_0 \) is a weak global solution of (1.1) beginning at \( u_0 \). This
semigroup is such that \( \mathbb{R}^+ \times H \ni (t, u_0) \mapsto T_s(t)u_0 \in H \) is a continuous map
and, if \( u_0 \in \mathcal{D}(A) \), then \( u_s(\cdot) := T_s(\cdot)u_0 \) is a Lipschitz continuous strong solution
of (1.1)

Considering \( h \equiv 0 \) and \( f : \mathbb{R} \to \mathbb{R} \) Lipschitz in [7] we get \( F = B : H \to H \)
globally Lipschitz. So, by Theorem 3.3 in [7] we have that problem (1.1) has a
global attractor \( \mathcal{A}_s \).

In order to prove the continuity of the flows (in Section 4) for problem (1.1) we
need the following result:

**Theorem 2.2.** If \( p \in C^1(\Omega) \), then \( C_0^\infty(\Omega) \subseteq \mathcal{D}(A) \).

**Proof:** If \( p \in C^1(\Omega) \) and \( u \in C_0^\infty(\Omega) \), then by Theorem 2.6 in [12] we have that
\( -\text{div}(x|p(x)^{-2}\nabla u) \in L^2(\Omega) \). The result follows observing that \( |u|^{p(x)-2}u \in L^2(\Omega) \)
if \( u \in C_0^\infty(\Omega) \).

### 3. Uniform estimates

Recall that we are considering \( p_s(\cdot) \in C^1(\bar{\Omega}) \) such that \( 2 < p \leq p^- \leq p^+ \leq a \),
for all \( s \in \mathbb{N} \), and \( p_s(\cdot) \to p \) in \( L^\infty(\Omega) \) as \( s \to \infty \). From now on, we denote
\( X_s := W^{1,p_s(\cdot)}(\Omega) \) and \( X := W^{1,p(\cdot)}(\Omega) \). It is a known result that \( X_s \subset H \) with
continuous and dense embedding (see [9]). Moreover,

**Lemma 3.1.** There exists a constant \( K = K(|\Omega|) > 0 \), independent of \( s \), such that
if \( u_s \in X_s, s \in \mathbb{N} \), then

\[
\|u_s\|_H \leq K\|u_s\|_{X_s}, \quad \forall s \in \mathbb{N}.
\]

**Proof:** We know that if \( p(x) > q(x) \) then \( L^{p(x)}(\Omega) \subset L^{q(x)}(\Omega) \) with \( \|u\|_{q(x)} \leq 2(|\Omega| + 1)\|u\|_{p(x)} \) for all \( u \in L^{p(x)}(\Omega) \) (see [6]). Thus, if \( u_s \in X_s \subset X \subset H \) we have

\[
\|u_s\|_H \leq 2(|\Omega| + 1)\|u_s\|_p \\
\leq 4(|\Omega| + 1)^2\|u_s\|_{p_s(x)} \\
\leq 4(|\Omega| + 1)^2(\|u_s\|_{p_s(x)} + \|\nabla u_s\|_{p_s(x)}) = K\|u_s\|_{X_s},
\]

where \( K = K(|\Omega|) := 4(|\Omega| + 1)^2 \).

We have the following uniform estimates on the solutions of (1.1):

**Lemma 3.2.** Let \( u_s \) be a solution of (1.1) with \( u_s(0) = u_0 \in H \). Given \( T_0 > 0 \),
there exists a positive number \( r_0 \) such that \( \|u_s(t)\|_H \leq r_0 \), for each \( t \geq T_0 \) and
\( s \in \mathbb{N} \). Furthermore, given a bounded set \( B \subset H \), there exists \( D_1 > 0 \) such that
\( \|u_s(t)\|_H \leq D_1 \) for all \( t \geq 0 \) and \( s \in \mathbb{N} \) such that \( u_0 \in B \).
Proof: It is enough to consider \( u_{0s} \in \mathcal{D}(A) \). Let \( \tau > 0 \), multiplying the equation in (1.1) by \( u_s(\tau) \) we have

\[
\left\langle \frac{d}{dt}u_s(\tau), u_s(\tau) \right\rangle - \langle \Delta_{p_s(x)}(u_s(\tau)) + \left| u_s(\tau) \right|^{p_s(x) - 2}u_s(\tau), u_s(\tau) \rangle = \langle B(u_s(\tau)), u_s(\tau) \rangle.
\]

Given \( T_0 > 0 \), if \( \| u_s(\tau) \|_{p_s(x)} \geq 1 \) and \( \| \nabla u_s(\tau) \|_{p_s(x)} \geq 1 \) then by Lemma 2.1, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(\tau) \|^2_H \leq - \frac{1}{2^{p_s-1}} \| u_s(\tau) \|_{p_s}^2 + \| B(u_s(\tau)) \|_H \| u_s(\tau) \|_H \\
\leq - \frac{1}{2^{p_s-1}} \| u_s(\tau) \|_{p_s}^2 + L \| u_s(\tau) \|_{\dot{H}^1}^2 + C_0 \| u_s(\tau) \|_H \\
\leq - \frac{1}{2^{p_s-1}} \| u_s(\tau) \|_{X_\alpha}^p + C_1 \| u_s(\tau) \|_{X_\alpha}^2 + C_2 \| u_s(\tau) \|_{X_\alpha},
\]

where \( C_0 = \| B(0) \|_H \geq 0 \), \( C_1 = LK^2 \) and \( C_2 = C_0K \), with \( K \) the constant independent of \( s \) of Lemma 3.1. We have \( C_2 = 0 \) if, and only if, \( C_0 = 0 \).

Now, we consider \( \epsilon > 0 \) arbitrary, \( \alpha := \frac{p}{2}, \frac{1}{\alpha} + \frac{1}{\alpha'} = 1 \) and \( \frac{1}{p} + \frac{1}{p'} = 1 \). Then using Young’s inequality we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(\tau) \|^2_H \leq - \frac{1}{2^{p_s-1}} \left( \frac{\alpha}{\alpha'} + \frac{1}{p} \right) \| u_s(\tau) \|_{p_s}^2 + \left( \frac{1}{\alpha'}C_1 + \frac{1}{p'}C_2 \right) \| u_s(\tau) \|_{X_\alpha}^2.
\]

Now, choose \( \epsilon_0 > 0 \) sufficiently small so that \( \frac{1}{\alpha} \epsilon_0^\alpha + \frac{1}{p} \epsilon_0^p < \frac{1}{2^{p_s-1}} \) in the case \( B(0) \neq 0 \) (\( C_0 \neq 0 \)) and for the case \( B(0) = 0 \), choose \( \epsilon_0 > 0 \) sufficiently small so that \( \frac{1}{\alpha} \epsilon_0^\alpha < \frac{1}{2^{p_s-1}} \). So, in both cases, we obtain

\[
\frac{1}{2} \frac{d}{dt} \| u_s(\tau) \|^2_H \leq - \frac{1}{2^{p_s-1}} \| u_s(\tau) \|_{p_s}^2 + C_3,
\]

where \( C_3 = C_3(L, K, \epsilon_0) > 0 \) is a constant. So,

\[
\frac{1}{2} \frac{d}{dt} \| u_s(\tau) \|^2_H \leq - \frac{1}{2^{p_s-1}} K^{-p} \| u_s(\tau) \|_{H}^p + C_3.
\]

Let \( I_s := \{ \tau \in (0, \infty); \| u_s(\tau) \|_{p_s(x)} \geq 1 \) and \( \| \nabla u_s(\tau) \|_{p_s(x)} \geq 1 \} \) and \( y_s : I_s \to \mathbb{R}, y_s(\tau) := \| u_s(\tau) \|^2_H \) satisfies the differential inequality

\[
y'_s(\tau) \leq - \frac{K^{-p}}{2^{p_s-1}} [y_s(\tau)]^2 + 2C_3.
\]

Therefore, from Lemma 5.1, p. 163 in [15], we get

\[
\| u_s(\tau) \|^2_H \leq \left( 2^p C_3 K^p \right)^{2/p} + \left[ 1 \cdot 2^{p-2} K^p (p-2)T_0 \right]^{1/p} := K_1, \forall \tau \geq T_0.
\]
Similarly for each of the cases: \(|u_s(\tau)|_{p_s(\tau)} \geq 1 \) and \(|\nabla u_s(\tau)|_{p_s(\tau)} \leq 1\); \(|u_s(\tau)|_{p_s(\tau)} \leq 1 \) and \(|\nabla u_s(\tau)|_{p_s(\tau)} \geq 1\); \(|u_s(\tau)|_{p_s(\tau)} \leq 1 \) and \(|\nabla u_s(\tau)|_{p_s(\tau)} \leq 1\), we obtain constants, \(K_2, K_3 \) and \(K_4\) such that

\[
|u_s(\tau)|_H^2 \leq K_i, \forall \tau \geq T_0,
\]

for \(i = 2, 3, 4\) respectively. So, taking \(r_0 := \max\{K_1^{1/2}, K_2^{1/2}, K_3^{1/2}, K_4^{1/2}\}\) we obtain

\[
|u_s(\tau)|_H \leq r_0, \forall \tau \geq T_0, \ s \in \mathbb{N},
\]

and the first part of the lemma is proved.

The second part of the lemma follows from the Gronwall-Bellman Lemma. \(\square\)

**Remark 3.3.** The constants \(r_0\) and \(D_1\) in the Lemma 3.2 depend neither on the initial data nor on \(s\).

**Corollary 3.4.** There exists a bounded set \(B_0\) in \(H\) such that \(A_s \subset B_0\) for all \(s \in \mathbb{N}\).

**Lemma 3.5.** Let \(u_s\) be a solution of (1.1). Given \(T_1 > 0\), there exists a positive constant \(r_1 > 0\), independent of \(s\), such that

\[
|u_s(t)|_{X_s} < r_1,
\]

for every \(t \geq T_1\) and \(s \in \mathbb{N}\).

**Proof:** Let \(u_s\) be a solution of (1.1) and consider \(T_1 > 0\). Take \(T_0 \in (0, T_1)\). Considering \(\varphi_{p_s(\tau)}\) as in (2.1), using the definition of subdifferential and Uniform Gronwall Lemma (see [15]), we obtain

\[
\varphi_{p_s(\tau)}(u_s(\tau)) \leq \tilde{r}_1,
\]

for all \(\tau \geq T_1\) and \(s \in \mathbb{N}\), where \(\tilde{r}_1 = \tilde{r}_1(T_1, T_0, L, r_0)\), with \(r_0\) as in Lemma 3.2. Therefore

\[
\int_{\Omega} \frac{1}{p_s(x)} |\nabla u_s(\tau,x)|^{p_s(\tau)} \, dx + \int_{\Omega} \frac{1}{p_s(x)} |u_s(\tau,x)|^{p_s(\tau)} \, dx \leq \tilde{r}_1,
\]

for all \(\tau \geq T_1\) and \(s \in \mathbb{N}\). So, considering \(\rho_s(v) := \int_{\Omega} |v(x)|^{p_s(\tau)} \, dx\), we have

\[
\rho_s(\nabla u_s(\tau)) + \rho_s(u_s(\tau)) \leq a\tilde{r}_1,
\]

for all \(\tau \geq T_1\) and \(s \in \mathbb{N}\). If \(\tau \geq T_1\) and \(|u_s(\tau)|_{X_s} > 1\) then we have four cases to analyze:

Case 1: If \(|\nabla u_s(\tau)|_{p_s(\tau)} \geq 1\) and \(|u_s(\tau)|_{p_s(\tau)} \geq 1\) then we know that

\[
|\nabla u_s(\tau)|_{p_s(\tau)} \leq \rho_s(\nabla u_s(\tau)) \leq |\nabla u_s(\tau)|_{p_s(\tau)}.
\]
and
\[ \|u_s(\tau)\|_{p_s(\tau)}^{p_s^-} \leq \rho_s(u_s(\tau)) \leq \|u_s(\tau)\|_{p_s(\tau)}^{p_s^+}. \]

Since \( p \leq p_s^- \leq p_s^+ \leq a \), we obtain by (3.1)
\[ \|u_s(\tau)\|_{X_s} \leq (ar_1)^{1/p}. \]

Case 2: If \( \|\nabla u_s(\tau)\|_{p_s(\tau)} \geq 1 \) and \( \|u_s(\tau)\|_{p_s(\tau)} \leq 1 \) then we know that
\[ \|\nabla u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(\nabla u_s(\tau)) \leq \|\nabla u_s(\tau)\|_{p_s(\tau)}, \]
and
\[ \|u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(u_s(\tau)) \leq \|u_s(\tau)\|_{p_s(\tau)}. \]

Using (3.1) we obtain in this case
\[ \|u_s(\tau)\|_{X_s} \leq (ar_1)^{1/p} + (ar_1)^{1/a}. \]

Case 3: If \( \|\nabla u_s(\tau)\|_{p_s(\tau)} \leq 1 \) and \( \|u_s(\tau)\|_{p_s(\tau)} \geq 1 \) then we know that
\[ \|\nabla u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(\nabla u_s(\tau)) \leq \|\nabla u_s(\tau)\|_{p_s(\tau)}, \]
and
\[ \|u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(u_s(\tau)) \leq \|u_s(\tau)\|_{p_s(\tau)}. \]

Then, by (3.1) we have that
\[ \|u_s(\tau)\|_{X_s} \leq (ar_1)^{1/a} + (ar_1)^{1/p}. \]

Case 4: If \( \|\nabla u_s(\tau)\|_{p_s(\tau)} \leq 1 \) and \( \|u_s(\tau)\|_{p_s(\tau)} \leq 1 \), then
\[ \|\nabla u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(\nabla u_s(\tau)) \leq \|\nabla u_s(\tau)\|_{p_s(\tau)}, \]
and
\[ \|u_s(\tau)\|_{p_s(\tau)} \leq \rho_s(u_s(\tau)) \leq \|u_s(\tau)\|_{p_s(\tau)}. \]

Using (3.1), we obtain
\[ \|u_s(\tau)\|_{X_s} \leq (ar_1)^{1/a}. \]

So considering \( r_1 := \max\{1, (ar_1)^{1/p} + (ar_1)^{1/a}\} \) we conclude that
\[ \|u_s(\tau)\|_{X_s} \leq r_1 \] for all \( \tau \geq T_1 \) and \( s \in \mathbb{N} \).

\( \square \)

**Corollary 3.6.** a) There exists a bounded set \( B^*_1 \) in \( X_s \) such that \( A_s \subset B^*_1 \).

b) Let \( u_s \) be a solution of problem (1.1). Given \( T_1 > 0 \) there exists a positive constant \( r_2 \), independent of \( s \), such that
\[ \|u_s(t)\|_X < r_2 \]
for all \( t \geq T_1 \) and \( s \in \mathbb{N} \).

c) \( A := \bigcup_{s \in \mathbb{N}} A_s \) is a compact subset of \( H \).
Proof: a) It follows from Lemma 3.5.

b) By Lemma 3.5 there exists \( r_1 > 0 \) such that

\[
\|u_s(t)\|_{X_s} < r_1 \quad \forall \ t \geq T_1, \ s \in \mathbb{N}.
\]

Thus

\[
\|u_s(t)\|_X = \|\nabla u_s(t)\|_p + \|u_s(t)\|_p \leq 2(|\Omega| + 1) (\|\nabla u_s(t)\|_{p_s(x)} + \|u_s(t)\|_{p_s(x)})
\]

\[
= 2(|\Omega| + 1)\|u_s(t)\|_{X_s} \leq 2(|\Omega| + 1)r_1
\]

for all \( t \geq T_1 \) and \( s \in \mathbb{N} \) and the result follows with \( r_2 := 2(|\Omega| + 1)r_1 \).

c) By b) there exists a bounded set \( B_1 \) in \( X \) such that \( A_s \subset B_1 \) for all \( s \in \mathbb{N} \).
Since \( X \subset H \) with continuous and compact embedding, the result is proved. \( \square \)

**Proposition 3.1.** Let \( u_s \) be a solution of (1.1) with initial value \( u_{0s} \). If there is \( C > 0 \) such that \( \|u_{0s}\|_{X_s} \leq C \) for all \( s \in \mathbb{N} \), then given \( T_1 > 0 \) there exists a positive constant \( R_1 \) such that

\[
\|u_s(t)\|_{X_s} \leq R_1, \quad \text{for all} \ t \in [0, T_1] \text{ and } s \in \mathbb{N}.
\]

Proof: Given \( T_1 > 0 \), if \( u_s \) is a solution of (1.1) then using the identity

\[
\frac{d}{dt} \varphi_{p_s(x)}(u_s(t)) = \langle \partial \varphi_{p_s(x)}(u_s(t)), \frac{\partial u_s}{\partial t}(t) \rangle
\]

and Lemma 3.2, we obtain

\[
\varphi_{p_s(x)}(u_s(\tau)) \leq \varphi_{p_s(x)}(u_{0s}) + C_1 T_1, \quad \text{for all } \tau \in [0, T_1], \ s \in \mathbb{N},
\]

where \( C_1 > 0 \) is a constant. Now, as \( \|u_{0s}\|_{X_s} \leq C \) for all \( s \in \mathbb{N} \) we obtain that

\[
\varphi_{p_s(x)}(u_{0s}) \leq \tilde{C} \text{ for all } s \in \mathbb{N}.
\]

So, the result follows as in the proof of Lemma 3.5. \( \square \)

**Corollary 3.7.** Let \( u_s \) be a solution of (1.1) with initial value \( u_{0s} \). If there is \( C > 0 \) such that \( \|u_{0s}\|_{X_s} \leq C \) for all \( s \in \mathbb{N} \), then given \( T_1 > 0 \) there exists a positive constant \( R_1 \) such that

\[
\|u_s(t)\|_X \leq \tilde{R}_1,
\]

for all \( t \in [0, T_1] \) and \( s \in \mathbb{N} \).

Proof: Since \( \|u_s(\tau)\|_X \leq 2(|\Omega| + 1)\|u_s(\tau)\|_{X_s} \) for all \( s \in \mathbb{N} \), the result follows from Proposition 3.1. \( \square \)
4. Continuity with respect to the initial values and upper semicontinuity of attractors

In this section we prove that, given \( T > 0 \), the solutions \( u_s \) of (1.1) go to the solution \( u \) of

\[
\begin{align*}
\frac{du}{dt} - \text{div}(|\nabla u(t)|^{p-2}\nabla u(t)) + |u|^{p-2}u &= B(u(t)), \quad t > 0, \\
u(0) &= u_0,
\end{align*}
\]

in \( C([0,T];H) \) and, after that, we obtain the upper semicontinuity on \( s \) in \( H \) of the family of global attractors \( \{A_s \subset H; s \in \mathbb{N}\} \) of (1.1) at \( p \).

**Lemma 4.1.** Given \( T > 0, M := \{u_s : s \in \mathbb{N}, u_s \text{ is a solution of (1.1) with } u_s(0) = u_{0s} \text{ and } u_{0s} \to u_0 \text{ in } H, \text{ as } s \to +\infty \} \) is compactly relatively in \( C([0,T];H) \).

**Proof:** We observe that it holds:

i) For each \( s \in \mathbb{N} \) the function \( [0,T] \ni t \mapsto B(u_s(t)) \in H \) is in \( L^1(0,T;H) \). Moreover, \( \{B(u_s(t))\}_{s \in \mathbb{N}} \) is uniformly bounded in \( L^1(0,T;H) \) and consequently uniformly integrable in \( L^1(0,T;H) \).

Indeed, as \( \int_0^T \|B(u_s(t))\|_H dt \leq \int_0^T (L\|u_s(t)\|_H + \|B(0)\|_H) dt \) the result follows from Lemma 3.2.

ii) The operator \( A^s, A^s u := -\Delta_{p_s(x)} u + |u|^{p_s(x)-2}u, \) is a maximal monotone operator in \( H, A^s u = \partial \varphi_{p_s(x)}(u) \) is the subdifferential of the convex, proper and lower semi continuous non negative map \( \varphi_{p_s(x)} \) and \( \cap_s D(\varphi_{p_s(x)}) = H \) since \( X_a \subset X_s \subset X, \) for all \( s \).

iii) For each \( u \in \cap_s D(\varphi_{p_s(x)}) \) there exists a constant \( k = k(u,a,\Omega) > 0 \) such that \( \varphi_{p_s(x)}(u) \leq k, \forall s \in \mathbb{N} \).

In fact, if \( u \in \cap_s D(\varphi_{p_s(x)}) = \cap_s X_s \) then for all \( s \)

\[
\varphi_{p_s(x)}(u) \leq \begin{cases}
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\end{cases}
\]

\[
\leq \begin{cases}
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\end{cases}
\]

\[
\leq \begin{cases}
\frac{1}{2} (\Omega(\Omega + 1)^p (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \leq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \leq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\frac{1}{2} (\|\nabla u\|_{p_s(x)}^p + \|u\|_{p_s(x)}^p), & \text{if } \|\nabla u\|_{p_s(x)} \geq 1 \text{ and } \|u\|_{p_s(x)} \geq 1 \\
\end{cases}
\]
So \( \varphi_{p,(x)}(u) \leq k \) for all \( s \in \mathbb{N} \), where \( k \) is the maximum between the values
\[
2^{-1}(\Omega + 1)^{p}\left(\|\nabla u\|_{p}^{p} + \|u\|_{p}^{p}\right), 2^{a-1}(\Omega + 1)^{a}\|\nabla u\|_{p}^{a} + 2^{b-1}(\Omega + 1)^{b}\|u\|_{p}^{b}, 2^{a-1}(\Omega + 1)^{a}\|\nabla u\|_{p}^{a} + 2^{b-1}(\Omega + 1)^{b}\|u\|_{p}^{b},
\]
iv) Let \( M(t) := \{u_{s}(t); u_{s} \in M\} \) and let \( \{S^{s}(t)\} \) be the semigroup generated by \( A^{s} \) in \( H \). For each \( t \in (0, T] \) and \( h > 0 \) such that \( t - h \in (0, T] \), the operator \( T_{h} : M(t) \to H \) defined by \( T_{h}u_{s}(t) = S^{s}(h)u_{s}(t - h) \) is compact. Moreover, \( M(0) \) is relatively compact in \( H \) once \( u_{0a} \to u_{0} \) in \( H \).

Thus, by Theorem 3.2 in \[8\], \( M \) is relatively compact in \( C([0, T]; H) \). \( \square \)

**Theorem 4.2.** For each \( s \in \mathbb{N} \) let \( u_{s} \) be a solution of (1.1) with \( u_{s}(0) = u_{0a} \). Suppose that there exists \( C > 0 \), independent of \( s \), such that \( \|u_{0a}\|_{X_{s}} \leq C \) for all \( s \in \mathbb{N} \) and \( u_{0a} \to u_{0} \) in \( H \) as \( s \to \infty \). Then, for each \( T > 0 \), \( u_{s} \to u \) in \( C([0, T]; H) \) as \( s \to \infty \) where \( u \) is a solution of (4.1) with \( u(0) = u_{0} \).

**Proof:** By Lemma 4.1 \( M \) is relatively compact in \( C([0, T]; H) \). So, \( \{u_{s}\} \) converges in \( C([0, T]; H) \) to a function \( u : [0, T] \to H \). Proposition 3.6 in \[3\] implies that
\[
\frac{1}{2}\|u_{s}(t) - \phi\|_{H}^{2} \leq \frac{1}{2}\|u_{s}(\tau) - \phi\|_{H}^{2} + \int_{\tau}^{t}\langle B(u_{s}(t')) + \Delta_{p,(x)}(\phi) - |\phi|^{p,(x)-2}\phi, u_{s}(t') - \phi\rangle dt'
\]
for every \( \phi \in D(A^{s}) \) and \( 0 \leq \tau \leq t \leq T \).

Now, the idea is to take the limit as \( s \to \infty \) (\( p_{a} \to p \)) on the last inequality. Since \( u_{s} \to u \) in \( C([0, T]; H) \) and \( B \) is globally Lipschitz, we have that \( u_{s} \to u \) and \( B \circ u_{s} \to B \circ u \) in \( C([\tau, t]; H) \) and, consequently \( u_{s} \to u \) and \( B \circ u_{s} \to B \circ u \) in \( L^{2}(\tau, t; H) \), \( \forall 0 \leq \tau \leq t \leq T \). Then,
\[
\langle B \circ u_{s} - h, u_{s} - \theta \rangle_{L^{2}(\tau, t; H)} \to \langle B \circ u - h, u - \theta \rangle_{L^{2}(\tau, t; H)}
\]
for all \( \theta, h \in H \).

Now consider \( \overline{\theta} \in C_{0}^{\infty}(\Omega) \subset D(A^{s}) \subset H \) arbitrarily fixed and let \( \overline{\theta} := -\Delta_{p}(\overline{\theta}) + |\overline{\theta}|^{p-2}\overline{\theta} \in H \). From (4.2)
\[
\frac{1}{2}\|u_{s}(t) - \overline{\theta}\|_{H}^{2} \leq \frac{1}{2}\|u_{s}(\tau) - \overline{\theta}\|_{H}^{2} + \int_{\tau}^{t}\langle B(u_{s}(t')) + \Delta_{p,(x)}(\overline{\theta}) - |\overline{\theta}|^{p,(x)-2}\overline{\theta}, u_{s}(t') - \overline{\theta}\rangle dt'
\]
\[
= \frac{1}{2}\|u_{s}(\tau) - \overline{\theta}\|_{H}^{2} + \int_{\tau}^{t}\langle B(u_{s}(t')) - \overline{\theta}, u_{s}(t') - \overline{\theta}\rangle dt' + \int_{\tau}^{t}\langle \overline{\theta} + \Delta_{p,(x)}(\overline{\theta}) - |\overline{\theta}|^{p,(x)-2}\overline{\theta}, u_{s}(t') - \overline{\theta}\rangle dt'.
\]
We claim that \( \int_{\tau}^{t}\langle \overline{\theta} + \Delta_{p,(x)}(\overline{\theta}) - |\overline{\theta}|^{p,(x)-2}\overline{\theta}, u_{s}(t') - \overline{\theta}\rangle dt' \to 0 \) as \( s \to +\infty \). In
fact, for each $t' > 0$

$$
|\bar{h} + \Delta_{p_s(x)}(\bar{\theta}) - |\bar{\theta}|^{p_s(x)-2}\bar{\theta}, u_s(t') - \bar{\theta}|\\
= |\bar{h}, u_s(t') - \bar{\theta} - (-\Delta_{p_s(x)}(\bar{\theta}) + |\bar{\theta}|^{p_s(x)-2}\bar{\theta}, u_s(t') - \bar{\theta})|\\
\leq \int_{\Omega} \left( |\nabla|^{p-1} - |\nabla|^{p_s(x)-1} \right) |\nabla u_s(t')| dx + \int_{\Omega} \left( |\nabla|^{p} - |\nabla|^{p_s(x)} \right) dx\\
+ \int_{\Omega} \left( |\bar{\theta}|^{p-1} - |\bar{\theta}|^{p_s(x)-1} \right) |u_s(t')| dx + \int_{\Omega} \left( |\bar{\theta}|^{p} - |\bar{\theta}|^{p_s(x)} \right) dx.
$$

Since $p_s(x) \to p$ for all $x \in I$ it follows by Dominated Convergence Theorem that

$$
\int_{\Omega} \left( |\nabla|^{p} - |\nabla|^{p_s(x)} \right) dx \to 0 \text{ as } s \to \infty,
$$

and

$$
\int_{\Omega} \left( |\bar{\theta}|^{p} - |\bar{\theta}|^{p_s(x)} \right) dx \to 0 \text{ as } s \to \infty.
$$

On the other hand, considering $\tilde{\Omega} := \{ x \in \Omega : \bar{\theta}(x) \neq 0 \}$, $\tilde{\Omega}_1 := \{ x \in \tilde{\Omega} : |\bar{\theta}(x)| \leq 1 \}$, $\tilde{\Omega}_2 := \{ x \in \tilde{\Omega} : |\bar{\theta}(x)| > 1 \}$, and using the Mean Value Theorem we obtain

$$
\int_{\tilde{\Omega}} \left( |\bar{\theta}|^{p-1} - |\bar{\theta}|^{p_s(x)-1} \right) |u_s(t')| dx = \int_{\tilde{\Omega}} \left( |\bar{\theta}|^{p-1} - |\bar{\theta}|^{p_s(x)-1} \right) |u_s(t')| dx\\
\leq \int_{\tilde{\Omega}} \left| \frac{\tau(s,x)}{\bar{\theta}} \right| (p_s(x) - p) |u_s(t')| dx\\
\leq \int_{\tilde{\Omega}_1} \left| p_s(x) - p \right| |u_s(t')| dx\\
+ \int_{\tilde{\Omega}_2} \left| p_s(x) - p \right| |u_s(t')| dx
$$

where $p - 1 < \tau(s,x) < p_s(x) - 1 \leq a - 1$. As $\bar{\theta} \in C^\infty_0(\Omega)$ there exist $K_{\bar{\theta}} > 0$ such that $|\bar{\theta}(x)| \leq K_{\bar{\theta}}$ for all $x \in \Omega$. So by the continuity of the functions $g_\alpha : [0, K_{\bar{\theta}}] \to \mathbb{R}$ given by

$$
g_\alpha(w) = \begin{cases} \quad \alpha \ln w & \text{if } w \in (0, K_{\bar{\theta}}] \\ \\ 0 & \text{if } w = 0,
\end{cases}
$$

for $\alpha = p - 1, a - 1$, we conclude that

$$
\int_{\tilde{\Omega}} \left( |\bar{\theta}|^{p-1} - |\bar{\theta}|^{p_s(x)-1} \right) |u_s(t')| dx \leq \|p_s - p\| \int_{\Omega} C |u_s(t')| dx\\
\leq \|p_s - p\| \left[ \int_{\tilde{\Omega}} \frac{1}{q_s(x)} C_{q_s(x)} dx + \int_{\Omega} \frac{1}{p_s(x)} |u_s(t')|^{p_s(x)} dx \right]\\
\leq \frac{1}{2} \int_{\Omega} C_{q_s(x)} dx + \frac{1}{2} \int_{\Omega} |u_s(t')|^{p_s(x)} dx
$$
where \( q_s(\cdot) \) is such that \( \frac{1}{p_\ast(x)} + \frac{1}{q_s(x)} = 1, \forall x \in \Omega \). By Proposition 3.1 there exists a constant \( C > 0 \) such that \( \int_\Omega |u_s(t')|^{p_s(x)} \, dx \leq C \) for every \( t' \in (\tau, t) \) and \( s \in \mathbb{N} \).

As \( 1 < q_s(x) < 2 \) we obtain that
\[
\int_\Omega \left( |\nabla U|^{p_s(x)-1} - |\nabla U|^{p_\ast(x)-1} \right) |u_s(t')| \, dx \leq \|u_s - p\|_\infty C' \to 0
\]
as \( s \to \infty \). Using the same arguments as above it follows that
\[
\int_\Omega \left( |\nabla U|^{p_s(x)-1} - |\nabla U|^{p_\ast(x)-1} \right) |\nabla u_s(t')| \, dx \to 0 \quad \text{as} \quad s \to \infty.
\]

Thus
\[
\int_\tau^t (\nabla + \Delta p_s(x) \nabla U) - |\nabla U|^{p_s(x)-2} \nabla U, u_s(t') - \nabla U) \, dt' \to 0 \quad \text{as} \quad s \to +\infty.
\]

So, taking the limit in (4.3) as \( s \to \infty \), we obtain
\[
\frac{1}{2} \|u(t) - \nabla \|^2_H \leq \frac{1}{2} \|u(\tau) - \nabla \|^2_H + \int_\tau^t \langle B(u(t')) + \Delta p(\nabla U) - |\nabla U|^{p_s(x)-2} \nabla U, u(t') - \nabla U \rangle \, dt'
\]
for every \( \nabla \in C_0^\infty(\Omega) \) and \( 0 \leq \tau \leq t \leq T \).

Now, we use a density argument to conclude that \( u \) is a solution of (4.1). Let \( \nabla \in \mathcal{D}(A^p) \subset W^{1,p}(\Omega) \), \( A^p u := -\Delta p u + |u|^{p-2} u \). So, there exists a sequence \( \{\nabla_j\}_{j \in \mathbb{N}} \subset C_0^\infty(\Omega) \) such that \( \|\nabla_j - \nabla\|_{W^{1,p}(\Omega)} \to 0 \) as \( j \to \infty \) and consequently \( \|\nabla_j - \nabla\|_H \to 0 \) as \( j \to \infty \). By (4.4),
\[
\frac{1}{2} \|u(t) - \nabla_j\|_H^2 \leq \frac{1}{2} \|u(\tau) - \nabla_j\|_H^2 + \int_\tau^t \langle B(u(t')) + \Delta p(\nabla_j) - |\nabla_j|^{p_s(x)-2} \nabla_j, u(t') - \nabla_j \rangle \, dt'
\]
for every \( j \in \mathbb{N} \) and \( 0 \leq \tau \leq t \leq T \). Obviously, \( \frac{1}{2} \|u(t) - \nabla_j\|_H^2 \to \frac{1}{2} \|u(t) - \nabla\|^2_H \) as \( j \to \infty \) and \( \frac{1}{2} \|u(\tau) - \nabla_j\|_H^2 \to \frac{1}{2} \|u(\tau) - \nabla\|_H^2 \) as \( j \to \infty \). With some computations and using the Dominated Convergence Theorem we obtain
\[
\langle B(u(t')) + \Delta p(\nabla_j) - |\nabla_j|^{p_s(x)-2} \nabla_j, u(t') - \nabla_j \rangle \to \langle B(u(t')) + \Delta p(\nabla) - |\nabla|^{p_s(x)-2} \nabla, u(t') - \nabla \rangle
\]
as \( j \to \infty \). So, taking the limit with \( j \to \infty \), we obtain
\[
\frac{1}{2} \|u(t) - \nabla\|_H^2 \leq \frac{1}{2} \|u(\tau) - \nabla\|_H^2 + \int_\tau^t \langle B(u(t')) + \Delta p(\nabla) - |\nabla|^{p_s(x)-2} \nabla, u(t') - \nabla \rangle \, dt'
\]
for every $\theta \in D(A^p)$ and $0 \leq \tau \leq t \leq T$. Thus, Proposition 3.6 in [3] implies that $u$ is a solution of (4.1).

Thus, following the same arguments as in Theorem 6 in [11] we conclude:

**Theorem 4.3.** The family of global attractors $\{A_s; s \in \mathbb{N}\}$ associated with problem (1.1) is upper semicontinuous on $s$ at infinity, in the topology of $H$.

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