Entropy Solutions For Nonlinear Parabolic Inequalities Involving Measure Data In Musielak-Orlicz-Sobolev Spaces

A. Talha, A. Benkirane, M.S.B. Elemine Vall

ABSTRACT: In this paper, we study an existence result of entropy solutions for some nonlinear parabolic problems in the Musielak-Orlicz-Sobolev spaces.

Key Words: Musielak-Orlicz-Sobolev spaces, parabolic equations, entropy solutions, truncations.

Contents

1 Introduction 199

2 Preliminary 200

2.1 Musielak-Orlicz-Sobolev spaces: 201

2.2 Inhomogeneous Musielak-Orlicz-Sobolev spaces: 204

3 Essential assumptions 205

4 Some technical Lemmas 206

5 Approximation and trace results 209

6 Compactness Results 212

7 Main results 214

1. Introduction

Let \( \Omega \) a bounded open subset of \( \mathbb{R}^N \) and let \( Q \) be the cylinder \( \Omega \times (0, T) \) with some given \( T > 0 \).

We consider the strongly nonlinear parabolic problem

\[
(P) \begin{cases}
\frac{\partial u}{\partial t} + A(u) + g(x, t, u, \nabla u) = f - \text{div}(F) & \text{in } Q, \\
u \equiv 0 & \text{on } \partial Q = \partial \Omega \times [0, T] \\
u(\cdot, 0) = u_0 & \text{on } \Omega,
\end{cases}
\]

where \( A : D(A) \subset W^{1, \varphi}_0(Q) \rightarrow W^{-1, \psi}(Q) \) (see section 2) defined by \( A(u) = -\text{div}(a(x, t, u, \nabla u)) \) is an operator of Leray-Lions type, where \( a \) is a Carathéodory function such that

\[
|a(x, t, s, \xi)| \leq \beta \left( h_1(x, t) + \psi_x^{-1}\gamma(x, \nu|s|) + \psi_x^{-1}\varphi(x, \nu|\xi|) \right)
\]

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\[ \left( a(x, t, s, \xi) - a(x, t, s, \xi') \right) (\xi - \xi') > 0 \]
\[ a(x, t, s, \xi) \geq \alpha \varphi(x, |\xi|) \]
with \( h_1 \in L^1(Q), \beta, \nu, \alpha > 0 \) and \( \gamma \) a Musielak function such that \( \gamma \ll \varphi \).
Let \( g \) be a Carathéodory function such that
\[ |g(x, t, s, \xi)| \leq b(|s|) \left( h_2(x, t) + \varphi(x, |\xi|) \right), \]
\[ g(x, t, s, \xi)s \geq 0, \]
is satisfied, where \( b \) a positive function in \( L^1(\mathbb{R}^+) \) and \( h_2 \in L^1(Q) \),
and \( f \in L^1(Q) \) and \( F \in (E_2(Q))^N \).
Under these assumptions, the above problem does not admit, in general, a weak solution since the field \( a(x, t, u, \nabla u) \) does not belong to \( (L^1_{loc}(Q))^N \) in general. To overcome this difficulty we use in this paper the framework of entropy solutions.
This notion was introduced by Bénilan and al. \[4\] for the study of nonlinear elliptic problems.
In the classical Sobolev spaces, the authors in \[9, 17\] proved the existence of solutions for the problem \((P)\) in the case where \( F \equiv 0 \), in \[7\] the authors had proved the existence of solutions for the problem \((P)\) in the elliptic case.
In the setting of Orlicz spaces, the solvability of \((P)\) was proved by Donaldson \[10\] and Robert \[18\], and by Elmahi \[12\] and Elmahi-Meskin \[13\]. In Musielak framework, recently M. L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in \[2\] had studied the problem \((P)\) in the Inhomogeneous case and the data belongs to \( L^1(Q) \), in the elliptic case the authors in \[1\] proved the existence of weak solutions for the problem \((P)\) where the data assume to be measure and \( g \equiv 0 \).
It is our purpose in this paper to prove the existence of entropy solutions for problem \((P)\) in the setting of Musielak Orlicz spaces for general Musielak function \( \varphi \) with a nonlinearity \( g(x, t, u, \nabla u) \) having natural growth with respect to the gradient.
Our result generalizes that of \[13, 1, 2\] to the case of inhomogeneous Musielak Orlicz Sobolev spaces.
The plan of the paper is as follows. Section 2 presents the mathematical preliminaries. Section 3 we make precise all the assumptions on \( a, g, f \) and \( u_0 \). Section 4 is devoted to some technical lemmas with be used in this paper. Section 5 we establish some compactness and approximation results. Final section is consecrate to define the entropy solution of \((P)\) and to prove existence of such a solution.

2. Preliminary

In this section we list briefly some definitions and facts about Musielak-Orlicz-Sobolev spaces. Standard reference is \[16\]. We also include the definition of inhomogeneous Musielak-Orlicz-Sobolev spaces and some preliminaries Lemmas to be used later.
2.1. Musielak-Orlicz-Sobolev spaces:

Let $\Omega$ be an open set in $\mathbb{R}^N$ and let $\varphi$ be a real-valued function defined in $\Omega \times \mathbb{R}_+$, and satisfying the following conditions:

a) $\varphi(x, \cdot)$ is an N-function (convex, increasing, continuous, $\varphi(x, 0) = 0$, $\varphi(x, t) > 0$, $\forall t > 0$, $\sup_{x \in \Omega} \frac{\varphi(x, t)}{t} \to 0$ as $t \to 0$, $\inf_{x \in \Omega} \frac{\varphi(x, t)}{t} \to \infty$ as $t \to \infty$).

b) $\varphi(\cdot, t)$ is a measurable function.

A function $\varphi$, which satisfies the conditions a) and b) is called Musielak-Orlicz function.

For a Musielak-orlicz function $\varphi$ we put $\varphi_x(t) = \varphi(x, t)$ and we associate its non-negative reciprocal function $\varphi_x^{-1}$, with respect to $t$ that is

$$\varphi_x^{-1}(\varphi(x, t)) = \varphi(x, \varphi_x^{-1}(t)) = t.$$  

The Musielak-orlicz function $\varphi$ is said to satisfy the $\Delta_2^-$-condition if for some $k > 0$ and a non negative function $h$ integrable in $\Omega$, we have

$$\varphi(x, 2t) \leq k \varphi(x, t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \quad (2.1)$$

When (2.1) holds only for $t \geq t_0 > 0$; then $\varphi$ said to satisfy $\Delta_2$ near infinity.

Let $\varphi$ and $\gamma$ be two Musielak-orlicz functions, we say that $\varphi$ dominate $\gamma$, and we write $\gamma \prec \varphi$, near infinity (resp. globally) if there exist two positive constants $c$ and $t_0$ such that for almost all $x \in \Omega$

$$\gamma(x, t) \leq \varphi(x, ct) \text{ for all } t \geq t_0, \quad (\text{resp. for all } t \geq 0 \text{ i.e. } t_0 = 0).$$

We say that $\gamma$ grows essentially less rapidly than $\varphi$ at $0$ (resp. near infinity), and we write $\gamma \ll \varphi$. If for every positive constant $c$ we have

$$\lim_{t \to 0} \left( \sup_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \right) = 0, \quad (\text{resp. } \lim_{t \to \infty} \left( \sup_{x \in \Omega} \frac{\gamma(x, ct)}{\varphi(x, t)} \right) = 0).$$

Remark 2.1. [6] If $\gamma \ll \varphi$ near infinity, then $\forall \varepsilon > 0$ there exist $k(\varepsilon) > 0$ such that for almost all $x \in \Omega$ we have

$$\gamma(x, t) \leq k(\varepsilon) \varphi(x, \varepsilon t), \quad \text{for all } t \geq 0. \quad (2.2)$$

We define the functional

$$\rho_{\varphi, \Omega}(u) = \int_\Omega \varphi(x, |u(x)|)dx.$$

where $u : \Omega \to \mathbb{R}$ a Lebesgue measurable function. In the following, the measurability of a function $u : \Omega \to \mathbb{R}$ means the Lebesgue measurability.

The set $K_{\varphi}(\Omega) = \{ u : \Omega \to \mathbb{R} \text{ measurable } : \rho_{\varphi, \Omega}(u) < +\infty \}$. 
is called the generalized Orlicz class. The Musielak-Orlicz space (or the generalized Orlicz spaces) \( L_\psi(\Omega) \) is the vector space generated by \( K_\psi(\Omega) \), that is, \( L_\psi(\Omega) \) is the smallest linear space containing the set \( K_\psi(\Omega) \).

Equivalently, 

\[
L_\psi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} : \rho_{\psi,\Omega}(\frac{u}{\lambda}) < +\infty, \text{ for some } \lambda > 0 \right\}.
\]

Let 

\[
\psi(x, s) = \sup_{t \geq 0} \{ st - \varphi(x, t) \}.
\]

that is, \( \psi \) is the Musielak-Orlicz function complementary to \( \varphi \) in the sens of Young with respect to the variable \( s \).

We define in the space \( L_\psi(\Omega) \) the following two norms

\[
\|u\|_{\psi,\Omega} = \inf \left\{ \lambda > 0 / \int_{\Omega} \varphi \left( x, \frac{|u(x)|}{\lambda} \right) dx \leq 1 \right\}.
\]

which is called the Luxemburg norm and the so called Orlicz norm by:

\[
\|u\|_{\varphi,\Omega} = \sup_{\|v\|_{\psi,\Omega} \leq 1} \int_{\Omega} |u(x)v(x)| dx.
\]

where \( \psi \) is the Musielak Orlicz function complementary to \( \varphi \). These two norms are equivalent \cite{16}.

The closure in \( L_\varphi(\Omega) \) of the bounded measurable functions with compact support in \( \Omega \) is denoted by \( E_\varphi(\Omega) \). A Musielak function \( \varphi \) is called locally integrable on \( \Omega \) if \( \rho_{\varphi}(t\chi_E) < \infty \) for all \( t > 0 \) and all measurable \( E \subset \Omega \) with \( \text{meas}(E) < \infty \). Note that local integrability in the previous definition differs from the one used in \( L^1_{\text{loc}}(\Omega) \), where we assume integrability over compact subsets.

**Lemma 2.1.** \cite{15} Let \( \varphi \) a Musielak function which is locally integrable. Then \( E_\varphi(\Omega) \) is separable.

We say that sequence of functions \( u_n \in L_\varphi(\Omega) \) is modular convergent to \( u \in L_\varphi(\Omega) \) if there exists a constant \( \lambda > 0 \) such that

\[
\lim_{n \to \infty} \rho_{\varphi,\Omega}(\frac{u_n - u}{\lambda}) = 0.
\]

For any fixed nonnegative integer \( m \) we define

\[
W^mL_\varphi(\Omega) = \left\{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in L_\varphi(\Omega) \right\}.
\]

and

\[
W^mE_\varphi(\Omega) = \left\{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, D^\alpha u \in E_\varphi(\Omega) \right\}.
\]
where \( \alpha = (\alpha_1, ..., \alpha_n) \) with nonnegative integers \( \alpha_i, |\alpha| = |\alpha_1| + ... + |\alpha_n| \) and \( D^\alpha u \) denote the distributional derivatives. The space \( W^m L_\varphi(\Omega) \) is called the Musielak Orlicz Sobolev space.

Let
\[
\gamma_{\varphi, \Omega}(u) = \sum_{|\alpha| \leq m} \rho_{\varphi, \Omega} \left( D^\alpha u \right) \quad \text{and} \quad \|u\|_{m, \varphi, \Omega}^m = \inf \left\{ \lambda > 0 : \gamma_{\varphi, \Omega} \left( \frac{u}{\lambda} \right) \leq 1 \right\}
\]
for \( u \in W^m L_\varphi(\Omega) \), these functionals are a convex modular and a norm on \( W^m L_\varphi(\Omega) \), respectively, and the pair \( \left( W^m L_\varphi(\Omega), \|\|_{m, \varphi, \Omega} \right) \) is a Banach space if \( \varphi \) satisfies the following condition \([16]\):

\[
\inf_{x \in \Omega} \varphi(x, 1) \geq c.
\]

The space \( W^m L_\varphi(\Omega) \) will always be identified to a subspace of the product \( \prod_{|\alpha| \leq m} L_\varphi(\Omega) = \Pi L_\varphi \), this subspace is \( \sigma(\Pi L_\varphi, \Pi E_\varphi) \) closed. We denote by \( D(\Omega) \) the space of infinitely smooth functions with compact support in \( \Omega \) and by \( D(\Omega) \) the restriction of \( D(\mathbb{R}^N) \) on \( \Omega \).

Let \( W^m_0 L_\varphi(\Omega) \) be the \( \sigma(\Pi L_\varphi, \Pi E_\varphi) \) closure of \( D(\Omega) \) in \( W^m L_\varphi(\Omega) \). Let \( W^m E_\varphi(\Omega) \) the space of functions \( u \) such that \( u \) and its distribution derivatives up to order \( m \) lie to \( E_\varphi(\Omega) \), and \( W^m_0 E_\varphi(\Omega) \) is the (norm) closure of \( D(\Omega) \) in \( W^m L_\varphi(\Omega) \).

The following spaces of distributions will also be used:

\[
W^{-m} L_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\psi(\Omega) \right\},
\]

and

\[
W^{-m} E_\psi(\Omega) = \left\{ f \in D'(\Omega); f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \right\}.
\]

We say that a sequence of functions \( u_n \in W^m L_\varphi(\Omega) \) is modular convergent to \( u \in W^m L_\varphi(\Omega) \) if there exists a constant \( k > 0 \) such that

\[
\lim_{n \to \infty} \gamma_{\varphi, \Omega} \left( \frac{u_n - u}{k} \right) = 0.
\]

For \( \varphi \) and her complementary function \( \psi \), the following inequality is called the Young inequality \([16]\):

\[
vt \leq \varphi(v, t) + \psi(v, s), \quad \forall t, s \geq 0, v \in \Omega.
\]

This inequality implies that

\[
\|u\|_{\varphi, \Omega} \leq \rho_{\varphi, \Omega}(u) + 1.
\]
In $L^\varphi(\Omega)$ we have the relation between the norm and the modular
\[
\|u\|_{\varphi, \Omega} \leq \rho_{\varphi, \Omega}(u) \text{ if } \|u\|_{\varphi, \Omega} > 1. \tag{2.6}
\]
\[
\|u\|_{\varphi, \Omega} \geq \rho_{\varphi, \Omega}(u) \text{ if } \|u\|_{\varphi, \Omega} \leq 1. \tag{2.7}
\]
For two complementary Musielak-Orlicz functions $\varphi$ and $\psi$, let $u \in L^\varphi(\Omega)$ and $v \in L^\psi(\Omega)$, then we have the Hölder inequality \cite{[16]}
\[
\left| \int_\Omega u(x)v(x)dx \right| \leq \|u\|_{\varphi, \Omega} \|v\|_{\psi, \Omega}. \tag{2.8}
\]

2.2. Inhomogeneous Musielak-Orlicz-Sobolev spaces:

Let $\Omega$ a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times [0,T]$ with some given $T > 0$. Let $\varphi$ be a Musielak function. For each $\alpha \in \mathbb{N}^N$, denote by $D^\alpha_x$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

\[
W^{1,x}_{\varphi}(Q) = \{ u \in L^\varphi(Q) : \forall |\alpha| \leq 1 \ D^\alpha_x u \in L^\varphi(Q) \}
\]

and

\[
W^{1,x}_{\varphi}(Q) = \{ u \in E^{\varphi}(Q) : \forall |\alpha| \leq 1 \ D^\alpha_x u \in E^\varphi(Q) \}
\]

The last space is a subspace of the first one, and both are Banach spaces under the norm
\[
\|u\| = \sum_{|\alpha| \leq m} \|D^\alpha_x u\|_{\varphi, Q}.
\]

We can easily show that they form a complementary system when $\Omega$ is a Lipschitz domain \cite{[5]}. These spaces are considered as subspaces of the product space $\Pi L^\varphi(Q)$ which has $(N + 1)$ copies. We shall also consider the weak topologies $\sigma(\Pi L^\varphi, \Pi E^\varphi)$ and $\sigma(\Pi L^\varphi, \Pi E^\psi)$. If $u \in W^{1,x}_{\varphi}(Q)$ then the function $t \mapsto \varphi(t) = u(t, \cdot)$ is defined on $[0,T]$ with values in $W^1 L^\varphi(\Omega)$. If, further, $u \in W^{1,x}_{\varphi}(Q)$ then this function is $W^1 E^\varphi(\Omega)$ valued and is strongly measurable. Furthermore the following imbedding holds: $W^{1,x}_{\varphi}(Q) \subset L^1([0,T], W^{1,x}_{\varphi}(\Omega))$. The space $W^{1,x}_{\varphi}(Q)$ is not in general separable, if $u \in W^{1,x}_{\varphi}(Q)$, we can not conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \mapsto \|u(t)\|_{\varphi, \Omega}$ is in $L^1(0,T)$. The space $W^{1,x}_{0,\varphi}(Q)$ is defined as the (norm) closure in $W^{1,x}_{\varphi}(Q)$ of $\mathcal{D}(Q)$.

We can easily show as in \cite{[5]} that when $\Omega$ a Lipschitz domain then each element $u$ of the closure of $\mathcal{D}(Q)$ with respect of the weak * topology $\sigma(\Pi L^\varphi, \Pi E^\varphi)$ is limit, in $W^{1,x}_{\varphi}(Q)$, of some subsequence $(u_i) \subset \mathcal{D}(Q)$ for the modular convergence; i.e., there exists $\exists \lambda > 0$ such that for all $|\alpha| \leq 1$,
\[
\int_Q \varphi(x, (\frac{D^2 u_i - D^2 u}{\lambda})) \, dx \, dt \to 0 \text{ as } i \to \infty,
\]
this implies that \((u_i)\) converges to \(u\) in \(W^{1,x}L_\varphi(Q)\) for the weak topology \(\sigma(\Pi L_\varphi, \Pi L_\psi)\). Consequently
\[
\overline{\text{D}(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}} = \overline{\text{D}(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}},
\]
this space will be denoted by \(W^{1,x}_0L_\varphi(Q)\). Furthermore, \(W^{1,x}_0E_\varphi(Q) = W^{1,x}_0L_\varphi(Q) \cap \Pi E_\varphi\).

We have the following complementary system
\[
\begin{pmatrix}
W^{1,x}_0L_\varphi(Q) & F \\
W^{1,x}_0E_\varphi(Q) & F_0
\end{pmatrix},
\]
\(F\) being the dual space of \(W^{1,x}_0E_\varphi(Q)\). It is also, except for an isomorphism, the quotient of \(\Pi L_\psi\) by the polar set \(W^{1,x}_0E_\varphi(Q)^\perp\), and will be denoted by \(F = W^{-1,x}L_\psi(Q)\) and it is shown that
\[
W^{-1,x}L_\psi(Q) = \left\{f = \sum_{|\alpha| \leq 1} D_\alpha f_\alpha : f_\alpha \in L_\psi(Q)\right\}.
\]
This space will be equipped with the usual quotient norm
\[
\|f\| = \inf \sum_{|\alpha| \leq 1} \|f_\alpha\|_{\psi,Q}
\]
where the inf is taken on all possible decompositions
\[
f = \sum_{|\alpha| \leq 1} D_\alpha f_\alpha, \quad f_\alpha \in L_\psi(Q).
\]
The space \(F_0\) is then given by
\[
F_0 = \left\{f = \sum_{|\alpha| \leq 1} D_\alpha f_\alpha : f_\alpha \in E_\psi(Q)\right\}
\]
and is denoted by \(F_0 = W^{-1,x}E_\psi(Q)\).

### 3. Essential assumptions

Let \(\Omega\) be a bounded open subset of \(\mathbb{R}^N\) satisfying the segment property and \(T > 0\) we denote \(Q = \Omega \times [0, T]\), and let \(\varphi\) and \(\gamma\) be two Musielak-Orlicz functions such that \(\gamma \prec \varphi\). Let \(A : D(A) \subset W^{1,x}_0L_\varphi(Q) \rightarrow W^{-1,x}L_\varphi(Q)\) be a mapping given by
\[
A(u) = -\text{div}(a(x, t, u, \nabla u)),
\]
where \(a : a(x, t, s, \xi) : \Omega \times [0, t] \times \mathbb{R} \times \mathbb{R}^N \rightarrow \mathbb{R}^N\) is a Carathéodory function satisfying, for a.e \((x, t) \in Q\) and for all \(s \in \mathbb{R}\) and all \(\xi, \xi' \in \mathbb{R}^N, \xi \neq \xi'\):
\[
|a(x, t, s, \xi)| \leq \beta \left(h_1(x, t) + \psi_\varphi^{-1} \gamma(x, |\xi|) + \psi_\varphi^{-1} \varphi(x, |\xi|)\right)
\]
\(3.1\)
\[
\left( a(x, t, s, \xi) - a(x, t, s, \xi') \right)(\xi - \xi') > 0
\] (3.2)

\[
a(x, t, s, \xi) \geq \alpha \varphi(x, |\xi|)
\] (3.3)

where \(c(x, t)\) a positive function, \(c(x, t) \in E_{\psi}(Q)\) and positive constants \(\nu, \alpha\).

Furthermore, let \(g(x, t, s, \xi) : \Omega \times ]0, T[ \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}\) be a Carathéodory function such that for a.e. \((x, t) \in \Omega \times ]0, T[\) and for all \(s \in \mathbb{R}, \xi \in \mathbb{R}^N\), the following conditions

\[
|g(x, t, s, \xi)| \leq b(|s|)\left(h_2(x, t) + \varphi(x, |\xi|)\right),
\] (3.4)

\[
g(x, t, s, \xi)s \geq 0,
\] (3.5)

are satisfied, where \(b : \mathbb{R}^+ \to \mathbb{R}^+\) is a continuous positive function which belongs to \(L^1(\mathbb{R})\) and \(h_2(x, t) \in L^1(Q)\).

\[
f \in L^1(Q) \quad \text{and} \quad F \in (E_{\psi}(Q))^N.
\] (3.6)

\[
u_0 \in L^1(\Omega).
\] (3.7)

4. Some technical Lemmas

Lemma 4.1. [5]. Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^N\) and let \(\varphi\) and \(\psi\) be two complementary Musielak-Orlicz functions which satisfy the following conditions:

i) There exist a constant \(c > 0\) such that \(\inf_{x \in \Omega} \varphi(x, 1) \geq c\).

ii) There exist a constant \(A > 0\) such that for all \(x, y \in \Omega\) with \(|x - y| \leq \frac{1}{2}\) we have

\[
\frac{\varphi(x, t)}{\varphi(y, t)} \leq \exp\left(4 \log\left(\frac{4}{|x - y|}\right)\right), \quad \forall t \geq 1.
\] (4.1)

iii)

If \(D \subset \Omega\) is a bounded measurable set, then \(\int_D \varphi(x, 1)\,dx < \infty\). (4.2)

iv) There exist a constant \(C > 0\) such that \(\psi(x, 1) \leq C\) a.e in \(\Omega\).

Under this assumptions, \(\mathcal{D}(\Omega)\) is dense in \(L^\varphi(\Omega)\) with respect to the modular topology, \(\mathcal{D}(\Omega)\) is dense in \(W^1_0 L^\varphi(\Omega)\) for the modular convergence and \(\mathcal{D}(\Omega)\) is dense in \(W^1 L^\varphi(\Omega)\) the modular convergence.

Consequently, the action of a distribution \(S\) in \(W^{-1} L^\psi(\Omega)\) on an element \(u\) of \(W^1_0 L^\varphi(\Omega)\) is well defined. It will be denoted by \(<S, u>_\).
Lemma 4.2. [6]. Let $F : \mathbb{R} \to \mathbb{R}$ be uniformly Lipschitzian, with $F(0) = 0$. Let $\varphi$ be a Musielak-Orlicz function and let $u \in W_0^1 L\varphi(\Omega)$. Then $F(u) \in W_0^1 L\varphi(\Omega)$. Moreover, if the set $D$ of discontinuity points of $F'$ is finite, we have

$$
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e in } \{x \in \Omega : u(x) \in D\}, \\
0 & \text{a.e in } \{x \in \Omega : u(x) \notin D\}.
\end{cases}
$$

Lemma 4.3. Let $(f_n), f \in L^1(\Omega)$ such that

i) $f_n \geq 0$ a.e in $\Omega$.

ii) $f_n \rightharpoonup f$ a.e in $\Omega$.

iii) $\int_\Omega f_n(x) dx \to \int_\Omega f(x) dx$.

then $f_n \rightharpoonup f$ strongly in $L^1(\Omega)$.

Lemma 4.4 (Jensen inequality). [19]. Let $\varphi : \mathbb{R} \to \mathbb{R}$ a convex function and $g : \Omega \to \mathbb{R}$ is function measurable, then

$$
\varphi \left( \int_\Omega g \, d\mu \right) \leq \int_\Omega \varphi \circ g \, d\mu.
$$

Lemma 4.5 (Poincaré inequality). [11]. Let $\varphi$ a Musielak Orlicz function which satisfies the assumptions of lemma 4.1, suppose that $\varphi(x,t)$ decreases with respect to one of coordinate of $x$.

Then, that exists a constant $c > 0$ depends only of $\Omega$ such that

$$
\int_\Omega \varphi(x, |u(x)|) dx \leq \int_\Omega \varphi(x, d |\nabla u(x)|) dx, \quad \forall u \in W^1_0 L\varphi(\Omega). \quad (4.3)
$$

Proof Since $\varphi(x,t)$ decreases with respect to one of coordinates of $x$, there exists $i_0 \in \{1, \ldots, N\}$ such that the function $\sigma \to \varphi(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N, t)$ is decreasing for every $x_1, \ldots, x_{i_0-1}, x_{i_0+1}, \ldots, x_N \in \mathbb{R}$ and $\forall t > 0$.

To prove our result, it suffices to show that

$$
\int_\Omega \varphi(x, |u(x)|) dx \leq \int_\Omega \varphi \left( x, 2d \left| \frac{\partial u}{\partial x_{i_0}}(x) \right| \right) dx, \quad \forall u \in W^1_0 L\varphi(\Omega). \quad (4.4)
$$

with $d = \max \left( \text{diam}(\Omega), 1 \right)$ and $\text{diam}(\Omega)$ is the diameter of $\Omega$.

First, suppose that $u \in D(\Omega)$, then so by the Jensen integral inequality we obtain

\begin{align*}
\varphi(x, |u(x_1, \ldots, x_N)|) & \leq \varphi \left( x, \int_{-\infty}^{\infty} \left| \frac{\partial u}{\partial x_{i_0}}(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N) \right| d\sigma \right), \\
& \leq \frac{1}{d} \int_{-\infty}^{\infty} \varphi \left( x, d \left| \frac{\partial u}{\partial x_{i_0}}(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N) \right| \right) d\sigma \\
& \leq \frac{1}{d} \int_{-\infty}^{\infty} f(\sigma) d\sigma,
\end{align*}
where \( f(\sigma) = \varphi(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N, d\frac{\partial u}{\partial x_{i_0}}(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N)) \).

By integrating with respect to \( x \), we get
\[
\int_{\Omega} \varphi(x, |u(x_1, \ldots, x_N)|)dx \\
\leq \int_{\Omega} \frac{1}{d} \int_{-\infty}^{+\infty} f(\sigma)d\sigma dx,
\]

since \( \varphi(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N, d\frac{\partial u}{\partial x_{i_0}}(x_1, \ldots, x_{i_0-1}, \sigma, x_{i_0+1}, \ldots, x_N)) \) independent of \( x_{i_0} \), we can get it out of the integral to respect of \( x_{i_0} \) and by the fact that \( \sigma \) is arbitrary, then by Fubini’s Theorem we get
\[
\int_{\Omega} \varphi(x, |u(x)|)dx \leq \int_{\Omega} \varphi(x, d\frac{\partial u}{\partial x_{i_0}}(x))dx, \quad \forall u \in D(\Omega). \tag{4.5}
\]

For \( u \in W^1_0L_2(\Omega) \) according to Lemma 4.1, we have the existence of \( u_n \in D(\Omega) \) and \( \lambda > 0 \) such that
\[
\overline{\Omega} \varphi, \Omega \left( \frac{u_n - u}{\lambda} \right) = 0, \quad \text{as } n \to +\infty,
\]

hence
\[
\left\{ \begin{array}{l}
\int_{\Omega} \varphi \left( x, \frac{|u_n - u|}{\lambda} \right)dx \to 0, \quad \text{as } n \to +\infty, \\
\int_{\Omega} \varphi \left( x, \frac{u_n - \nabla u}{\lambda} \right)dx \to 0, \quad \text{as } n \to +\infty,
\end{array} \right.
\]

\( u_n \to u \) a.e in \( \Omega \), \ (for a subsequence still denote \( u_n \)).

Then, we have
\[
\int_{\Omega} \varphi \left( x, \frac{|u(x)|}{2d\lambda} \right)dx \leq \liminf_{n \to +\infty} \int_{\Omega} \varphi \left( x, \frac{|u_n(x)|}{2d\lambda} \right)dx \\
\leq \liminf_{n \to +\infty} \int_{\Omega} \varphi \left( x, \frac{1}{2d\lambda} \frac{\partial u_n}{\partial x_{i_0}}(x) \right)dx \\
= \liminf_{n \to +\infty} \int_{\Omega} \varphi \left( x, \frac{1}{2d\lambda} \frac{\partial u_n}{\partial x_{i_0}}(x) - \frac{\partial u}{\partial x_{i_0}}(x) \right)dx \\
\leq \frac{1}{2} \liminf_{n \to +\infty} \int_{\Omega} \varphi \left( x, \frac{1}{d\lambda} \frac{\partial u_n}{\partial x_{i_0}}(x) \right)dx \\
+ \frac{1}{2} \int_{\Omega} \varphi \left( x, \frac{1}{d\lambda} \frac{\partial u}{\partial x_{i_0}}(x) \right)dx \\
\leq \int_{\Omega} \varphi \left( x, \frac{1}{d\lambda} \frac{\partial u}{\partial x_{i_0}}(x) \right)dx.
\]

Hence
\[
\int_{\Omega} \varphi \left( x, |u(x)| \right)dx \leq \int_{\Omega} \varphi \left( x, 2d\frac{\partial u}{\partial x_{i_0}}(x) \right)dx, \quad \forall u \in W^1_0L_2(\Omega).
\]

□
Lemma 4.6 (The Nemytskii Operator). Let $\Omega$ be an open subset of $\mathbb{R}^N$ with finite measure and let $\varphi$ and $\psi$ be two Musielak-Orlicz functions. Let $f: \Omega \times \mathbb{R}^p \rightarrow \mathbb{R}^q$ be a Carathéodory function such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}^p$:

$$|f(x, s)| \leq c(x) + k_1 \psi^{-1}_x \varphi(x, k_2 |s|). \quad (4.6)$$

where $k_1$ and $k_2$ are real positive constants and $c(.) \in E_\varphi(\Omega)$.

Then the Nemytskii Operator $N_f$ defined by $N_f(u)(x) = f(x, u(x))$ is continuous from $P\left(E_\varphi(\Omega), \frac{1}{k_2}\right) = \prod \left\{ u \in L_\varphi(\Omega) : d(u, E_\varphi(\Omega)) < \frac{1}{k_2} \right\}$. into $(L_\psi(\Omega))^q$ for the modular convergence.

Furthermore if $c(.) \in E_\varphi(\Omega)$ and $\gamma \rightarrow \psi$ then $N_f$ is strongly continuous from $P\left(E_\varphi(\Omega), \frac{1}{k_2}\right) \rightarrow (E_\gamma(\Omega))^q$ to $(E_\gamma(\Omega))^q$.

5. Approximation and trace results

In this section, $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$ with the segment property and $f$ is a subinterval of $\mathbb{R}$ (both possibly unbounded) and $Q = \Omega \times I$.

It is easy to see that $Q$ also satisfies Lipschitz domain. We say that $u_n \rightarrow u$ in $W^{-1,\infty} L_\psi(Q) + L^1(Q)$ for the modular convergence if we can write

$$u_n = \sum_{|\alpha| \leq 1} D^\alpha u_n^\alpha + u_n^0 \quad \text{and} \quad u = \sum_{|\alpha| \leq 1} D^\alpha u^\alpha + u^0,$$

with $u_n^\alpha \rightarrow u^\alpha$ in $L_\psi(Q)$ for the modular convergence for all $|\alpha| \leq 1$, and $u_n^0 \rightarrow u^0$ strongly in $L^1(Q)$. We shall prove the following approximation theorem, which plays a fundamental role when the existence of solutions for parabolic problems is proved. [2] Let $\varphi$ be an Musielak-Orlicz function satisfies the assumption (4.1).

If $u \in W^{1,\infty} L_\psi(Q)$ (respectively $u \in W^{0,\infty} L_\psi(Q)$) and $\frac{\partial u}{\partial t} \in W^{-1,\infty} L_\psi(Q) + L^1(Q)$, then there exists a sequence $(v_j) \in \mathcal{D}(\Omega)$ (respectively $\mathcal{D}(\Omega)$) such that $v_j \rightarrow u$ in $W^{1,\infty} L_\psi(Q)$ and $\frac{\partial v_j}{\partial t} \rightarrow \frac{\partial u}{\partial t}$ in $W^{-1,\infty} L_\psi(Q) + L^1(Q)$ for the modular convergence.

Lemma 5.1. [2] Let $a < b \in \mathbb{R}$ and let $\Omega$ be a bounded Lipschitz domain in $\mathbb{R}^N$.

Then

$$\left\{ u \in W^{1,\infty} L_\psi(\Omega \times \{a, b]\) : \frac{\partial u}{\partial t} \in W^{-1,\infty} L_\psi(\Omega \times \{a, b]\) + L^1(\Omega \times \{a, b]\) \right\}$$

is a subset of $C[\{a, b\}, L^1(\Omega))]$.

In order to deal with the time derivative, we introduce a time mollification of a function $u \in W^{1,\infty} L_\psi(Q)$.

Thus we define, for all $\mu > 0$ and all $(x, t) \in Q$

$$u_\mu(x, t) = \int_{-\infty}^{t} \tilde{u}(x, \sigma) \exp(\mu(\sigma - t))d\sigma \quad (5.1)$$
where $\tilde{u}(x,t) = u(x,t)\chi_{[0,T]}(t)$.

Throughout the paper the index $\mathfrak{i}$ always indicates this mollification.

**Lemma 5.2.** \([\mathfrak{2}]\) If $u \in L_{\varphi}(Q)$ then $u_\mu$ is measurable in $Q$ and $\frac{\partial u_\mu}{\partial t} = \mu(u - u_\mu)$ and if $u \in K_{\varphi}(Q)$ then

$$
\int_Q \varphi(x,u_\mu)dxdt \leq \int_Q \varphi(x,u)dxdt.
$$

**Lemma 5.3.**

1. If $u \in L_{\varphi}(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $L_{\varphi}(Q)$ as $\mu \longrightarrow \infty$.

2. If $u \in W^{1,1}_0 L_{\varphi}(Q)$ then $u_\mu \longrightarrow u$ for the modular convergence in $W^{1,1}_0 L_{\varphi}(Q)$ as $\mu \longrightarrow \infty$.

**Proof**

1. Let $(v_k)_k \subset D(Q)$ such that $v_k \longrightarrow u$ in $L_{\varphi}(Q)$ for the modular convergence.

Let $\lambda > 0$ large enough such that

$$
\frac{u}{\lambda} \in K_{\varphi}(Q), \quad \int_Q \varphi\left(x,\frac{|v_k - u|}{\lambda}\right)dxdt \longrightarrow 0 \quad \text{as} \quad k \longrightarrow +\infty.
$$

On the one hand, for a.e $(x,t) \in Q$, we have

$$
\left|(v_k)_\mu(x,t) - v_k(x,t)\right| = \frac{1}{\mu} \left|\frac{\partial v_k}{\partial t}(x,t)\right| \leq \left\|\frac{\partial v_k}{\partial t}\right\|_{L^\infty(Q)}
$$

On the other hand, one has

$$
\int_Q \varphi\left(x,\frac{|u_\mu - u|}{3\lambda}\right)dxdt \leq \frac{1}{3} \int_Q \varphi\left(x,\frac{|u_\mu - (v_k)_\mu|}{\lambda}\right)dxdt
$$

$$
+ \frac{1}{3} \int_Q \varphi\left(x,\frac{|(v_k)_\mu - v_k|}{\lambda}\right)dxdt
$$

$$
+ \frac{1}{3} \int_Q \varphi\left(x,\frac{|v_k - u|}{\lambda}\right)dxdt
$$

$$
\leq \frac{1}{3} \int_Q \varphi\left(x,\frac{|u - v_k|}{\lambda}\right)dxdt
$$

$$
+ \frac{1}{3} \int_Q \varphi\left(x,\frac{|(v_k)_\mu - v_k|}{\lambda}\right)dxdt
$$

$$
+ \frac{1}{3} \int_Q \varphi\left(x,\frac{|v_k - u|}{\lambda}\right)dxdt.
$$

This implies that

$$
\int_Q \varphi\left(x,\frac{|u_\mu - u|}{3\lambda}\right)dxdt \leq \frac{2}{3} \int_Q \varphi\left(x,\frac{|v_k - u|}{\lambda}\right)dxdt
$$

$$
+ \int_Q \varphi\left(x,\frac{1}{\mu} \left|\frac{\partial v_k}{\partial t}\right| \right\|_{L^\infty(Q)}\right)dxdt.
$$
Let $\varepsilon > 0$ there exists $k_0 > 0$ such that $\forall k > k_0$, we have

$$\int_Q \varphi \left( x, \frac{|u_k - u|}{\lambda} \right) dxdt < \varepsilon$$

and there exists $\mu_0 > 0$ such that $\forall \mu > \mu_0$ and for all $k > k_0$

$$\frac{1}{\lambda \mu} \left\| \frac{\partial u_k}{\partial t} \right\|_{L^\infty(Q)} \leq 1$$

Then, we get

$$\int_Q \varphi \left( x, \frac{|u_{\mu} - u|}{3\lambda} \right) dxdt \leq \varepsilon + \frac{1}{\lambda \mu} \left\| \frac{\partial u_k}{\partial t} \right\|_{L^\infty(Q)} T \int_\Omega \varphi(x, 1) dxdt$$

Finely, by using $(iii)$ of Lemma 4.1 and by letting $\mu \to +\infty$, there exits $\mu_1 > 0$ such that

$$\int_Q \varphi \left( x, \frac{|u_{\mu} - u|}{3\lambda} \right) dxdt \leq \varepsilon, \quad \text{for all } \mu > \mu_1.$$ 

2. Since for all indice $\alpha$ such that $|\alpha| \leq 1$, we have $D_x^\alpha(u_{\mu}) = (D_x^\alpha u)_{\mu}$, consequently, the first part above applied on each $D_x^\alpha u$, gives the result.

\[ \square \]

**Remark 5.1.** If $u \in E^\phi(Q)$, we can choose $\lambda$ arbitrary small since $\mathcal{D}(Q)$ is (norm) dense in $E^\phi(Q)$.

Thus, for all $\lambda > 0$, we have

$$\int_Q \varphi \left( x, \frac{|u_{\mu} - u|}{\lambda} \right) dxdt \quad \text{as } \mu \to +\infty.$$

and $u_{\mu} \to u$ strongly in $E^\phi(Q)$,Idem for $W^{1,\infty}E^\phi(Q)$.

**Lemma 5.4.** If $u_n \to u$ in $W^{1,\infty}_0 L^\phi(Q)$ strongly (resp., for the modular convergence), then $(u_{\mu})_{\mu} \to u_{\mu}$ strongly (resp., for the modular convergence).

**Proof** For all $\lambda > 0$ (resp., for some $\lambda > 0$),

$$\int_Q \varphi \left( x, \frac{|D_x^\phi((u_n))_{\mu} - D_x^\phi(u)_{\mu}|}{\lambda} \right) dxdt \to \int_Q \varphi \left( x, \frac{|D_x^\phi u_n - D_x^\phi u|}{\lambda} \right) dxdt \to 0,$$

as $n \to +\infty$. Then $(u_{n})_{\mu} \to u_{\mu}$ in $W^{1,\infty}L^\phi(Q)$ strongly (resp., for the modular convergence). \[ \square \]
6. Compactness Results

For each \( h > 0 \), define the usual translated \( \tau_h f \) of the function \( f \) by \( \tau_h f(t) = f(t + h) \).

If \( f \) is defined on \([0, T]\) then \( \tau_h f \) is defined on \([-h, T - h]\).

First of all, recall the following compactness results proved by the authors in [2].

**Lemma 6.1.** Let \( \varphi \) be a Musielak function. Let \( Y \) be a Banach space such that the following continuous imbedding holds \( L^1(\Omega) \subset Y \). Then for all \( \varepsilon > 0 \) and all \( \lambda > 0 \), there is \( C_\varepsilon > 0 \) such that for all \( u \in W_0^1, L^1(\varphi, Q) \) with \( \frac{|\nabla u|}{\lambda} \in K_\varphi(Q) \), we have

\[
\|u\|_1 \leq \varepsilon \lambda \left( \int_Q \varphi(x, \frac{|\nabla u|}{\lambda}) \, dx \right) dt + C_\varepsilon \|u\|_{L^1(0, T; Y)}.
\]

**Proof** Since \( W_0^1, L^1(\varphi, \Omega) \subset L^1(\varphi) \) with compact imbedding, then for all \( \varepsilon > 0 \), there is \( C_\varepsilon > 0 \) such that for all \( v \in W_0^1, L^1(\varphi, \Omega) \)

\[
\|v\|_{L^1(\Omega)} \leq \varepsilon \|\nabla u\|_{L^1(\varphi, \Omega)} + C_\varepsilon \|v\|_Y. \tag{6.1}
\]

Indeed, if the above assertion holds false, there is \( \varepsilon_0 > 0 \) and \( v_n \in W_0^1, L^1(\varphi, \Omega) \) such that

\[
\|v_n\|_{L^1(\Omega)} \geq \varepsilon_0 \|\nabla v_n\|_{L^1(\varphi, \Omega)} + n \|v_n\|_Y.
\]

This gives, by setting \( w_n = \frac{\nabla v_n}{\|\nabla v_n\|_{L^1(\varphi, \Omega)}} \),

\[
\|w_n\|_{L^1(\Omega)} \geq \varepsilon_0 + n \|w_n\|_Y, \quad \|\nabla w_n\|_{L^1(\varphi, \Omega)} = 1.
\]

Since \((w_n)_n\) is bounded in \( W_0^1, L^1(\varphi, \Omega) \) then for a subsequence

\[
w_n \to w \text{ in } W_0^1, L^1(\varphi, \Omega) \text{ for } \sigma(\Pi L_\varphi, \Pi E_\varphi) \text{ and strongly in } L^1(\Omega).
\]

Thus, \( \|w_n\|_{L^1(\Omega)} \) is bounded and \( \|w_n\|_Y \to 0 \) as \( n \to +\infty \).

We conclude \( w_n \to 0 \) in \( Y \) and that \( w = 0 \) implying that \( \varepsilon_0 \leq \|w_n\|_{L^1(\Omega)} \to 0 \), a contradiction.

Using \( v = u(t) \) in (6.1) for all \( u \in W_0^1, L^1(\varphi, Q) \) with \( \frac{|\nabla u|}{\lambda} \in K_\varphi(Q) \) and a.e. \( t \in [0, T] \), we have

\[
\|u(t)\|_{L^1(\Omega)} \leq \varepsilon \|\nabla u(t)\|_{L^1(\varphi)} + C_\varepsilon \|u(t)\|_Y.
\]

Since \( \int_Q \varphi(x, \frac{|\nabla u(x, t)|}{\lambda}) \, dx < \infty \), we have thanks to Fubini’s theorem

\[
\int_\Omega \varphi(x, \frac{|\nabla u(x, t)|}{\lambda}) \, dx < \infty \text{ for a.e. } t \in [0, T] \text{ and then}
\]

\[
\|\nabla u(t)\|_{L^1(\varphi)} \leq \lambda \left( \int_\Omega \varphi(x, \frac{|\nabla u(x, t)|}{\lambda}) \, dx + 1 \right),
\]

which implies that

\[
\|u(t)\|_{L^1(\varphi, \Omega)} \leq \varepsilon \lambda \left( \int_\Omega \varphi(x, \frac{|\nabla u(x, t)|}{\lambda}) \, dx + 1 \right) + C_\varepsilon \|u(t)\|_Y.
\]
Integrating this over \([0,T]\) yields

\[
\|u\|_1 \leq \varepsilon \lambda \left( \int_Q \varphi(x, \frac{\left| \nabla u \right|}{\lambda}) \, dx \, dt + T \right) + C \|u\|_{L^1(0,T,Y)}.
\]

We also prove the following lemma which allows us to enlarge the space \(Y\) whenever necessary.

**Lemma 6.2.** If \(F\) is bounded in \(W^{1,x}_0 L\varphi(Q)\) and is relatively compact in \(L^1(0,T,Y)\) then \(F\) is relatively compact in \(L^1(Q)\) (and also in \(E_\gamma(Q)\) for all Musielak function \(\gamma \ll \varphi\)).

**Proof** Let \(\varepsilon > 0\) be given. Let \(C > 0\) be such that \(\int_Q \varphi(x, \frac{\left| \nabla f \right|}{C}) \, dx \, dt \leq 1\) for all \(f \in F\).

By the previous lemma, there exists \(C_\varepsilon > 0\) such that for all \(u \in W^{1,x}_0 L\varphi(Q)\) with \(\frac{\left| \nabla u \right|}{C} \in K\varphi(Q)\),

\[
\|u\|_{L^1(Q)} \leq \frac{2\varepsilon C}{4C(1+T)} \left( \int_Q \varphi(x, \frac{\left| \nabla u \right|}{2C}) \, dx \, dt + C \|u\|_{L^1(0,T,Y)} \right).
\]

Moreover, there exists a finite sequence \((f_i)_i\) in \(F\) satisfying

\[
\forall f \in F, \exists f_i \text{ such that } \|f - f_i\|_{L^1(0,T,Y)} \leq \frac{\varepsilon}{2C\varepsilon}.
\]

So that,

\[
\|f - f_i\|_{L^1(Q)} \leq \frac{\varepsilon}{2(1+T)} \left( \int_Q \varphi(x, \frac{\left| \nabla f - \nabla f_i \right|}{2C}) \, dx \, dt + T \right) + C \|f - f_i\|_{L^1(0,T,Y)} \leq \varepsilon.
\]

and hence \(F\) is relatively compact in \(L^1(Q)\).

Since \(\gamma \ll \varphi\) then by using Vitali’s theorem, it is easy to see that \(F\) is relatively compact in \(E_\gamma(Q)\). \(\square\)

**Remark 6.1.** If \(F \subset L^1(0,T,B)\) is such that \(\left\{ \frac{\partial f}{\partial t} : f \in F \right\}\) is bounded in \(F \subset L^1(0,T,B)\) then \(\|\tau_h f - f\|_{L^1(0,T,B)} \rightarrow 0\) as \(h \rightarrow 0\) uniformly with respect to \(f \in F\).

**Lemma 6.3.** Let \(\varphi\) be a Musielak function. If \(F\) is bounded in \(W^{1,x} L\varphi(Q)\) and \(\left\{ \frac{\partial f}{\partial t} : f \in F \right\}\) is bounded in \(W^{-1,x} L\varphi(Q)\), then \(F\) is relatively compact in \(L^1(Q)\).

**Proof** Let \(\gamma\) and \(\theta\) be Musielak functions such that \(\gamma \ll \varphi\) and \(\theta \ll \varphi\) near infinity.
For all $0 < t_1 < t_2 < T$ and all $f \in F$, we have
\[
\| \int_{t_1}^{t_2} f(t) dt \|_{W^{1,\infty}_0 E_\gamma(\Omega)} \leq \int_0^T \| f(t) \|_{W^{1,\infty}_0 E_\gamma(\Omega)} dt \\
\leq C_1 \| f \|_{W^{1,\infty}_0 E_\gamma(\Omega)} \\
\leq C_2 \| f \|_{W^{1,\infty}_0 E_\gamma(\Omega)} \\
\leq C.
\]
where we have used the following continuous imbedding
\[
W^{1,\infty}_0 E_\gamma(\Omega) \subset W^{1,\infty}_0 E_\gamma(\Omega) \subset L^1(0, T, W^{1,\infty}_0 E_\gamma(\Omega)).
\]
Since the imbedding $W^{1,\infty}_0 E_\gamma(\Omega) \subset L^1(0, T, W^{1,\infty}_0 E_\gamma(\Omega))$ is compact we deduce that $(f(t))_{t \in F}$ is relatively compact in $L^1(\Omega)$ and $W^{-1,1}(\Omega)$ as well.

On the other hand, \( \{ \partial f \partial t : f \in F \} \) is bounded in $W^{-1,1}_0 E_\gamma(Q)$ and $L^1(0, T, W^{-1,1}_0 E_\gamma(\Omega))$ as well, since
\[
W^{-1,1}_0 E_\gamma(Q) \subset W^{-1,1}_0 E_\gamma(Q) \subset L^1(0, T, W^{-1,1}_0 E_\gamma(\Omega)) \subset L^1(0, T, W^{-1,1}_0 E_\gamma(\Omega)),
\]
with continuous imbedding. By Remark 3 of [12], we deduce that $\| \tau h f - f \|_{L^1(0, T, W^{-1,1}_0 E_\gamma(\Omega))} \rightarrow 0$ uniformly in $f \in F$ when $h \rightarrow + \infty$ and by using Theorem 2 of [12], $F$ is relatively compact in $L^1(0, T, W^{-1,1}_0 E_\gamma(\Omega))$. Since $L^1(\Omega) \subset W^{-1,1}_0 E_\gamma(\Omega)$ with continuous imbedding we can apply Lemma 6.2 to conclude that $F$ is relatively compact in $L^1(Q)$.

Lemma 6.4. Let $\varphi$ be a Musielak function.

Let $(u_n)$ be a sequence of $W^{1,\infty}_0 L_\varphi(Q)$ such that
\[
u_n \rightharpoonup u \text{ weakly in } W^{1,\infty}_0 L_\varphi(Q) \text{ for } \sigma(PL_\varphi, PL_\psi)
\]

and
\[
\frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } D'(Q)
\]

with $(h_n)_n$ bounded in $W^{-1,\infty}_0 L_\psi(Q)$ and $(k_n)_n$ bounded in the space $M(Q)$ set of measures on $Q$.

then $u_n \rightarrow u$ strongly in $L^1_\text{loc}(Q)$.

If further $u_n \in W^{1,\infty}_0 L_\varphi(Q)$ then $u_n \rightarrow u$ strongly in $L^1(Q)$.

Proof It is easily adapted from that given in [8] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [20].

7. Main results

For $k > 0$ we define the truncation at height $k$: $T_k : \mathbb{R} \rightarrow \mathbb{R}$ by:
\[
T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  \frac{s}{|s|} & \text{if } |s| > k.
\end{cases}
\]
We note also
\[ S_k(r) = \int_0^r T_k(\sigma)d\sigma = \begin{cases} \frac{r^2}{2} & \text{if } |r| \leq k, \\ k|r| - \frac{r^2}{2} & \text{if } |r| > k. \end{cases} \] (7.2)

We define
\[ T_0^1,\psi(Q) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1,\infty}_0(Q) \forall k > 0 \right\} \]

We consider the following boundary value problem
\[ (P) \begin{cases} \frac{\partial u}{\partial t} + A(u) + g(x,t,u,\nabla u) = f - \text{div}(F) & \text{in } Q, \\ u \equiv 0 & \text{on } \partial Q = \partial \Omega \times [0,T], \\ u(.,0) = u_0 & \text{on } \Omega. \end{cases} \]

We will prove the following existence theorem.

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \), \( \varphi \) and \( \psi \) be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.1 and \( \varphi(x,t) \) decreases with respect to one of coordinate of \( x \), we assume also that (3.1)-(3.6) and (3.7) hold true. Then the problem \((P)\) has at least one entropy solution of the following sense
\[ \begin{align*}
&\{ u \in T_0^{1,\varphi}(Q) \cap W_0^{1,\infty}(Q), S_k(u) \in L^1(Q), g(.,u,\nabla u) \in L^1(Q) \\
&\int_\Omega S_k(u(T) - v(T))dx + \int_\Omega \langle \frac{\partial \psi}{\partial t}, T_k(u - v) \rangle + \int_\Omega a(x,t,u,\nabla u) \cdot \nabla T_k(u - v)dxdt \\
&+ \int_\Omega g(x,t,u,\nabla u)T_k(u - v)dxdt \\
&\leq \int_\Omega fT_k(u - v)dxdt + \int_\Omega F \cdot \nabla T_k(u - v)dxdt + \int_\Omega S_k(u_0 - v(0))dx \\
&\forall v \in W^{1,\infty}_0(Q) \cap L^\infty(Q) \text{ such that } \frac{\partial v}{\partial t} \in W^{-1,\infty}_0(Q) + L^1(Q). \}
\]

**Proof**

**Step 1 : Approximate problems**

Consider the following approximate problem
\[ (P_n) \begin{cases} u_n \in W_0^{1,\varphi}(Q), \quad u_n(.,0) = u_{0n} \text{ in } \partial Q = \partial \Omega \times [0,T], \\
\frac{\partial u_n}{\partial t} - \text{div}(a(x,t,u_n,\nabla u_n)) + g_n(x,t,u_n,\nabla u_n) = f_n - \text{div}(F) & \text{in } Q, \end{cases} \]

where we have set \( g_n(x,t,s,\xi) = T_n(g(x,t,s,\xi)) \). Moreover, the sequence \((f_n) \subset \mathcal{D}(Q)\) is such that \( f_n \to f \) strongly in \( L^1(Q) \) and \( \|f_n\|_{L^1(Q)} \leq \|f\|_{L^1(Q)} \) and \((u_{0n}) \subset \mathcal{D}(\Omega)\) is such that \( u_{0n} \to u_0 \) strongly in \( L^1(\Omega) \) and \( \|u_{0n}\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)} \). Thanks to theorem 5.1 of [2], there exists at least one solution \( u_n \) of problem \((P_n)\).
**Step 2 : A priori estimates**

In this section we denote by $c_i$, $i = 1, 2, \ldots$ a constants not depends on $k$ and $n$.

For $k > 0$, consider the test function $T_k(u_n)$ in $(\mathbb{P}_n)$, we have

$$
\int_{Q} \frac{\partial u_n}{\partial t} T_k(u_n) dx dt + \int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \\
+ \int_{Q} g_n(x, t, u_n, \nabla u_n) T_k(u_n) dx dt = \int_{Q} f_n T_k(u_n) dx dt + \int_{Q} F \cdot \nabla T_k(u_n) dx dt \\
\leq \|f\|_{L^1(Q)} k + \int_{Q} F \cdot \nabla T_k(u_n) dx dt.
$$

(7.3)

On the one hand, let $0 < p < \min(\alpha, 1)$, (where $\alpha$ is the constant of (3.3)), then by using the Young’s inequality, we have

$$
\int_{Q} F \cdot \nabla T_k(u_n) dx dt = \int_{Q} \frac{1}{p} F \cdot p \nabla T_k(u_n) dx dt \\
\leq \int_{Q} \psi \left( x, \frac{1}{p} |F| \right) dx dt + p \int_{Q} \varphi \left( x, |\nabla T_k(u_n)| \right) dx dt.
$$

(7.4)

Combining (7.3) and (7.4), we obtain

$$
\int_{Q} \frac{\partial u_n}{\partial t} T_k(u_n) dx dt + \int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \\
+ \int_{Q} g_n(x, t, u_n, \nabla u_n) T_k(u_n) dx dt \leq c_1 k + c_2 + p \int_{Q} \varphi \left( x, |\nabla T_k(u_n)| \right) dx dt.
$$

(7.5)

Using now (3.5) and (3.3) which implies that

$$
\int_{Q} \frac{\partial u_n}{\partial t} T_k(u_n) dx dt + \frac{\alpha - p}{\alpha} \int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \leq c_1 k + c_2.
$$

(7.6)

In other hand, the first term of the left hand side of the last inequality, reads as

$$
\int_{Q} \frac{\partial u_n}{\partial t} T_k(u_n) dx dt = \int_{\Omega} S_k(u_n(T)) dx - \int_{\Omega} S_k(u_{n0}) dx,
$$

Hence

$$
\int_{\Omega} S_k(u_n(T)) dx + \frac{\alpha - p}{\alpha} \int_{Q} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \leq c_1 k + c_2 + \int_{\Omega} S_k(u_{n0}) dx.
$$
Using the fact that $S_k(\sigma) > 0$, $|S_k(u_{0n})| \leq k|u_{0n}|$, then (7.6) can be written as

$$\frac{\alpha - p}{\alpha} \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdot \nabla T_k(u_n) dx dt \leq c_3 k + c_2. \quad (7.7)$$

Hence by using (3.3), we have

$$\int_Q \varphi(x, |\nabla T_k(u_n)|) dx dt \leq c_4 k + c_5.$$  

By using the Lemma 4.5, we have

$$\int_Q \varphi(x, |T_k(u_n)|) dx \leq \int_Q \varphi(x, 1/\lambda |T_k(u_n)|) dx \leq c_4 k + c_5,$$

where $c$ is the constant of Lemma 4.5.

Then $(T_k(u_n))_n$ and $(\nabla T_k(u_n))_n$ are bounded in $L^\infty(\Omega)$, hence $(T_k(u_n))_n$ is bounded in $W^{1,0}_\infty L^\infty(\Omega)$, there exist some $v_k \in W^{1,0}_\infty L^\infty(\Omega)$ such that

\begin{align*}
& T_k(u_n) \rightharpoonup v_k \text{ weakly in } W^{1,0}_\infty L^\infty(\Omega) \text{ for } \sigma(\Pi L^\phi, \Pi E^\psi) \\
& T_k(u_n) \rightarrow v_k \text{ strongly in } E^\phi(\Omega). 
\end{align*}

(7.9)

**Step 3 : Convergence in measure of $(u_n)_n$**

Let $k > 0$ large enough, by using (7.8), we have

$$\text{meas} \{|u_n| > k\} \leq \frac{1}{\inf_{x \in \Omega} \varphi(x, k/\lambda)} \int_{\{|u_n| > k\}} \varphi(x, 1/\lambda) dx dt$$

$$\leq \frac{1}{\inf_{x \in \Omega} \varphi(x, k/\lambda)} \int_Q \varphi(x, 1/\lambda |T_k(u_n)|) dx dt$$

$$\leq \frac{c_4 k + c_5}{\inf_{x \in \Omega} \varphi(x, k/\lambda)} \forall n, \forall k \geq 0.$$  

Where $c_4$ is a constant not dependent on $k$, hence

$$\text{meas} \{|u_n| > k\} \leq \frac{c_4 k + c_5}{\inf_{x \in \Omega} \varphi(x, k/\lambda)} \rightarrow 0 \text{ as } k \rightarrow \infty.$$  

For every $\lambda > 0$ we have

$$\text{meas} \{|u_n - u_m| > \lambda\} \leq \text{meas} \{|u_n| > k\} \quad + \quad \text{meas} \{|u_m| > k\}$$

$$+ \quad \text{meas} \{|T_k(u_n) - T_k(u_m)| > \lambda\}. \quad (7.10)$$

Consequently, by (7.8) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in $Q$.

Let $\epsilon > 0$, then by (7.10) there exists some $k = k(\epsilon) > 0$ such that

$$\text{meas} \{|u_n - u_m| > \lambda\} < \epsilon, \quad \text{for all } n, m \geq h_0(k(\epsilon), \lambda).$$
Which means that \((u_n)_n\) is a Cauchy sequence in measure in \(Q\), thus converge almost every where to some measurable functions \(u\). Then
\[
\left\{\begin{array}{l}
T_k(u_n) \rightharpoonup T_k(u) \quad \text{weakly in } W^{1,1}_0(Q) \\
T_k(u_n) \to T_k(u) \quad \text{strongly in } E^\varphi(Q).
\end{array}\right.
\] (7.11)

**Step 4 : Boundedness of** \((u(\cdot, \cdot), T_k(u_n); \nabla T_k(u_n))_n \text{ in } (L^\varphi(Q))^N\)

Let \(w \in (E^\varphi(Q))^N\) be arbitrary such that \(\|w\|_{E; Q} \leq 1\), by (3.2) we have
\[
\left( a(x, t, T_k(u_n)), \nabla T_k(u_n) \right) - a(x, t, T_k(u_n), \frac{w}{\nu})((\nabla T_k(u_n) - \frac{w}{\nu}) > 0.
\]

hence
\[
\int_Q a(x, t, T_k(u_n)), \nabla T_k(u_n) \frac{w}{\nu} \, dxdt \leq \int_Q a(x, t, T_k(u_n)), \nabla T_k(u_n) \nabla T_k(u_n) \, dxdt
\]
\[
- \int_Q a(x, t, T_k(u_n), \frac{w}{\nu})(\nabla T_k(u_n) - \frac{w}{\nu}) \, dxdt.
\]

Thanks to (7.7), we have
\[
\int_Q a(x, t, T_k(u_n)), \nabla T_k(u_n) \nabla T_k(u_n) \, dxdt \leq c_3 k + c_2.
\]

On the other hand, for \(\lambda\) large enough \((\lambda > \beta)\), we have by using (3.1).
\[
\int_Q \psi_x \left( \frac{a(x, t, T_k(u_n), \frac{w}{\nu})}{3\lambda} \right) \, dxdt
\]
\[
\leq \frac{\beta}{3\lambda} \int_Q \psi_x \left( \frac{h_1(x, t) + \psi^{-1}_x(\gamma(x, \nu|T_k(u_n)|)) + \psi^{-1}_x(\varphi(x, |w|))}{3} \right) \, dxdt
\]
\[
\leq \frac{\beta}{3\lambda} \left( \int_Q \psi_x(h_1(x, t)) \, dxdt + \int_Q \varphi(x, |T_k(u_n)|) \, dxdt + \int_Q \varphi(x, |w|) \, dxdt \right)
\]
\[
\leq \frac{\beta}{3\lambda} \left( \int_Q \psi_x(h_1(x, t)) \, dxdt + \int_Q \gamma(x, \nu k) \, dxdt + \int_Q \varphi(x, |w|) \, dxdt \right).
\]

Now, since \(\gamma\) grows essentially less rapidly than \(\varphi\) near infinity ad by using the Remark 2.1, there exists \(r(k) > 0\) such that \(\gamma(x, \nu k) \leq r(k) \varphi(x, 1)\) and so we have
\[
\int_Q \psi_x \left( \frac{a(x, t, T_k(u_n), \frac{w}{\nu})}{3\lambda} \right) \, dxdt
\]
\[
\leq \frac{\beta}{3\lambda} \left( \int_Q \psi_x(h_1(x, t)) \, dxdt + r(k) \int_Q \varphi(x, 1) \, dxdt + \int_Q \varphi(x, |w|) \, dxdt \right).
hence \(a(x, t, T_k(u_n), \frac{\alpha_j^k}{\alpha_j^k})\) is bounded in \((L^\psi(Q))^N\). Which implies that second term of the right hand side of (7.12) is bounded, consequently we obtain

\[
\int_Q a(x, t, T_k(u_n), \nabla T_k(u_n))w dx dt \leq c_0(k), \quad \text{for all } w \in (L^\psi(Q))^N \text{ with } \|w\|_{\psi, Q} \leq 1.
\]

Hence by the theorem of Banach Steinhous the sequence \((a(x, t, T_k(u_n), \nabla T_k(u_n)))_n\) remains bounded in \((L^\psi(Q))^N\). Which implies that, for all \(k > 0\) there exists a function \(h_k \in (L^\psi(Q))^N\) such that

\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup h_k \text{ weakly-star in } (L^\psi(Q))^N \text{ for } \sigma(\Pi L_{\psi}, \Pi E \varphi),
\]

(7.13)

Step 5: Modular convergence of truncations

For the sake of simplicity, we will write only \(\varepsilon(n, j, \mu, s)\) to mean all quantities (possibly different) such that

\[
\lim_{n \to +\infty} \lim_{j \to +\infty} \lim_{\mu \to +\infty} \lim_{s \to +\infty} \varepsilon(n, j, \mu, s) = 0.
\]

Since \(T_k(u) \in W_0^{1, x} L^\varphi(Q)\) then there exists a sequence \((\alpha_j^k) \subset D(Q)\) such that \((\alpha_j^k) \rightharpoonup T_k(u)\) for the modular convergence in \(W_0^{1, x} L^\varphi(Q)\). For the remaining of this article, \(\chi_j\) and \(\chi_{j, s}\) will denoted respectively the characteristic functions of the sets \(Q_\alpha = \{(x, t) \in Q : |\nabla T_k(u, x, t)| \leq s\}\) and \(Q_{j, s} = \{(x, t) \in Q : |\nabla T_k(u, x, t)| \leq s\}\). Taking now \(T_\eta(u_n - T_k(\alpha_j^k))\) as test function in \(\mathcal{P}_n\), we get

\[
\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_j^k)) dx dt + \int_Q a(x, t, u_n, \nabla u_n) \cdot \nabla T_\eta(u_n - T_k(\alpha_j^k)) dx dt + \int_Q g_n(x, t, u_n, \nabla u_n) T_\eta(u_n - T_k(\alpha_j^k)) dx dt \leq \|f\|_1 \eta + \int_{\{|T_\eta(u_n - T_k(\alpha_j^k))| < \eta\}} |F| \cdot \nabla T_\eta(u_n - T_k(\alpha_j^k)) dx dt.
\]

Let \(0 < p < \min(1, \alpha)\), by Young’s inequality, we have

\[
\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_j^k)) dx dt + \int_Q a(x, t, u_n, \nabla u_n) \cdot \nabla T_\eta(u_n - T_k(\alpha_j^k)) dx dt + \int_Q g_n(x, t, u_n, \nabla u_n) T_\eta(u_n - T_k(\alpha_j^k)) dx dt \leq \|f\|_1 \eta + \int_{\{|T_\eta(u_n - T_k(\alpha_j^k))| < \eta\}} \psi(x, |F|) dx dt + p \int_Q \varphi(x, |\nabla T_\eta(u_n - T_k(\alpha_j^k))|) dx dt.
\]
Using now (3.3) on the last term of the last inequality, we get
\[
\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt + \int_Q a(x, t, u_n, \nabla u_n) \cdot \nabla T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt \\
+ \int_Q g_n(x, t, u_n) T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt \\
\leq \|J\|_p \eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_{\mu})| < \eta\}} \psi(x, \frac{|F|}{p}) dx dt \\
+ \frac{p}{\alpha} \int_Q a(x, t, T_{k+\eta}(u_n), \nabla T_{k+\eta}(u_n)) \nabla u_n dx dt.
\]

Which implies that,
\[
\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt + \frac{\alpha - p}{\alpha} \int_Q a(x, t, T_k+\eta(u_n), \nabla T_{k+\eta}(u_n)) \nabla u_n dx dt \\
+ \int_Q g_n(x, t, u_n, \nabla u_n) T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt \\
\leq c_1 \eta + \int_{\{|T_\eta(u_n - T_k(\alpha_k^j)_{\mu})| < \eta\}} \psi(x, \frac{|F|}{p}) dx dt.
\]

The first term of the left hand side of the last equality reads as
\[
\int_Q \frac{\partial u_n}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt = \int_Q \left( \frac{\partial u_n}{\partial t} - \frac{\partial T_\eta}{\partial t}(\alpha_k^j)_{\mu} \right) T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt \\
+ \int_Q \frac{\partial T_k(\alpha_k^j)_{\mu}}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt.
\]

The second term of the last equality can be easily to see that is positive and the third term can be written as
\[
\int_Q \frac{\partial T_k(\alpha_k^j)_{\mu}}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt = \mu \int_Q (T_k(\alpha_k^j) - T_k(\alpha_k^j)_{\mu}) T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt,
\]

thus by letting \( n, j \rightarrow +\infty \), and since \( (\alpha_k^j) \rightarrow T_k(u) \) a.e. \( \text{in } Q \) and by using Lebesgue Theorem,
\[
\int_Q (T_k(\alpha_k^j) - T_k(\alpha_k^j)_{\mu}) T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt = \int_Q (T_k(u) - T_k(u)_{\mu}) \cdots \\
\cdots T_\eta(u - T_k(u)_{\mu}) dx dt + \varepsilon(n, j).
\]

Consequently
\[
\int_Q \frac{\partial T_k(\alpha_k^j)_{\mu}}{\partial t} T_\eta(u_n - T_k(\alpha_k^j)_{\mu}) dx dt \geq \varepsilon(n, j).
\]
Then, (7.14) can be written as

\[
\frac{\alpha - p}{\alpha} \int_Q a(x, t, u_n, \nabla u_n) T_\eta (u_n - T_k(\alpha_k^j)_\mu) \, dx \, dt \\
+ \int_Q g_n(x, t, u_n, \nabla u_n) T_\eta (u_n - T_k(\alpha_k^j)_\mu) \, dx \, dt \leq c_1 \eta \\
+ \int_{\{|T_\eta (u_n - T_k(\alpha_k^j)_\mu)| < \eta\}} \psi(x, \frac{|F|}{p}) \, dx \, dt + \varepsilon(n, j).
\] (7.16)

On the other hand,

\[
\int_Q a(x, t, u_n, \nabla u_n) T_\eta (u_n - T_k(\alpha_k^j)_\mu) \, dx \, dt \\
= \int_{\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, T_k(\alpha_k^j)_\mu, \nabla T_k(\alpha_k^j)_\mu) (\nabla T_k(\alpha_k^j)_\mu - \nabla T_k(\alpha_k^j)_\mu \chi_{|\nabla T_k(\alpha_k^j)| > s}) \, dx \, dt \\
+ \int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dx \, dt \\
- \int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, u_n, \nabla T_k(\alpha_k^j)_\mu \chi_{|\nabla T_k(\alpha_k^j)| > s}) \, dx \, dt
\]

Thus, by using the fact that

\[
\int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \, dx \, dt \geq 0
\]

We have

\[
\frac{\alpha - p}{\alpha} \int_{\{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, T_k(\alpha_k^j)_\mu, \nabla T_k(\alpha_k^j)_\mu) (\nabla T_k(\alpha_k^j)_\mu - \nabla T_k(\alpha_k^j)_\mu \chi_{|\nabla T_k(\alpha_k^j)| > s}) \, dx \, dt \\
+ \int_Q g_n(x, t, u_n, \nabla u_n) T_\eta (u_n - T_k(\alpha_k^j)_\mu) \, dx \, dt \\
\leq c_1 \eta + \int_{\{|T_\eta (u_n - T_k(\alpha_k^j)_\mu)| < \eta\}} \psi(x, \frac{|F|}{p}) \, dx \, dt \\
+ \frac{\alpha - p}{\alpha} \int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha_k^j)_\mu| < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_k^j)_\mu \chi_{|\nabla T_k(\alpha_k^j)| > s}) \, dx \, dt \\
+ \varepsilon(n, j)
\] (7.17)

Now, using (3.5) and the fact that \(T_\eta (u_n - T_k(\alpha_k^j)_\mu)\) has the same sign of \(u_n\) on
the set \( \{|u_n| > k\} \), we get

\[
\frac{\alpha - p}{\alpha} \int_{\{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_n \mu \chi_{\left\{|\nabla T_k(\alpha_k^j)_n|>s\}\right\}) dx \, dt
\]

\[
+ \int_{\{|u_n| \leq k\}} g_n(x, t, u_n, \nabla u_n) T_\eta(u_n - T_k(\alpha_k^j)_n) dx \, dt
\]

\[
\leq c_1 \eta + \int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_n)|<\eta\}} \psi(x, \frac{|F|}{p}) dx \, dt
\]

\[
+ \frac{\alpha - p}{\alpha} \int_{\{|u_n| > k\} \cap \{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k) \mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx \, dt
\]

\[
+ \epsilon(n, j)
\]

(7.18)

Hence, by using (3.4), we get

\[
\frac{\alpha - p}{\alpha} \int_{\{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_n \mu \chi_{\left\{|\nabla T_k(\alpha_k^j)_n|>s\}\right\}) dx \, dt
\]

\[
\leq c_1 \eta + \int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_n)|<\eta\}} \psi(x, \frac{|F|}{p}) dx \, dt
\]

\[
+ \frac{\alpha - p}{\alpha} \int_{\{|u_n| > k\} \cap \{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k) \mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx \, dt
\]

\[
+ \epsilon(n, j)
\]

\[
+ \int_{\{|u_n| \leq k\}} b_k \left( h_2(x, t) + \varphi(x, |\nabla T_k(u_n)|) \right) |T_\eta(u_n - T_k(\alpha_k^j)_n)| dx \, dt,
\]

(7.19)

where \( b_k = \sup \{ b(s) : |s| \leq k \} \).

Using now (7.8), there exists a constant \( c_3 > 0 \) depends on \( k \) such that

\[
\int_{\{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha_k^j)_n \mu \chi_{\left\{|\nabla T_k(\alpha_k^j)_n|>s\}\right\}) dx \, dt
\]

\[
\leq c_3 \eta + \int_{\{|T_\eta(u_n-T_k(\alpha_k^j)_n)|<\eta\}} \psi(x, \frac{|F|}{p}) dx \, dt
\]

\[
+ \int_{\{|u_n| > k\} \cap \{|u_n-T_k(\alpha_k^j)_n|<\eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha_j^k) \mu \chi_{\{|\nabla T_k(\alpha_j^k)|>s\}} dx \, dt
\]

\[
+ \epsilon(n, j).
\]

(7.20)

Since \( a(x, t, T_k+\eta(u_n), \nabla T_k+\eta(u_n)) \to h_k+\eta \) weakly-star in \( (L_\psi(Q))^{\mathcal{N}} \) for \( \sigma(\Pi L_\psi, \Pi E_\varphi) \),
then
\[
\int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha^k_j) \mu \chi(|\nabla T_k(\alpha^k_j)| > s) \, dx \, dt
\]
\[
= \int_{\{|u| > k\} \cap \{|u - T_k(\alpha^j)\}_\mu < \eta\}} h_{k+n} \cdot \nabla T_k(\alpha^k_j) \mu \chi(|\nabla T_k(\alpha^k_j)| > s) \, dx \, dt + \varepsilon(n).
\]
Now, letting \( j \) to infinity, we obtain
\[
\int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha^k_j) \mu \chi(|\nabla T_k(\alpha^k_j)| > s) \, dx \, dt
\]
\[
= \int_{\{|u| > k\} \cap \{|u - T_k(u)\}_\mu < \eta\}} h_{k+n} \cdot \nabla T_k(u) \mu \chi(|\nabla T_k(u)| > s) \, dx \, dt + \varepsilon(n, j).
\]
Hence, we get
\[
\int_{\{|u_n| > k\} \cap \{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(\alpha^k_j) \mu \chi(|\nabla T_k(\alpha^k_j)| > s) \, dx \, dt
\]
\[
= \int_{\{|u| > k\} \cap \{|u - T_k(u)\}_\mu < \eta\}} h_{k+n} \cdot \nabla T_k(u) \mu \chi(|\nabla T_k(u)| > s) \, dx \, dt + \varepsilon(n, j, \mu)
\]
Then (7.20) becomes
\[
\int_{\{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha^k_j) \mu \chi_{j,s}) \, dx \, dt
\]
\[
\leq c_3 \eta + \int_{\{|T_k(u_n - T_k(\alpha^j)\}_\mu < \eta\}} \psi(x, |F|) \, dx \, dt + \varepsilon(n, j, \mu, s).
\] (7.21)
On the other hand, remark that
\[
\int_{\{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha^k_j) \mu \chi_{j,s}) \, dx \, dt
\]
\[
= \int_{\{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(u_n) - \nabla T_k(\alpha^k_j) \mu \chi_{j,s}) \, dx \, dt
\]
\[
+ \int_{\{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) \cdots
\]
\[
\cdots (\nabla T_k(\alpha^k_j) \mu \chi_{j,s} - \nabla T_k(\alpha^k_j) \mu \chi_{j,s}) \, dx \, dt
\] (7.22)
for the second term of the last inequality, we have obviously that
\[
\int_{\{|u_n - T_k(\alpha^j)\}_\mu < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n)) (\nabla T_k(\alpha^k_j) - \nabla T_k(\alpha^k_j) \mu \chi_{j,s}) \, dx \, dt
\]
\[
= \varepsilon(n, j, \mu, s).
\]
Then (7.21) becomes
\[
\int_{\{u_n - T_k(\alpha^k_n, u) < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\alpha^k_n)\chi_{j,s})dxdt \\
\leq c\eta + \int_{\{T_k(u_n - T_k(\alpha^k_n, u)) < \eta\}} \psi(x, \frac{|F|}{p})dxdt + \varepsilon(n, j, \mu, s).
\tag{7.23}
\]

Hence by letting \(\eta\) to zero, we get
\[
\int_{\{u_n - T_k(\alpha^k_n, u) < \eta\}} a(x, t, T_k(u_n), \nabla T_k(u_n))(\nabla T_k(u_n) - \nabla T_k(\alpha^k_n)\chi_{j,s})dxdt \\
\leq \varepsilon(n, j, \mu, s, \eta).
\tag{7.24}
\]

Now, let \(0 < \theta < 1\), by applying the Young’s inequality with \(p = \frac{1}{\theta}\) and \(\frac{1}{1-\theta}\), \(y_n = (x, t, T_k(u_n), \nabla T_k(u_n))\), \(y = (x, t, T_k(u), \nabla T_k(u))\), we get
\[
\int_{Q} \chi_{\|T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta} \left( a(y_n) - a(y) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \theta dxdt \\
= \int_{Q} \left( a(y_n) - a(y) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \theta dxdt \\
\leq c \max \left\{ \|T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta \right\} \frac{1}{\eta^\theta} \\
+ c \left( \int_{Q} \chi_{\|T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta} \left( a(y_n) - a(y) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) dxdt \right)^\theta.
\tag{7.25}
\]

But we have for \(s > \tau\), \(y_\chi = (x, t, T_k(u_n), \nabla T_k(u)\chi_s)\) and \(y_\alpha = (x, t, T_k(u_n), \nabla T_k(\alpha^k_n)\chi_{j,s})\), we have
\[
\int_{Q} \chi_{\|T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta} \left( a(y_n) - a(y_\chi) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(u) \chi_s \right) dxdt \\
\leq \int_{\{T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta\}} \left( a(y_n) - a(y_\chi) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(\chi_s) \chi_{j,s} \right) dxdt \\
\leq \int_{\{T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta\}} \left( a(y_n) - a(y_\alpha) \right) \times \left( \nabla T_k(u_n) - \nabla T_k(\alpha^k_n) \chi_{j,s} \right) dxdt \\
+ \int_{\{T_k(u_n) - T_k(\alpha^k_n, u)\| < \eta\}} a(y_n) \left( \nabla T_k(\alpha^k_n) \chi_{j,s} - \nabla T_k(u) \chi_s \right) dxdt.
\]
We shall go to limit as \( n, j, \mu \) and \( s \) to infinity in the last fifth integrals of the last side. Starting by \( J_1 \), one has

\[
J_1 \leq \varepsilon(n, j, \mu, \eta) - \int_{|T_k(u_n) - T_k(\alpha_j^k)| < \eta} a(y_n) \left[ \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \right] dxdt.
\]

Since \( a(y_n) \) converge strongly to \( a(x, t, T_k(u), \nabla T_k(\alpha_j^k)\chi_{j,s}) \) in \( (E^\varphi(Q))^N \) and \( \nabla T_k(u_n) \rightarrow \nabla T_k(u) \) weakly in \( (L^\varphi(Q))^N \), then

\[
\int_{|T_k(u_n) - T_k(\alpha_j^k)| < \eta} a(y_n) \left[ \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \right] dxdt
\]

which gives by letting \( j \rightarrow \infty, \mu \rightarrow \infty \) and \( s \rightarrow \infty \) respectively

\[
\int_{|T_k(u_n) - T_k(\alpha_j^k)| < \eta} a(y_n) \left[ \nabla T_k(u_n) - \nabla T_k(\alpha_j^k) \right] dxdt
\]

Finally, we get

\[
J_1 = \varepsilon(n, j, \mu, \eta). \tag{7.27}
\]

Similarly, we get

\[
J_2 = J_3 = J_4 = J_5 = \varepsilon(n, j, \mu, \eta). \tag{7.28}
\]

Combining (7.25)-(7.28), we get

\[
\lim_{n \rightarrow +\infty} \int_{Q_\varepsilon} \left( \left[ a(y_n) - a(y) \right] \times \left[ \nabla T_k(u_n) - \nabla T_k(u) \right] \right)^6 dxdt = 0.
\]
and, like a same argument in [3], we have

$$\nabla T_k(u_n) \to \nabla T_k(u)$$

as $n \to +\infty$ for the modular convergence, \(7.29\)

**Step 6 : Compactness of the nonlinearities**

In this step, we need to prove that

$$g_n(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u)$$

strongly in $L^1(Q)$. \(7.30\)

By virtue of \((7.29)\), one has

$$g_n(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u) \text{ a.e. in } Q. \quad (7.31)$$

Let $E$ be measurable subset of $Q$ and let $m > 0$. Using \((3.3)\) and \((3.4)\), we can write

$$\int_E |g_n(x, t, u_n, \nabla u_n)| dx dt$$

$$= \int_{E \cap \{ |u_n| \leq m \}} |g_n(x, t, u_n, \nabla u_n)| dx dt + \int_{E \cap \{ |u_n| > m \}} |g_n(x, t, u_n, \nabla u_n)| dx dt$$

$$\leq b(m) \int_E h_2(x, t) dx dt + b(m) \int_E a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx dt$$

$$+ \frac{1}{m} \int_E g_n(x, t, u_n, \nabla u_n) u_n dx dt.$$ 

Taking $u_n$ as a test function in $(\mathcal{P}_n)$ and using the same argument as in step 2, there exists a constant $c > 0$ such that

$$\int_E g_n(x, t, u_n, \nabla u_n) u_n dx dt \leq c.$$ 

Then, we have

$$\lim_{m \to +\infty} \frac{1}{m} \int_E g_n(x, t, u_n, \nabla u_n) u_n dx dt = 0.$$ 

Thanks to \((7.29)\) the sequence $(a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n))_n$ is equi-integrable, the fact which allows us to get

$$\lim_{|E| \to 0} \sup_n \int_E a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) dx dt = 0.$$ 

This shows that $g_n(x, t, u_n, \nabla u_n)$ is equi-integrable. Thus, Vitali’s theorem implies that $g(x, t, u, \nabla u) \in L^1(Q)$ and

$$g_n(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u)$$

strongly in $L^1(Q)$.

**Step 7 : Passage to the limit**
Let $v \in W^{1,p}_0(Q)$ such that $\frac{\partial v}{\partial t} \in W^{-1,q}_{\psi}(Q) + L^1(Q)$.

There exists a prolongation $\overline{v}$ of $v$ such that (see the proof of lemma )

\[
\begin{align*}
\overline{v} = v & \quad \text{on } Q, \\
\overline{v} \in W^{1,p}_0(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}), \\
\text{and } \frac{\partial \overline{v}}{\partial t} & \in W^{-1,q}_{\psi}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}).
\end{align*}
\]

By theorem, there exists a sequence $(w_j)_j$ in $D(\Omega \times \mathbb{R})$ such that $w_j \rightharpoonup v$ in $W^{1,p}_0(\Omega \times \mathbb{R})$ and $\frac{\partial w_j}{\partial t} \rightharpoonup \frac{\partial \overline{v}}{\partial t}$ in $W^{-1,q}_{\psi}(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R})$ for the modular convergence and $\|w_j\|_{\infty,Q} \leq (N + 2)\|v\|_{\infty,Q}$.

Using $T_k(u_n - w_j)\chi_{[0,\tau]}$ as a test function in $(P_n)$, then for every $\tau \in [0,T]$, one has

\[
\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j) dx dt + \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(u_n - w_j) dx dt \leq \int_{Q_\tau} f_n T_k(u_n - w_j) dx dt + \int_{Q_\tau} F \cdot \nabla T_k(u_n - w_j) dx dt. \tag{7.32}
\]

For the first term of (7.32), we get

\[
\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j) dx dt = \left[ \int_{\Omega} T_k(u_n - w_j) dx \right]^\tau_0 + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u_n - w_j) dx dt = \left[ \int_{\Omega} T_k(u - w_j) dx \right]^\tau_0 + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u - w_j) dx dt + \varepsilon(n) = \int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j) dx dt.
\]

for the second term of (7.32), we have if $|u_n| > \lambda$ then $|u_n - w_j| \geq |u_n| - \|w_j\|_{\infty} > k$, for some $k > 0$. 


therefore \( \{|u_n - w_j| \leq k\} \subseteq \{|u_n| \leq k + (N + 2)\|v\|_\infty\} \), which implies that, we get

\[
\liminf_{n \to +\infty} \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - w_j) \, dx \, dt \\
\geq \int_Q a(y\|v\|)(\nabla T_{k+(N+2)\|v\|_\infty}(u) - \nabla w_j) \chi_{\{|u-v| \leq k\}} \, dx \, dt,
\]

(7.33)

\[
= \int_Q a(x, t, u, \nabla u)(\nabla u - \nabla w_j) \chi_{\{|u-w_j| \leq k\}} \, dx \, dt
\\
= \int_Q a(x, t, u, \nabla u) \nabla T_k(u - w_j) \, dx \, dt,
\]

where \( y\|v\| = (x, t, T_{k+(N+2)\|v\|_\infty}(u), \nabla T_{k+(N+2)\|v\|_\infty}(u)) \). Consequently, y using the strong convergence of \((g_n(x, t, u_n, \nabla u_n))_n\) and \((f_n)_n\), one has

\[
\int_{Q_T} \frac{\partial u}{\partial t} T_k(u - w_j) \, dx \, dt \\
+ \int_{Q_T} a(x, t, u, \nabla u) \cdot \nabla T_k(u - w_j) \, dx \, dt
\\
+ \int_{Q_T} g(x, t, u, \nabla u) T_k(u - w_j) \, dx \, dt
\]

\[
\leq \int_{Q_T} f T_k(u - w_j) \, dx \, dt \\
+ \int_{Q_T} F \cdot \nabla T_k(u - w_j) \, dx \, dt.
\]

(7.34)

Thus, by using the modular convergence of \( j \), we achieve this step.

As a conclusion of Step 1 to Step 7, the proof of Theorem 7 is complete.

\[\square\]

References


A. Talha, A. Benkirane, M.S.B. Elemine Vall, Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, University Sidi Mohamed Ben Abdellah, P.O. Box 1796 Atlas, Fes 30000, Morocco. E-mail address: talha.abdous@gmail.com
E-mail address: abd.benkirane@gmail.com
E-mail address: saad2012bouh@gmail.com