The characterization of generalized Jordan centralizers on algebras

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Abstract: In this paper, it is shown that if $A$ is a CSL subalgebra of a von Neumann algebra and $\phi$ is a continuous mapping on $A$ such that $(m + n + k + l)\phi(A^2) - (m\phi(A)A + n\phi(A)A + k\phi(I)A^2 + lA^2\phi(I)) \in F$ for any $A \in A$, where $F$ is the real field or the complex field, then $\phi$ is a centralizer. It is also shown that if $\phi$ is an additive mapping on $A$ such that $(m + n + k + l)\phi(A^2) = m\phi(A)A + n\phi(A)A + k\phi(I)A^2 + lA^2\phi(I)$ for any $A \in A$, then $\phi$ is a centralizer.

Key Words: Jordan centralizers; centralizers, CSL subalgebras of von Neumann algebras

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1. Introduction

Throughout the paper, $F$ will denote the real field or the complex field. Let $H$ be a complex Hilbert space and $\mathcal{L}$ be a subspace lattice of $H$. Denote by $\text{Alg}\mathcal{L}$ the algebra of all bounded operators in $B(H)$ which leave every subspace in $\mathcal{L}$ invariant. Dually, for a subalgebra $A$ of $B(H)$, denote by $\text{Lat}A$ the lattice of all closed subspaces left invariant under every operator in $A$. For convenience we shall disregard the distinction between a closed subspace of $H$ and the orthogonal projection onto it. A totally ordered subspace lattice is called a nest. If each pair of projections in $\mathcal{L}$ commute, then the subspace lattice $\mathcal{L}$ is called a commutative subspace lattice, or a CSL. If $\mathcal{L}$ is a CSL, whose projections are contained in a von Neumann algebra $N$ acting on the Hilbert space $H$, then $A = N \cap \text{Alg}\mathcal{L}$ is called a CSL subalgebra of the von Neumann algebra $N$.

Let $\mathcal{R}$ be a ring or an algebra and $\phi$ be an additive mapping on $\mathcal{R}$. If $\phi(AB) = \phi(A)B$ (resp. $\phi(AB) = A\phi(B)$) for any $A, B \in \mathcal{R}$, then $\phi$ is called a left centralizer (resp. a right centralizer). A centralizer of $\mathcal{R}$ is an additive mapping which is a left centralizer.

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as well as a right centralizer. An additive mapping \( \phi : \mathcal{R} \rightarrow \mathcal{R} \) is called a left (resp. right) Jordan centralizer, if \( \phi(A^2) = \phi(A)A \) (resp. \( \phi(A^2) = A\phi(A) \)) for any \( A \in \mathcal{R} \). A Jordan centralizer of \( \mathcal{R} \) is an additive mapping which is a left Jordan as well as a right Jordan centralizer. An \((m,n)-\) Jordan centralizer is defined in (\cite{16}) as follows: An additive mapping \( \phi : \mathcal{R} \rightarrow \mathcal{R} \) is called an \((m,n)-\) Jordan centralizer if \((m+n)\phi(A^2) = m\phi(A)A + nA\phi(A) \) for any \( A \in \mathcal{R} \), where \( m, n \in \mathbb{N} \) with \( m + n \neq 0 \). Obviously, every centralizer is a Jordan centralizer and any Jordan centralizer is an \((m,n)-\) Jordan centralizer, but the converse is not true in general.

The characterization of centralizers on algebras or rings is a subject in various areas. Bresar and Zalar (\cite{22}) have proved that if \( \mathcal{R} \) is a prime ring and \( \phi \) is an additive mapping on \( \mathcal{R} \) such that \( \phi(A^2) = \phi(A)A \) (resp. \( \phi(A^2) = A\phi(A) \)) for any \( A \in \mathcal{R} \), then \( \phi \) is a left (resp. a right) centralizer. Zalar (\cite{23}) generalized the result to 2-torsion free semi-prime rings as follows: if \( \mathcal{R} \) is a 2-torsion free semi-prime ring and \( \phi \) is an additive mapping on \( \mathcal{R} \) such that \( \phi(A^2) = \phi(A)A \) (resp. \( \phi(A^2) = A\phi(A) \)) for any \( A \in \mathcal{R} \), then \( \phi \) is a left (resp. a right) centralizer. Vukman (\cite{15}) has proved that if \( \mathcal{R} \) is a 2-torsion free semi-prime ring and \( \phi \) is an additive mapping on \( \mathcal{R} \) such that \( 2\phi(A^2) = \phi(A)A + A\phi(A) \) for any \( A \in \mathcal{R} \), then \( \phi \) is a centralizer. Benkovic and Eremita (\cite{1}) proved that if \( \mathcal{R} \) is a prime ring with \( Ch(\mathcal{R}) = 0 \) or \( Ch(\mathcal{R}) \geq n \), where \( n \) is a fixed positive integer and \( n \geq 2 \), and \( \phi \) is an additive mapping on \( \mathcal{R} \) such that \( \phi(A^n) = \phi(A)A^{n-1} \) for any \( A \in \mathcal{R} \), then \( \phi \) is a centralizer. Vukman and Kosi-Ubl (\cite{17}) proved that if \( X \) is a Banach space over the field \( \mathbb{F} \), and \( \mathcal{A} \) is a standard subalgebra of \( \mathcal{B}(X) \) and \( \phi : \mathcal{A} \rightarrow \mathcal{B}(X) \) is an additive mapping such that \( \phi(A^{m+n+1}) = A^m\phi(A)A^n \) for any \( A \in \mathcal{A} \), where \( m, n \in \mathbb{Z}^+ \) and then \( \phi \) is a centralizer. Qi etc. (\cite{14}) proved that if \( \mathcal{A} \) is a standard subalgebra of \( \mathcal{B}(X) \) with the identity \( I \) and \( \phi : \mathcal{A} \rightarrow \mathcal{B}(X) \) is an additive mapping such that \( \phi(A^{m+n+1}) = A^m\phi(A)A^n \in \mathbb{F}I \) for any \( A \in \mathcal{A} \), where \( X \) is a Banach space over the field \( \mathbb{F} \) and \( m, n \in \mathbb{Z}^+ \), then \( \phi \) is a centralizer. Yang and Zhang (\cite{22}) proved that, if \( \phi : \tau(\mathcal{N}) \rightarrow \tau(\mathcal{N}) \) is an additive mapping on a nest algebra \( \tau(\mathcal{N}) \), such that \((m+n)\phi(A^{p+1}) = m\phi(A)A^p + nA^p\phi(A) \) or \( \phi(A^{m+n+1}) = A^m\phi(A)A^n \) for any \( A \in \tau(\mathcal{N}) \), where \( \mathcal{N} \) is a non-trivial nest on \( \tau(\mathcal{N}) \), then \( \phi \) is a centralizer. J. Vukman (\cite{16}) proved that an \((m,n)-\) Jordan centralizer on a prime ring with \( Ch(\mathcal{R}) \neq 6mn(m+n) \) is a centralizer. Li etc. (\cite{12}) proved that a Jordan centralizer on a CSL subalgebra of a von Neumann algebra is a centralizer.

Motivated by these results, we are concerned with an additive mapping \( \phi \) on \( \mathcal{A} \), a CSL subalgebra of a von Neumann algebra, which is not a semi-prime ring. It is shown that if \( \phi \) is a continuous mapping on \( \mathcal{A} \) such that \((m+n+k+l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I \) for any \( A \in \mathcal{A} \), then \( \phi \) is a centralizer (Theorem 3.1). It is also shown that if \( \phi \) is an additive mapping on \( \mathcal{A} \) such that \((m+n+k+l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) \) for any \( A \in \mathcal{A} \), then \( \phi \) is a centralizer (Theorem 3.2). It follows that an \((m,n)-\) Jordan centralizer on \( \mathcal{A} \) is a centralizer (Corollary 3.1). Furthermore, it is shown that if \( \phi \) is an additive mapping on \( \mathcal{A} \) such that \((m+n)\phi(A^{p+1}) = m\phi(A)A^p + nA^p\phi(A) \) or \( \phi(A^{m+n+1}) = A^m\phi(A)A^n \) for any \( A \in \mathcal{A} \), then \( \phi \) is a centralizer (Theorem 3.3 and Theorem 3.4).
2. Preliminaries: some lemmas

In this section, let $\mathcal{A}$ be a unital algebra. We discuss an additive mapping $\phi$ on $\mathcal{A}$ such that

$$(m + n + k + l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in FI,$$  \hspace{1cm} (2.1)

that is, for any $A \in \mathcal{A}$, there is $\mu_A \in F$ (depending on $A$) such that

$$(m + n + k + l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) + \mu_A I,$$

where $m > 0, n > 0, k \geq 0, l \geq 0$.

Lemma 2.1. Suppose that $\phi$ is an additive mapping on $\mathcal{A}$ as above. Then, for any $A, B \in \mathcal{A}$,

$$(1) \hspace{1cm} (m + n + k + l)\phi(AB + BA) = m\phi(A)B + nA\phi(B) + m\phi(B)A$$

$$+ k\phi(I)AB + k\phi(I)BA + lAB\phi(I) + lBA\phi(I)$$

$$+ (\mu_{A+B} - \mu_A - \mu_B)I;$$

$$(2) \hspace{1cm} (m + n + 2k + 2l)\phi(A) = (m + 2k)\phi(I)A + (n + 2l)A\phi(I) + (\mu_{A+I} - \mu_A)I.$$  \hspace{1cm} (2.2)

Proof: For any $A, B \in \mathcal{A}$,

$$(m + n + k + l)\phi(A + B)^2 = m\phi(A + B)(A + B) + n(A + B)\phi(A + B)$$

$$+ k\phi(I)(A + B)^2 + l(A + B)^2\phi(I) + \mu_{A+B}I$$

$$= m\phi(A)A + m\phi(A)B + m\phi(B)A + m\phi(B)B$$

$$+ nA\phi(A) + nA\phi(B) + nB\phi(A) + nB\phi(B)$$

$$+ k\phi(I)A^2 + k\phi(I)BA + k\phi(I)AB + k\phi(I)B^2$$

$$+ lA^2\phi(I) + lAB\phi(I) + lBA\phi(I) + lB^2\phi(I) + \mu_{A+B}I.$$

On the other hand,

$$(m + n + k + l)\phi(A + B)^2 = (m + n + k + l)\phi(A^2 + AB + BA + B^2)$$

$$= m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I) + nB\phi(B)$$

$$+ m\phi(B)B + k\phi(I)B^2 + lB^2\phi(I)$$

$$+ (m + n + k + l)\phi(AB + BA) + \mu_A I + \mu_B I.$$

Comparing above two equalities, we obtain that

$$(m + n + k + l)\phi(AB + BA) = m\phi(A)B + nA\phi(B) + m\phi(B)A + nB\phi(A)$$

$$+ k\phi(I)AB + k\phi(I)BA + lAB\phi(I)$$

$$+ lBA\phi(I) + (\mu_{A+B} - \mu_A - \mu_B)I.$$

\hspace{1cm} (2.2)
Putting in (2.2) \( B = I \), it follows from \( \mu_I = 0 \) that
\[
(m + n + 2k + 2l)\phi(A) = (m + 2k)\phi(I)A + (n + 2l)A\phi(I) + (\mu_{A,I} - \mu_A)I. \tag{2.3}
\]

\[ \square \]

**Lemma 2.2.** Let \( \phi \) be an additive mapping on \( A \) as above. If \( A_0 \in A \) with \( A_0 \notin F I \) such that \( A_0\phi(I) = \phi(I)A_0 \), then \( \mu_{A_0 + I} - \mu_{A_0} = 0 \) and \( \phi(A_0) = A_0\phi(I) = \phi(I)A_0 \).

**Proof:** Since \( A_0\phi(I) = \phi(I)A_0 \), \( A_0^2\phi(I) = \phi(I)A_0^2 = A_0\phi(I)A_0 \). By (2.3), we have that
\[
\phi(A_0) = \phi(I)A_0 + \frac{1}{m + n + 2k + 2l}((\mu_{A_0 + I} - \mu_{A_0})I)
\]
and
\[
\phi(A_0^2) = \phi(I)A_0^2 + \frac{1}{m + n + 2k + 2l}((\mu_{A_0^2 + I} - \mu_{A_0^2})I).
\]

Hence
\[
(m + n + k + l)\phi(A_0^2) = (m + n + k + l)\phi(I)A_0^2 + \frac{m + n + k + l}{m + n + 2k + 2l}((\mu_{A_0^2 + I} - \mu_{A_0^2})I).
\]

On the other hand,
\[
(m + n + k + l)\phi(A_0^2) = m\phi(A_0)A_0 + nA_0\phi(A_0) + k\phi(I)A_0^2 + lA_0^2\phi(I) + \mu_{A_0}I
\]
\[
= m(\phi(I)A_0 + \frac{1}{m + n + 2k + 2l}((\mu_{A_0 + I} - \mu_{A_0})I)A_0
\]
\[
+ nA_0(\phi(I)A_0 + \frac{1}{m + n + 2k + 2l}((\mu_{A_0 + I} - \mu_{A_0})I)
\]
\[
+ k\phi(I)A_0^2 + lA_0^2\phi(I) + \mu_{A_0}I.
\]

Comparing the two equalities, we have that \( \frac{m+n}{m+n+2k+2l}(\mu_{A_0 + I} - \mu_{A_0})A_0 \in F I \). Since \( A_0 \notin FI \) and \( m + n > 0 \), \( \mu_{A_0 + I} - \mu_{A_0} = 0 \) and \( \phi(A_0) = A_0\phi(I) = \phi(I)A_0 \). \( \square \)

**Lemma 2.3.** Let \( \phi \) be an additive mapping on \( A \) as above. If \( P \in A \) with \( P^2 = P \), then
1) \( \phi(P) = P\phi(I) = \phi(I)P = \phi(P)P = \phi(P); \)
2) \( \mu_{P + I} = \mu_P = 0 \).

**Proof.** If \( P = 0 \) or \( P = I \), the result is trivial.

Let \( P \) be a non-trivial idempotent, that is, \( P \neq 0 \) and \( P \neq I \). By (2.1),
\[
(m + n + k + l)\phi(P) = m\phi(P)P + nP\phi(P) + k\phi(I)P + lP\phi(I) + \mu_P I. \tag{2.4}
\]

By (2.3),
\[
(m + n + 2k + 2l)\phi(P) = (m + 2k)\phi(I)P + (n + 2l)P\phi(I) + (\mu_{P + I} - \mu_P)I. \tag{2.5}
\]

Multiplying (2.4) by \( P \) from the left and the right sides, gives that
\[
(\mu_P + P)\phi(P) = (k + l)P\phi(P) + \mu_P P. \tag{2.6}
\]
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Multiplying (2.5) by $P$ from the left and the right sides, we have that

$$P \phi(P) P = P \phi(I) P + \frac{1}{m + n + 2k + 2l} (\mu_{P+i} - \mu_P) P.$$

(2.7)

By comparing (2.6) with (2.7),

$$(m + n + 2k + 2l) \mu_P = (k + l) (\mu_{P+i} - \mu_P).$$

(2.8)

Multiplying (2.4) by $P$ from the left side gives that

$$(m + n + k + l) P \phi(P) = m P \phi(P) P + n P \phi(P) + k P \phi(I) P + l P \phi(I) + \mu_P P,$$

that is,

$$(m + k + l) P \phi(P) = m P \phi(P) P + k P \phi(I) P + l P \phi(I) + \mu_P P.$$

It follows from (2.7) that

$$(m + k + l) P \phi(P) = (m + k) P \phi(I) P + l P \phi(I) + \frac{m}{m + n + 2k + 2l} (\mu_{P+i} - \mu_P + \mu_P) P.$$

(2.9)

Thus

$$(m + n + 2k + 2l)(m + k + l) P \phi(P) = (m + k)(m + n + 2k + 2l) P \phi(I) P$$

$$+ l(m + n + 2k + 2l) P \phi(I) + (m + n + 2k + 2l) \mu_P P. $$

(2.9)’

Multiplying (2.5) by $P$ from the left side, yields that

$$(m + n + 2k + 2l) P \phi(P) = (m + 2k) P \phi(I) P + (n + 2l) P \phi(I) + (\mu_{P+i} - \mu_P) P.$$

(2.10)

Comparing (2.9)’ and (2.10), we obtain that

$$(m + k + l)(m + 2k) P \phi(I) P + (n + 2l)(m + k + l) P \phi(I) + (m + k + l)(\mu_{P+i} - \mu_P) P$$

$$= (m + k)(m + n + 2k + 2l) P \phi(I) P + l(m + n + 2k + 2l) P \phi(I)$$

$$+ (m \mu_{P+i} - \mu_P) + (m + n + 2k + 2l) \mu_P P.$$

It follows from (2.8) that

$$(m + k + l)(\mu_{P+i} - \mu_P) P = (m \mu_{P+i} - \mu_P) + (m + n + 2k + 2l) \mu_P P$$

and

$$P \phi(I) = P \phi(I) P.$$

(2.11)

It follows from (2.9) that

$$P \phi(P) = P \phi(I) P + \frac{1}{m + n + 2k + 2l} (\mu_{P+i} - \mu_P) P.$$

(2.12)

Similarly,

$$\phi(I) P = P \phi(I) P$$

(2.13)

and

$$\phi(P) P = P \phi(I) P + \frac{1}{m + n + 2k + 2l} (\mu_{P+i} - \mu_P) P.$$

(2.14)
(2.11) and (2.13) yield that $\phi(I)P = P\phi(I)$. And

$$\phi(P)P = P\phi(P) = P\phi(P)P$$ (2.15)

by (2.12) and (2.14). By Lemma 2.2 and $\phi(I)P = P\phi(I)$, it follows that $\phi(P) = \phi(I)P = P\phi(I)$ and $\mu_{P+I} - \mu_P = 0$. And by (2.8), $\mu_P = 0$ and $\mu_{P+I} = \mu_P = 0$. Identity (2.4) yields that

$$\phi(P) = \phi(I)P = P\phi(I)P = P\phi(P)P = \phi(P)P = P\phi(P).$$

\[\square\]

**Lemma 2.4.** Let $\phi$ be an additive mapping on $A$ as above. If $A, P \in A$ with $P^2 = P$, then (1) $\phi(AP) = \phi(A)P + \mu(\phi(I)P - \mu(A)P$, (2) $\phi(PA) = P\phi(A) + \mu(PA)I - \mu(A)P$, where $\mu(A) = \frac{1}{m+n+2k+2l}(\mu_{A+I} - \mu_A)$.

**Proof:** By (2.3),

$$\phi(AP) = \frac{m+2k}{m+n+2k+2l}\phi(I)AP + \frac{n+2l}{m+n+2k+2l}AP\phi(I) = \left(\frac{m+2k}{m+n+2k+2l}\phi(I)A + \frac{n+2l}{m+n+2k+2l}\phi(I)\right)P + \frac{1}{m+n+2k+2l}(\mu_{AP+I} - \mu_{AP})I$$

$$= \phi(A)P + \mu(\phi(I)P - \mu(A)P).$$

Similarly, $\phi(PA) = P\phi(A) + \mu(PA)I - \mu(A)P$. \[\square\]

**Lemma 2.5.** Let $\phi$ be an additive mapping on $A$ as above. If $A, P \in A$ with $P^2 = P$, then

$$\phi(PAP) = \phi(P\phi(P)P = P\phi(P)P = P\phi(PAP).$$

**Proof:** If $P = 0$ or $P = I$, the result is trivial.

Let $P$ be a non-trivial idempotent, that is, $P \neq 0$ and $P \neq I$. It follows from Lemma 2.4 that

$$\phi(PAP) = \phi(P\phi(P)P = P\phi(PAP) + \mu(PAP)I - \mu(PAP)P, \quad \text{(2.16)}$$

$$\phi(PAP) = \phi(PPAP) = P\phi(PAP) + \mu(PAP)I - \mu(PAP)P. \quad \text{(2.17)}$$

Comparing (2.16) and (2.17), we have that

$$P\phi(PAP) = \phi(PAP)P, \quad \text{(2.18)}$$

It follows from Lemma 2.1(2) that

$$(m+n+2k+2l)\phi(PAP) = (m+2k)\phi(I)PAP + (n+2l)PAP\phi(I) + (\mu_{AP+I} - \mu_{PAP})I. \quad \text{(2.19)}$$
By Lemma 2.3, we have that $\phi(I)PAP = \phi(P)PAP$, $PAP\phi(I) = PAP\phi(P)$ and $\mu_P = 0$. Putting $PAP$ for $A$ and $P$ for $B$ in (2.2), we have that
\[
2(m + n + k + l)\phi(PAP) = (m + n + k + l)\phi((PAP)P + P(PAP)) = m\phi(PAP)P + nP\phi(PAP) + nPAP\phi(P) + m\phi(P)PAP + 2k\phi(I)PAP + 2lPAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP} - \mu_P)I \tag{2.20}
\]
from the left side yields that
\[
2(m + n + k + l)(m + n + 2k + 2l)\phi(PAP) = (m + n + k + l)P\phi(PAP) + (m + n + 2k + 2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})I + (m + n)(\mu_{PAP+I} - \mu_{PAP}P). \tag{2.21}
\]
It follows from (2.19) that
\[
2(m + n + k + l)(m + n + 2k + 2l)\phi(PAP) = 2(m + n + k + l)\phi(I)PAP + 2(n + 2l)(m + n + k + l)PAP\phi(I) + 2(m + n + k + l)(\mu_{PAP+P} - \mu_{PAP})I + (m + n)(\mu_{PAP+I} - \mu_{PAP}P). \tag{2.22}
\]
Comparing (2.21) and (2.22), we have that
\[
(m + n + 2k + 2l)(\mu_{PAP+P} - \mu_{PAP})I + (m + n)(\mu_{PAP+I} - \mu_{PAP}P) = 2(m + n + k + l)(\mu_{PAP+I} - \mu_{PAP})I. \tag{2.23}
\]
Multiplying (2.19) by $P$ from the left and the right sides gives that
\[
(m + n + 2k + 2l)P\phi(PAP)P = (m + n + 2k)\phi(I)PAP + (n + 2l)PAP\phi(I) + (\mu_{PAP+I} - \mu_{PAP}P). \tag{2.24}
\]
Multiplying (2.20) by $P$ from the left side yields that
\[
2(m + n + k + l)P\phi(PAP)P = (m + n)P\phi(PAP)P + (m + 2k)\phi(I)PAP + (n + 2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})P. \tag{2.25}
\]
It follows that
\[
(m + n + 2k + 2l)P\phi(PAP)P = (m + 2k)\phi(I)PAP + (n + 2l)PAP\phi(I) + (\mu_{PAP+P} - \mu_{PAP})P. \tag{2.26}
\]
Comparing (2.24) and (2.25), we have that
\[
\mu_{PAP+P} - \mu_{PAP} = \mu_{PAP+I} - \mu_{PAP}. \tag{2.26}
\]
It follows from (2.23) and (2.26) that \((m+n)(\mu_{PAP+P} - \mu_{PAP}) = 0\). Since \(m+n > 0\), \(\mu_{PAP+P} - \mu_{PAP} = 0\) and
\[
\mu_{PAP+P} - \mu_{PAP} = \mu_{PAP+I} - \mu_{PAP} = 0. \tag{2.27}
\]

By (2.20) and (2.27),
\[
2(m + n + k + l)\phi(PAP) = (m + n)P\phi(PAP) + (m + 2k)\phi(I)PAP + (n + 2l)PA\phi(I). \tag{2.28}
\]

Combating it with (2.28), we have that
\[
(m + n + 2k + 2l)\phi(PAP) = (m + 2k)\phi(I)PAP + (n + 2l)PA\phi(I). \tag{2.29}
\]

Combining (2.28) and (2.29),
\[
\phi(PAP) = P\phi(PAP) = \phi(PAP)P = P\phi(PAP)P. \tag{2.30}
\]

\[\square\]

3. Generalized Jordan centralizers on CSL subalgebras of von Neumann algebras

In this section, we discuss an additive mapping \(\phi\) on \(A\), a CSL subalgebra of a von Neumann algebra, such that \((m + n + k + l)\phi(A^2) = (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I\) for any \(A \in A\), where \(\mathbb{F}\) is the real field or the complex field. The main result is as follows:

**Theorem 3.1.** Let \(N\) be a von Neumann algebra on a Hilbert space \(H\), and let \(L\) be a CSL, whose projections are contained in \(N\), and \(A = N \cap \text{Alg}L\) be the CSL subalgebra of the von Neumann algebra \(N\). If \(\phi : A \to A\) is a continuous mapping on \(A\) such that
\[
(m + n + k + l)\phi(A^2) = (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathbb{F}I
\]
for any \(A \in A\), where \(m, n, k, l \geq 0\) with \(mn \neq 0\), then \(\phi\) is a centralizer. That is, \(\phi(A) = \phi(I)A = A\phi(I)\) for any \(A \in A\).

The proof of Theorem 3.1 will proceed through several lemmas, each of which we maintain the same notation.

**Proposition 3.1** ([12]). Suppose that \(\bar{A} = N \cap \text{Alg}L\) is a CSL subalgebra of the von Neumann algebra \(N\). Let \(Q_1(H)\), or \(Q_1\) simply, be the orthogonal projection onto the linear span of the set \(\{PAP^\perp x : P \in L, \ A \in A, \ x \in H\}\); and let \(Q_2(H)\), or \(Q_2\) simply, be the orthogonal projection onto the linear span of the set \(\{P^\ast A^\ast P x : P \in L, \ A \in A, \ x \in H\}\), and \(Q = Q_1(H) \lor Q_2(H)\). Then

1. \(Q_1, Q_2\) and \(Q \in L' \cap N \subseteq A\), where \(L'\) is the commutant of \(L\). And \(Q_1\) commutes with \(Q_2\), and \(Q_1 \in \text{Lat}A\). Furthermore, \(Q^\ast AQ = QAQ^\perp = 0\) for any \(A \in A\), so that \(\bar{A} = QAQ \oplus Q^\ast AQ\).

2. If \(Q \neq I\), then \(Q^\ast AQ\) is a von Neumann algebra on \(Q^\perp H\).
Proof: By (2.30),
\[
\phi(A_i) = \phi(P_iA_iP_i) = P_i\phi(P_iA_iP_i)P_i \in A_i, (i = 1, 2)
\]
Let \( A_{12} = PAP^\perp \). Since \( A_{12} = P - (P - PAP^\perp) \) is the difference of two idempotents, it follows from Lemma 2.3 that
\[
\phi(A_{12}) = \phi(I)A_{12} = \phi(I)P_1A_{12} = \phi(P_1)A_{12} \in A_{12},
\]
\[
\phi(A_{12}) = A_{12}\phi(I) = A_{12}P_2\phi(I) = A_{12}\phi(P_2) \in A_{12}.
\]
\[\square\]

Lemma 3.2. For any \( A \in \mathcal{A}, B \in \mathcal{A}, A_{ij} \in A_{ij}, B_{ij} \in A_{ij}, \) \( 1 \leq i \leq j \leq 2 \),
(1) \( \phi(A_{11}B_{12}) = \phi(A_{11})B_{12} = A_{11}\phi(B_{12}) \).
(2) \( \phi(A_{12}B_{22}) = \phi(A_{12})B_{22} = A_{12}\phi(B_{22}) \).
(3) \( \phi(AB_{12}) = \phi(A)B_{12} = \phi(B_{12}) = \phi(I)AB_{12} = AB_{12}\phi(I) = A\phi(B_{12}) \).
(4) \( \phi(A_{12}B) = \phi(A_{12})B = A_{12}\phi(B) = \phi(I)A_{12}B = A_{12}B\phi(I) = A_{12}\phi(I)B \).

Proof: By (2.2),
\[
(m + n + k + l)\phi(A_{11}B_{12} + B_{12}A_{11}) = m\phi(A_{11})B_{12} + nA_{11}\phi(B_{12}) + mB_{12}\phi(A_{11}) + nB_{12}\phi(A_{11}) + k\phi(I)A_{11}B_{12} + k\phi(I)B_{12}A_{11} + lA_{11}B_{12}\phi(I) + lB_{12}A_{11}\phi(I) + (\mu_{A_{11}B_{12}} - \mu_{A_{11}} - \mu_{B_{12}})I. \tag{3.1}
\]
Since \( B_{12}A_{11} = 0, \phi(B_{12})A_{11} \in A_{12}A_{11} = 0 \) and \( B_{12}\phi(A_{11}) \in B_{12}A_{11} = 0 \), it follows that
\[
(m + n + k + l)\phi(A_{11}B_{12}) = m\phi(A_{11})B_{12} + nA_{11}\phi(B_{12}) + k\phi(I)A_{11}B_{12} + lA_{11}B_{12}\phi(I) + (\mu_{A_{11}B_{12}} - \mu_{A_{11}} - \mu_{B_{12}})I. \tag{3.2}
\]
Multiplying (3.2) by \( P_1 \) from the right sides, using the fact that \( \phi(B_{12}), \phi(A_{11}B_{12}) \in A_{12} \), yields \( (\mu_{A_{11}+B_{12}} - \mu_{A_{11}} - \mu_{B_{12}})P_1 = 0 \) and \( \mu_{A_{11}+B_{12}} - \mu_{A_{11}} - \mu_{B_{12}} = 0 \). Since \( B_{12} \in A_{12} \), it follows from Lemma 3.1(2) that
\[
\phi(B_{12}) = \phi(I)B_{12} = B_{12}\phi(I). \tag{3.3}
\]
Since \( A_{11}B_{12} \in A_{12} \),
\[
\phi(A_{11}B_{12}) = \phi(I)A_{11}B_{12} = A_{11}B_{12}\phi(I) = A_{11}\phi(B_{12}), \tag{3.4}
\]
Combining it with (3.2), we have that \( m\phi(A_{11}B_{12}) = m\phi(A_{11})B_{12} \). Since \( m \neq 0 \),
\[
\phi(A_{11})B_{12} = A_{11}B_{12}\phi(I) = \phi(A_{11}B_{12}) = A_{11}\phi(B_{12}) = \phi(I)A_{11}B_{12}. \tag{3.5}
\]
(2) By (2.1),
\[
(m + n + k + l)\phi(A_{12}B_{22} + B_{22}A_{12}) = m\phi(A_{12})B_{22} + nA_{12}\phi(B_{22}) + m\phi(B_{22})A_{12} + nB_{22}\phi(A_{12}) + k\phi(I)A_{12}B_{22} + k\phi(I)B_{22}A_{12} + lA_{12}B_{22}\phi(I) + lB_{22}A_{12}\phi(I) + (\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})I,
\]
Using the fact that \( \phi(B_{22})A_{12}, B_{22}\phi(A_{12}) \in A_{22}A_{12} = 0 \), yields that
\[
(m + n + k + l)\phi(A_{12}B_{22}) = m\phi(A_{12})B_{22} + nA_{12}\phi(B_{22}) + k\phi(I)A_{12}B_{22} + lA_{12}B_{22}\phi(I) + (\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})I. \tag{3.6}
\]
Multiplying (3.6) by \( P_1 \) from the right side, we have that \( (\mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}})P_1 = 0 \) and \( \mu_{A_{12}+B_{22}} - \mu_{A_{12}} - \mu_{B_{22}} = 0 \). It follows from Lemma 3.1(2) that \( \phi(A_{12}) = \phi(I)A_{12} = A_{12}\phi(I) \) and
\[
\phi(A_{12}B_{22}) = A_{12}B_{22}\phi(I) = \phi(I)A_{12}B_{22} = \phi(A_{12})B_{22}.
\]
Combining it with (3.6), we have that \( n\phi(A_{12}B_{22}) = nA_{12}\phi(B_{22}) \). Since \( n \neq 0 \),
\[
\phi(A_{12}B_{22}) = \phi(I)A_{12}B_{22} = A_{12}\phi(B_{22}) = \phi(I)A_{12}B_{22} = A_{12}B_{22}\phi(I). \tag{3.7}
\]
(3) Let \( B_{12} = PBP^\perp \). Then \( AB_{12} = PAPB^\perp \in A_{12} \). It follows from Lemma 3.1(2) that \( \phi(AB_{12}) = \phi(I)AB_{12} = AB_{12}\phi(I) = A\phi(B_{12}) \). It follows from (1) that
\[
\phi(AB_{12}) = \phi(PAPB^\perp) = \phi(PAP)PBP^\perp.
\]
By Lemma 3.1, \( \phi(PBP^\perp) \in A_{12} \), \( \phi(PAP) \in A_{11} \), \( \phi(PAP^\perp) \in A_{12} \), and \( \phi(P^\perp AP^\perp) \in A_{22} \). Therefore,
\[
\phi(A)B_{12} = \phi(A)PBP^\perp = \phi(PAP)PBP^\perp + \phi(PAP^\perp)PBP^\perp + \phi(P^\perp AP^\perp)PBP^\perp = \phi(PAP)PBP^\perp = \phi(APB^\perp) = \phi(AB_{12})
\]
It follows that
\[
\phi(AB_{12}) = \phi(A)B_{12} = A\phi(B_{12}) = \phi(I)AB_{12} = AB_{12}\phi(I) \tag{3.8}
\]
for any \( A, B \in A \).
(4) The proof is similar to the proof of (3). \( \square \)
Lemma 3.3. For any $A, B \in A$,

1. $(\phi(AB) - A\phi(B))Q_1(H) = 0,$ $(\phi(AB) - \phi(A)B)Q_1(H) = 0.$
2. $Q_2(H)(\phi(AB) - A\phi(B)) = 0,$ $Q_2(H)(\phi(AB) - \phi(A)B) = 0.$

Proof: (1) Let $T \in A, P \in \mathcal{L}$. It follows from Lemma 3.2 that

\[ \phi(AB)P T P^\perp = \phi(AB)P B P T P^\perp = \phi(AB)P T P^\perp = A\phi(AB)P T P^\perp = A\phi(P T P^\perp) = A\phi(B)P T P^\perp \]  
(3.9)

for any $A, B \in A$. So that $(\phi(AB) - A\phi(B))P T P^\perp = 0$ and $(\phi(AB) - \phi(A)B)P T P^\perp = 0$. It follows that

\[ (\phi(AB) - A\phi(B))Q_1(H) = 0, \quad (\phi(AB) - \phi(A)B)Q_1(H) = 0. \]  
(3.10)

(2) Similarly, for any $T \in A, P \in \mathcal{L},$

\[ P T P^\perp \phi(AB) = \phi(PT P^\perp AB) = \phi(PT P^\perp AP^\perp B) = P TP^\perp A\phi(B) = \phi(P T P^\perp A)B = P T P^\perp \phi(AB). \]  
(3.11)

Thus $P T P^\perp (\phi(AB) - A\phi(B)) = 0$ and $P T P^\perp (\phi(AB) - \phi(A)B) = 0$. Thus

\[ Q_2(H)(\phi(AB) - A\phi(B)) = 0, \quad Q_2(H)(\phi(AB) - \phi(A)B) = 0. \]  
(3.12)

\[ \square \]

Lemma 3.4. If $Q_1(H) \vee Q_2(H) = I$, then $\phi$ is a centralizer, that is, $\phi(A) = A\phi(I) = \phi(I)A$ for any $A \in A$.

Proof: Let $Q_1 = Q_1(H), \quad Q_2 = Q_2(H)$ for simplicity. By lemma 3.2(3), for any $A \in A, T \in A, P \in \mathcal{L},$

\[ \phi(PT P^\perp) = \phi(I)A PT P^\perp = \phi(A)PT P^\perp = APT P^\perp \phi(I) = A\phi(I)P T P^\perp. \]

It follows that $\phi(I)AQ_1 = \phi(A)Q_1 = A\phi(I)Q_1$. Since $Q_1 \in A$ is an idempotent, we have that $A\phi(I)Q_1 = AQ_1\phi(I)$ and $\phi(I)AQ_1 = AQ_1\phi(I)$. It follows from Lemma 2.2 that $\phi(AQ_1) = \phi(I)AQ_1 = AQ_1\phi(I)$ for any $A \in A$. And

\[ \phi(Q_1AQ_1) = \phi(I)AQ_1 = Q_1AQ_1\phi(I). \]  
(3.13)

If $Q_1(H) \vee Q_2(H) = I$, then $Q_1 Q_2 = Q_1$ and

\[ Q_1(\phi(AB) - \phi(A)B) = Q_1 Q_2(\phi(AB) - \phi(A)B) = 0 \]

and

\[ Q_1(\phi(AB) - A\phi(B)) = Q_1 Q_2(\phi(AB) - A(\phi(B))) = 0 \]

for any $A, B \in A$. In particular, $Q_1(\phi(A) = Q_1^+ \phi(AL) = Q_1^+ A\phi(I), \quad Q_1(\phi(A) = Q_1^+ \phi(I)A = \phi(I)Q_1A,$ so $Q_1^+ A\phi(I) = \phi(I)Q_1A$. By Lemma 2.2, $\phi(Q_1^+ A) = Q_1^+ A\phi(I) = \phi(I)Q_1^+ A$ and

\[ \phi(Q_1^+ AQ_1) = Q_1^+ A\phi(I) = \phi(I)Q_1^+ AQ_1. \]  
(3.14)
Since $Q_1AQ_1^\perp = Q_1 - (Q_1 - Q_1AQ_1^\perp)$ is the difference of two idempotents, it follows from Lemma 2.3 that

$$\phi(Q_1AQ_1^\perp) = Q_1AQ_1^\perp \phi(I) = \phi(I)Q_1AQ_1^\perp. \quad (3.15)$$

By (3.13), (3.14) and (3.15),

$$\phi(A) = \phi(Q_1AQ_1 + Q_1AQ_1^\perp + Q_1AQ_1^\perp) = \phi(Q_1AQ_1) + \phi(Q_1AQ_1^\perp) + \phi(Q_1AQ_1^\perp) = Q_1AQ_1\phi(I) + Q_1AQ_1\phi(I) + Q_1AQ_1\phi(I) = \phi(I)Q_1AQ_1 + \phi(I)Q_1AQ_1^\perp + \phi(I)Q_1AQ_1^\perp = A\phi(I) = \phi(I)A.$$

\[\square\]

**Lemma 3.5.** Let $A_1$ be a von Neumann algebra and $\phi : A_1 \to A_1$ a continuous mapping such that

$$(m + n + k + l)\phi(A^2) - (m\phi(A)A + nA\phi(A) + k\phi(I)A^2 + lA^2\phi(I)) \in \mathcal{F}$$

for any $A \in A_1$, where $m, n, k, l \geq 0$ with $mn \neq 0$. Then $\phi$ is a centralizer, that is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in A_1$.

**Proof:** Since a von Neumann algebra is the norm-closure of the subalgebra generated by the idempotents in it, the result follows from lemma 2.3. \[\square\]

**Proof of Theorem 3.1** By Proposition 3.1(1), we have that $A = QAQ \oplus Q^{-1}AQ^{-1}$. Let $\phi_1, \phi_2$ be the restriction of $\phi$ on $QAQ$, $Q^{-1}AQ^{-1}$ respectively. By Lemma 2.5, we have that $\phi(QAQ) = Q\phi(QAQ)Q$ and $\phi(Q^{-1}AQ^{-1}) = Q^{-1}\phi(Q^{-1}AQ^{-1})Q^{-1}$. So that $\phi_1$ is an additive mapping from $QAQ$ to itself, and $\phi_2$ is an additive mapping from $Q^{-1}AQ^{-1}$ to $Q^{-1}AQ^{-1}$. Since $QAQ = QAQQAQ$ and $Q^{-1}AQ^{-1} = QAQ^{-1}AQ^{-1}$, $\phi_1, \phi_2$ both satisfy the equality: $(m + n + k + l)\phi_1(A_i^2) - (m\phi_1(A_i)A_i + nA_i\phi_1(A_i) + k\phi_1(I_i)A_i^2 + lA_i^2\phi_1(I_i)) \in \mathcal{F} (i = 1, 2)$ for any $A_1 \in QAQ$ and $A_2 \in Q^{-1}AQ^{-1}$, where $I_1 = Q$ is the identity element of $QAQ$ and $I_2 = Q^{-1}$ is the identity element of $Q^{-1}AQ^{-1}$. Since

$$QAQ = \{ T \in \mathcal{L}Q : (Q - IP)TQ = 0 \text{ for any } P \in \mathcal{L} \} = QNQ \cap Alg(Q\mathcal{L}),$$

we have that $QAQ$ is a CSL subalgebra of the von Neumann algebra $Q\mathcal{L}$. For any $P \in \mathcal{L}$, $A \in A$ and $x \in H$, we have that $QAQ^2 = 0$ and

$$PAP^\perp x = QPA^\perp x = QPA(Q - IP)x = QPA(Q - IP)x.$$
we have that \( Q_1(H) = Q_1(QH) \) and \( Q_2(H) = Q_2(QH) \). It follows that \( Q_1(QH) \lor Q_2(QH) = Q \) is the identity element of \( Q \). All the conditions for Lemma 3.4 are satisfied, so we have that \( \phi_1 \) is a centralizer on \( QAQ \).

Since \( \phi_2 \) is a continuous mapping on the von Neumann algebra \( QAQ \) such that 
\[
(m + n + k + l)\phi_2(A^2) = (m\phi_2(A)A + nA\phi_2(A) + k\phi_2(I_2)A^2 + lA^2\phi_2(I_2)) \in \mathbb{F} \text{ for any } A \in QAQ^+ \text{, } \phi_2 \text{ is a centralizer by Lemma 3.5. It follows that } \phi \text{ is a centralizer on } \mathbb{F} \text{, that is, } \phi(A) = \phi(I)A = A\phi(I) \text{ for any } A \in \mathcal{A}.
\]

**Theorem 3.2.** Let \( \mathcal{N} \) be a von Neumann algebra on a Hilbert space \( H \), and \( \mathcal{L} \) be a CSL, whose projections are contained in \( \mathcal{N} \). And let \( \mathcal{A} = \mathcal{N} \cap \text{Alg} \mathcal{L} \) be the CSL subalgebra of the von Neumann algebra \( \mathcal{N} \). If \( \phi \) is an additive mapping on \( \mathcal{A} \) such that 
\[
(m + n + k + l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I_2)A^2 + lA^2\phi(I)
\]
for any \( A \in \mathcal{A} \), where \( m, n, k, l \geq 0 \) with \( mn \neq 0 \), then \( \phi \) is a centralizer. That is, \( \phi(A) = \phi(I)A = A\phi(I) \) for any \( A \in \mathcal{A} \).

In order to prove Theorem 3.2, we need a Lemma.

**Lemma 3.6.** Let \( \mathcal{A} \) be a unital \( C^* \)-algebra with the unity element \( I \). If \( \phi : \mathcal{A} \to \mathcal{A} \) is an additive mapping on \( \mathcal{A} \) such that 
\[
(m + n + k + l)\phi(A^2) = m\phi(A)A + nA\phi(A) + k\phi(I_2)A^2 + lA^2\phi(I)
\]
for any \( A \in \mathcal{A} \). Putting \( A + I \) for \( A \) in (3.16), we have that 
\[
(m + n + 2k + 2l)\phi(A) = (m + 2k)\phi(I)A + (n + 2l)A\phi(I).
\]  
(3.17)

By (3.16),
\[
(m + n + k + l)(m + n + 2k + 2l)\phi(A^2)
= m(m + n + 2k + 2l)\phi(A)A + nA(m + n + 2k + 2l)\phi(A)
+ k(m + n + 2k + 2l)\phi(I)A^2 + l(m + n + 2k + 2l)A^2\phi(I).
\]  
(3.18)

By (3.17) and (3.18),
\[
(m + n + k + l)(m + n + 2k + 2l)\phi(A^2)
= m(m + 2k)\phi(I)A + (m + 2l)A\phi(I)A + nA(m + 2k)\phi(I)A + (n + 2l)A\phi(I)
+ k(m + n + 2k + 2l)\phi(I)A^2 + l(m + n + 2k + 2l)A^2\phi(I)
= (k(m + n + k + l) + m(m + 2k))\phi(I)A^2 + (l(m + n + 2k + 2l)
+ n(n + 2l))A^2\phi(I) + (m + 2l)nA\phi(I)A.
\]  
(3.19)
On the other hand, putting $A^2$ for $A$ in (3.17), we have that

$$
(m + n + k + l)(m + n + 2k + 2l)\phi(A^2) \\
= (m + n + k + l)(m + 2k)\phi(I)A^2 + (m + n + k + l)(n + 2l)A^2\phi(I).
$$

Comparing (3.19) with (3.20), we have that $(mn + ml + nk)\phi(I)A^2 + (mn + ml + nk)A^2\phi(I) = 2(mn + ml + nk)A\phi(I)A$. Since $(mn + ml + nk) \neq 0$, $\phi(I)A^2 + A^2\phi(I) = 2A\phi(I)A$, that is, $[[\phi(I), A], A] = 0$. Then we have that $\phi(I)A = A\phi(I)$.

Indeed, let $\Delta(A) = [\phi(I), A]$ ($A \in A$), where $[A, B] = AB - BA$ is the commutator. Then $\Delta$ is an inner derivation on $A$, and $[\Delta(A), A] = 0$ for any $A \in A$. In particular, $[\Delta(A + B), A + B] = 0$ for any $A, B \in A$. It follows that $[\Delta(A), B] + [\Delta(B), A] = 0$. In the identity, putting $\phi(I)$ for $B$, we get that $[\Delta(A), \phi(I)] = 0$, that is, $\Delta^2(A) = 0$ for any $A \in A$. For any $A, B \in A$, $\Delta^2(AB) = \Delta^2(A)B + 2\Delta(A)\Delta(B) + A\Delta^2(B)$ and $\Delta^2(AB) = \Delta^2(A) = \Delta^2(B) = 0$. So we have that $\Delta(A)\Delta(B) = 0$ for any $A, B \in A$. Thus $\Delta(A)[\Delta(A), A] = 0$ for any $D \in A$, that is, $\Delta(A)\Delta(D)A + \Delta(A)D\Delta(A) = 0$. So that $\Delta(A)D\Delta(A) = 0$. Since $D$ is arbitrary, we have that $\Delta(A)\Delta(A) = 0$. By the truth that every unital $C^*$-algebra is a semi-prime ring, we have that $\Delta = 0$, that is, $\phi(I)A = A\phi(I)$ for any $A \in A$. By (3.17), $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in A$.

**Proof of Theorem 3.2** By Proposition 3.1(1), $A = QAQ \oplus Q^+AQ^\perp$. Let $\phi_1, \phi_2$ be the restrictions of $\phi$ on $QAQ$, $Q^+AQ^\perp$ respectively. By Lemma 2.5, $\phi(QAQ) = Q\phi(QAQ)Q$ and $\phi(Q^+AQ^\perp) = Q^+\phi(Q^+AQ^\perp)Q^\perp$. So $\phi_1$ is an additive mapping from $QAQ$ to $QAQ$, and $\phi_2$ is an additive mapping from $Q^+AQ^\perp$ to $Q^+AQ^\perp$. Since $QAQ^2 = QAQQAQ$ and $Q^+AQ^\perp = Q^+AQ^\perp,Q^+AQ^\perp$, $\phi_1, \phi_2$ both satisfy the equality: $(m + n + k + l)\phi_2(A^2) = m\phi_2(A)A + n\phi_2(A)A + k\phi_2(I)A^2 + lA^2\phi_2(I)$ for any $A \in QAQ$ and $A_2 \in Q^+AQ^\perp$, where $I_1 = Q$ is the identity element of $QAQ$ and $I_2 = Q^\perp$ is the identity element of $Q^+AQ^\perp$. Similar to the proof of Theorem 3.1, $\phi_1$ is a centralizer on $QAQ$.

Since $\phi_2$ is an additive mapping on the von Neumann algebra $Q^+AQ^\perp$ such that $(m + n + k + l)\phi_2(A^2) = m\phi_2(A)A + n\phi_2(A)A + k\phi_2(I)A^2 + lA^2\phi_2(I)$ for any $A \in QAQ^\perp$, it follows from Lemma 3.6 that $\phi_2$ is a centralizer. Therefore, $\phi$ is a centralizer, that is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in A$.

**Corollary 3.1.** Let $N$ be a von Neumann algebra on a Hilbert space $H$, and $L$ be a CSL, whose projections are contained in $N$, and $A = N \cap \text{Alg}L$ be the CSL subalgebra of the von Neumann algebra $N$. If $\phi : A \to A$ is an additive mapping on $A$ such that

$$(m + n)\phi(A^2) = m\phi(A)A + n\phi(A)$$

for any $A \in A$, where $m, n > 0$, then $\phi$ is a centralizer. That is, $\phi(A) = \phi(I)A = A\phi(I)$ for any $A \in A$.

The following theorems characterize the generalized Jordan centralizer. Zhang et al. ([22]) have proved them for the nest algebra. They are also true for the CSL subalgebra of a von Neumann algebra.
The characterization of generalized Jordan centralizers on algebras

Theorem 3.3. Let $N$ be a von Neumann algebra on a Hilbert space $H$, and $\mathcal{L}$ be a CSL, whose projections are contained in $N$, and $\mathcal{A} = N \cap \text{Alg}\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $N$. If $\phi: \mathcal{A} \rightarrow \mathcal{A}$ is an additive mapping $\mathcal{A}$ such that

$$(m + n)\phi(A^{p+1}) = m\phi(A)A^p + nA \phi(A)$$

for any $A \in \mathcal{A}$, where $m, n > 0$, $p \in \mathbb{Z}^+$, then $\phi$ is a centralizer. That is, $\phi(A) = \phi(I)A = A \phi(I)$ for any $A \in \mathcal{A}$.

Theorem 3.4. Let $N$ be a von Neumann algebra on a Hilbert space $H$, and $\mathcal{L}$ be a CSL, whose projections are contained in $N$, and $\mathcal{A} = N \cap \text{Alg}\mathcal{L}$ be the CSL subalgebra of the von Neumann algebra $N$. If $\phi: \mathcal{A} \rightarrow \mathcal{A}$ is an additive mapping such that

$$\phi(A^{m+n+1}) = A^m \phi(A)A^n$$

for any $A \in \mathcal{A}$, where $m, n$ are two positive integers, then $\phi$ is a centralizer. That is, $\phi(A) = \phi(I)A = A \phi(I)$ for any $A \in \mathcal{A}$.

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