Coefficient Inequalities For A Class Of Analytic Functions Associated With The Lemniscate Of Bernoulli

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ABSTRACT: In this paper, a new subclass of analytic functions $ML^*_λ$ associated with the right half of the lemniscate of Bernoulli is introduced. The sharp upper bound for the Fekete-Szegő functional $|a_3 - μa_2^2|$ for both real and complex $μ$ are considered. Further, the sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the function $f$ in the class $ML^*_λ$ using Toeplitz determinant is studied. Relevances of the main results are also briefly indicated.

Key Words: Starlike Function, Fekete-Szegő Inequality, Hankel Determinant, Lemniscate of Bernoulli.

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1. Introduction and Motivation

Let $A$ be the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in $U := \{z \in \mathbb{C} : |z| < 1\}$.

Let $S$ be the subclass of $A$ consisting of univalent functions in $U$. A function $f \in A$ is said to be starlike of order $α$, $(0 \leq α < 1)$, denoted by $S(α)$ if and only if

$$\Re \left\{ \frac{zf'(z)}{f(z)} \right\} > α \quad (z \in U). \quad (1.2)$$

It may be noted that for $α = 0$, the class $S(α) = S^*$, the familiar subclass of starlike functions in $U$. Similarly, a function $f \in A$ is said to be in the class $\tilde{R}(α)$, $α > 0$, if it satisfies the inequality

$$|(f'(z))^2 - α| < α \quad (z \in U). \quad (1.3)$$

The class $\tilde{R}(1) = \tilde{R}$ was considered by Sahoo and Patel [28].
Let $f$ and $g$ be two analytic functions in $U$. We say $f$ is subordinate to $g$, written $f(z) \prec g(z)$ $(z \in U)$, if and only if there exists an analytic function $w$ in $U$ such that $w(0) = 0$ and $|w(z)| < 1$ for $|z| < 1$ and $f(z) = g(w(z))$. In particular, if $g$ is univalent in $U$, we have the following (see [19]):

$$f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).$$

In 1966, Pommerenke [26] defined the $q$ th Hankel determinant of $f$ for $q \geq 1$ and $n \geq 1$ as

$$H_q(n) = \begin{vmatrix}
   a_n & a_{n+1} & \cdots & a_{n+q-1} \\
   a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\
   \vdots & \vdots & \ddots & \vdots \\
   a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2}
\end{vmatrix}.$$ 

A good amount of literature is available about the importance of Hankel determinant. It is useful in the study of power series with integral coefficients (see [5]), meromorphic functions (see [32]) and also singularities (see [7]). Noonan and Thomas [22] studied about the second Hankel determinant of a really mean p-valent functions. Noor [23] determined the rate of growth of $H_q(n)$ as $n \to \infty$ for the functions in $S$ with a bounded boundary. Ehrenborg [9] studied the Hankel determinant of exponential polynomials.

For $q = 2, n = 1, a_1 = 1$ and $q = 2, n = 2$, the Hankel determinant simplifies respectively to

$$H_2(1) = \begin{vmatrix}
   1 & a_2 \\
   a_2 & a_3
\end{vmatrix} = a_3 - a_2^2$$

and

$$H_2(2) = \begin{vmatrix}
   a_2 & a_3 \\
   a_3 & a_4
\end{vmatrix} = a_2a_4 - a_3^2.$$ 

It is well-known that for $f \in S$ and given by (1.1) (see [8]), the sharp inequality $|a_3 - a_2^2| \leq 1$ holds. This corresponds to the Hankel determinant with $q = 2$ and $n = 1$. Fekete-Szegő (see [10]) problem is to estimate $|a_3 - \mu a_2^2|$ with $\mu$ real and $f \in S$. For details, see [6,24,25]. Given family $\mathcal{F}$ of the functions in $A$, the functional $|H_2(2)|$ is popularly known as the second Hankel determinant. Second Hankel determinant for various subclasses of analytic functions were obtained by different researchers including Janteng et al. [14], Mishra and Gochhayat [20] and Murugusundaramoorthy and Magesh [21]. For some more recent works see [1,3,4,11,12,13,15,31].

Sokól and Stankiewicz [29](also see [2,30]) introduced the class $SL^*$ consisting of normalized analytic functions $f$ in $U$ satisfying the condition $\left|\frac{zf'(z)}{f(z)^2} - 1\right| < 1, \ (z \in U)$. We called such function as Sokól-Stankiewicz starlike function. Recently, Raza and Malik [27] determined the upper bound of third Hankel determinant $H_3(1)$ for the class $SL^*$. Further, Sahoo and Patel [28] obtained the upper
bound to the second Hankel determinant for the class \( \tilde{R} = \{ f \in A : |f'(z)|^2 - 1 | < 1, z \in \mathbb{U} \} \).

Motivated by the above mentioned works obtained by earlier researchers, we introduce the following subclass of analytic function as below:

**Definition 1.1.** A function \( f \in A \) is said to be in the class \( ML^*_\lambda \), \( 0 \leq \lambda \leq 1 \), if it satisfies the condition

\[
\left| \frac{zf'(z)}{(1-\lambda)f(z)+\lambda z} \right|^2 - 1 < 1 \quad (z \in \mathbb{U}).
\]  

(1.4)

Note that for \( \lambda = 0 \), the class \( ML^*_0 \) reduces to the class \( SL^* \), studied by Raza and Malik [27] and while for \( \lambda = 1 \), the class \( ML^*_1 \) reduces to \( \tilde{R} \) studied by Sahoo and Patel [28]. In term of subordination, relation (1.4) can be written as.

\[
\frac{zf'(z)}{(1-\lambda)f(z)+\lambda z} \prec q(z) = \sqrt{1+z} \quad (z \in \mathbb{U}),
\]

(1.5)

where \( q(0) = 1 \). To state the geometrical significance of the class \( ML^*_\lambda \), consider

\[
w = q(e^{i\theta}) = \sqrt{1+e^{i\theta}} \quad (0 \leq \theta \leq 2\pi).
\]

(1.6)

It follows from (1.6) that \( w^2 - 1 = e^{i\theta} \), which implies \( |w^2 - 1| = 1 \). Taking \( w = u+iv \) and simplifying we get

\[
(u^2 + v^2)^2 = 2(u^2 - v^2).
\]

Therefore, \( q(\mathbb{U}) \) is the region bounded by the right half of the lemniscate of Bernoulli given by \( (u^2 + v^2)^2 - 2(u^2 - v^2) = 0 \).

In this paper, following the techniques devised by Libera and Zlotkiewicz [16, 17], we solve the Fekete-Szegő problem and also determine the upper bounds of the Hankel determinant \( |H_2(1)| \) for a subclass \( ML^*_\lambda \).

**2. Preliminaries**

Let \( P \) be the class of analytic functions \( p \) normalized by

\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n,
\]

(2.1)

such that

\[\Re \{p(z)\} > 0 \quad (z \in \mathbb{U}).\]

Each of the following results will be required in our present investigation.

**Lemma 2.1.** [18] Let \( p \in P \) and of the form (2.1). Then

\[
|p_2 - \nu p_1^2| \leq \begin{cases} -4\nu + 2, & \nu < 0 \\ 2, & 0 \leq \nu \leq 1 \\ 4\nu - 2, & \nu > 1. \end{cases}
\]
When $\nu < 0$ or $\nu > 1$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$ or one of its rotations. If $0 < \nu < 1$, then the equality holds if and only if $p(z) = \frac{(\frac{1}{2} + \frac{\eta}{2})}{1 - \frac{\eta}{2}}$, $(0 \leq \eta \leq 1)$, or one of its rotations. If $\nu = 0$, the equality holds if and only if $p(z) = \frac{1+z}{1-z}$, or one of its rotations. If $\nu = 1$, the equality holds if and only if $p(z)$ is the reciprocal of one of the functions such that the equality holds in the case of $\nu = 0$. Although the above upper bound is sharp, when $0 < \nu < 1$, it can be improved as follows:

$$|p_2 - \nu p_1^2| + \nu |p_1|^2 \leq 2 \left(0 < \nu \leq \frac{1}{2}\right),$$

and

$$|p_2 - \nu p_1^2| + (1-\nu)|p_1|^2 \leq 2 \left(\frac{1}{2} < \nu \leq 1\right).$$

**Lemma 2.2.** [18] Let $p \in \mathcal{P}$ be of the form (2.1), then for any complex number $\nu$,

$$|p_2 - \nu p_1^2| \leq 2 \max(1, |2\nu - 1|). \quad (2.2)$$

This result is sharp for the functions

$$p(z) = \frac{1+z^2}{1-z^2}, \quad p(z) = \frac{1+z}{1-z}.$$

**Lemma 2.3.** ([16], [17, p. 254]) Let the function $p \in \mathcal{P}$ be given by the power series (2.1). Then

$$2p_2 = p_1^2 + x(4-p_1^2) \quad (2.3)$$

and

$$4p_3 = p_1^3 + 2(4-p_1^2)p_1 x - (4-p_1^2)p_1 x^2 + 2(4-p_1^2)(1-|x|^2)z \quad (2.4)$$

for some complex numbers $x$, $z$ satisfying $|x| \leq 1$ and $|z| \leq 1$.

3. Main Results

The first two theorems give the results related to Fekete-Szegő functional, which is a special case of the Hankel determinant.

**Theorem 3.1.** Let the function $f$ given by (1.1) be in the class $ML^*_\lambda$. Then for real $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \begin{cases} \frac{1-3\lambda^2-2\lambda \mu-4\mu^2}{4(2+\lambda)(1+\lambda)^2}, & \mu < \delta_1, \\ \frac{1}{2(2+\lambda)}, & \delta_1 \leq \mu \leq \delta_2, \\ \frac{1-3\lambda^2-2\lambda \mu-4\mu^2}{8(2+\lambda)(1+\lambda)^2}, & \mu > \delta_2, \end{cases} \quad (3.1)$$

Furthermore, for $\delta_1 < \mu \leq \delta_1 + \beta$,

$$|a_3 - \mu a_2^2| + (\mu - \delta_1)|a_2|^2 \leq \frac{1}{2(2+\lambda)}, \quad (3.2)$$
and for $\delta_1 + \beta < \mu < \delta_1 + 2\beta$,

$$|a_3 - \mu a_2^2| + (\delta_1 + 2\beta - \mu)|a_2|^2 \leq \frac{1}{2(2 + \lambda)},$$  \hspace{1cm} (3.3)

where

$$\delta_1 = -\left[\frac{3 + 10\lambda + 7\lambda^2}{2(2 + \lambda)}\right],$$  \hspace{1cm} (3.4)

$$\delta_2 = \frac{5 + 6\lambda + \lambda^2}{2(2 + \lambda)}$$  \hspace{1cm} (3.5)

and

$$\beta = \frac{2(1 + \lambda)^2}{\lambda + 2}.$$  \hspace{1cm} (3.6)

These results are sharp.

**Proof.** Let $f \in ML^\lambda_*$. In view of Definition 1.1, there exists an analytic function $w(z)$ satisfying the condition of Schwarz lemma such that

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z} = \sqrt{1 + w(z)} \quad (z \in \mathbb{U}).$$  \hspace{1cm} (3.7)

Define a function

$$p(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in \mathbb{U}).$$  \hspace{1cm} (3.8)

Clearly $p \in \mathcal{P}$. From (3.8), we get

$$w(z) = \frac{p(z) - 1}{p(z) + 1} \quad (z \in \mathbb{U}).$$  \hspace{1cm} (3.9)

From (3.7) and (3.9), we have

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z} = \sqrt{\frac{p(z) - 1}{p(z) + 1} + 1} = \sqrt{\frac{2p(z)}{1 + p(z)}}.$$  \hspace{1cm} (3.10)

Now, by substituting the series expansion of $p(z)$ from (3.8) in (3.10), it follows that

$$\sqrt{\frac{2p(z)}{1 + p(z)}} = 1 + \frac{1}{4}p_1 z + \left(\frac{p_2}{4} - \frac{5}{32}p_1^2\right)z^2 + \left(\frac{p_3}{4} - \frac{5}{16}p_1p_2 + \frac{13}{128}p_1^3\right)z^3 + \cdots.$$  \hspace{1cm} (3.11)

Using series expansions for $f(z)$ and $f'(z)$ from (1.1) give

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z} = 1 + (1 + \lambda)a_2 z + \{(2 + \lambda)a_4 - (1 - \lambda^2)a_2^2\}z^2 + \{(3 + \lambda)a_4 - (1 - \lambda)(2\lambda + 3)a_2 a_3 + (1 + \lambda)(1 - \lambda^2)a_2^3\}z^3 + \cdots.$$  \hspace{1cm} (3.12)
Making use of (3.11) and (3.12) in (3.10) and equating the coefficients of $z$, $z^2$ and $z^3$ in the resulting equation, we deduce that

$$a_2 = \frac{p_1}{4(1 + \lambda)},$$  \hspace{1cm} (3.13)

$$a_3 = \frac{1}{4(2 + \lambda)} \left[ p_2 - \frac{7\lambda + 3}{8(1 + \lambda)} p_1 \right],$$  \hspace{1cm} (3.14)

and

$$a_4 = \frac{1}{4(3 + \lambda)} \left[ p_3 - \frac{7\lambda^2 + 16\lambda + 7}{4(1 + \lambda)(2 + \lambda)} p_1 p_2 + \frac{13 + 40\lambda + 25\lambda^2}{32(1 + \lambda)(2 + \lambda)} p_1^2 \right].$$  \hspace{1cm} (3.15)

For real $\mu$, it follows from (3.13) and (3.14) that

$$|a_3 - \mu a_2^2| = \frac{1}{4(2 + \lambda)} |p_2 - \nu p_1^2|,$$  \hspace{1cm} (3.16)

where

$$\nu = \frac{3 + 10\lambda + 7\lambda^2 + 4\mu + 2\lambda \mu}{8(1 + \lambda)^2}.$$  

In view of (3.16) and by an application of Lemma 2.1, we obtain the desired assertion.

The results are sharp for the functions $\psi_i(z)$, $i = 1, 2, 3, 4$ such that

$$\frac{z\psi'_1(z)}{(1 - \lambda)\psi_1(z) + \lambda z} = \sqrt{1 + z} \quad (\mu < \delta_1 \text{ or } \mu > \delta_2),$$

$$\frac{z\psi'_2(z)}{(1 - \lambda)\psi_2(z) + \lambda z} = \sqrt{1 + z^2} \quad (\delta_1 < \mu < \delta_2),$$

$$\frac{z\psi'_3(z)}{(1 - \lambda)\psi_3(z) + \lambda z} = \sqrt{1 + \phi(z)} \quad (\mu = \delta_1),$$

and

$$\frac{z\psi'_4(z)}{(1 - \lambda)\psi_4(z) + \lambda z} = \sqrt{1 - \phi(z)} \quad (\mu = \delta_2),$$

where

$$\phi(z) = \frac{z(z + \eta)}{1 + \eta z} \quad (0 \leq \eta \leq 1).$$

Thus, the proof of Theorem 3.1 is completed. \hspace{1cm} \square

**Remark 3.2.** Putting $\lambda = 1$ in Theorem 3.1, we get the result due to Sahoo and Patel (see [28, Corollary 2.2]).

**Remark 3.3.** Putting $\lambda = 0$ in Theorem 3.1, we get the Fekete-Szegö functional for the class $SL^*$ due to Raza and Malik (see [27, Theorem 2.1]).
Theorem 3.4. Let the function $f$ given by (1.1) be in the class $ML^*_\lambda$. Then, for a complex number $\mu$, we have

$$|a_3 - \mu a_2^2| \leq \frac{1}{2(2 + \lambda)} \max \left\{ 1, \left| \frac{3\lambda^2 + 2\lambda + 2\mu + 4\mu - 1}{4(1 + \lambda)^2} \right| \right\}. \quad (3.17)$$

The estimate in (3.17) is sharp.

Proof. From (3.16), we have

$$|a_3 - \mu a_2^2| = \frac{1}{4(2 + \lambda)} |p_2 - \nu p_1^2|.$$

Therefore, by virtue of Lemma 2.2, we obtain the desired assertion. The result is sharp for the function

$$zf' \left( \frac{1}{(1 - \lambda)f \left( \frac{1}{z} \right) + \lambda z} \right) = \sqrt{1 + z},$$

or

$$zf' \left( \frac{1}{(1 - \lambda)f \left( \frac{1}{z} \right) + \lambda z} \right) = \sqrt{1 + z^2}.$$ 

\[\square\]

Remark 3.5. Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 3.4, we get the result of Raza and Malik (see [27, Theorem 2.2]) and Sahoo and Patel (see [28, Thoerem 2.1]) respectively.

Taking $\lambda = 0$ and $\mu = 1$ in Theorem 3.4, we get the result for $|H_2(1)|$ as follows.

**Corollary 3.6.** [27] If the function $f$, given by (1.1) belongs to the class $SL^*$, then

$$|a_3 - a_2^2| \leq \frac{1}{4}.$$ 

Further, putting $\lambda = \mu = 1$ and $\lambda = 1, \mu = 0$ in Theorem 3.4, we have the following results due to Sahoo and Patel [28].

**Corollary 3.7.** [28, Corollary 2.1] If the function $f$, given by (1.1) belongs to the class $\mathcal{R}$, then

$$|a_3 - a_2^2| \leq \frac{1}{6} \text{ and } |a_3| \leq \frac{1}{6}. \quad (3.18)$$

The estimates are sharp.

Now, we determine the sharp upper bound to the second Hankel determinant $|H_2(1)|$ for the class $ML^*_\lambda$.

**Theorem 3.8.** Let $f \in A$ given by (1.1) be in the class $ML^*_\lambda$. Assume that its coefficients $a_2, a_3$ and $a_4$ are given by (3.13), (3.14) and (3.15), with $p_1 > 0$. Then

$$|a_2 a_4 - a_3^2| \leq \frac{1}{4(2 + \lambda)^2}. \quad (3.19)$$

The estimate in (3.19) is sharp.
Proof. From (3.13), (3.14) and (3.15), we have

\[
a_{2}a_{4} - a_{3}^2 = \frac{p_{1}}{16(1 + \lambda)(3 + \lambda)} \left( p_{3} - \frac{7\lambda^2 + 16\lambda + 7}{4(1 + \lambda)(2 + \lambda)}p_{1}p_{2} + \frac{13 + 40\lambda + 25\lambda^2}{32(1 + \lambda)(2 + \lambda)}p_{1}^3 \right) \\
- \left[ \frac{1}{4(2 + \lambda)} \left( p_{2} - \frac{3 + 7\lambda}{8(1 + \lambda)}p_{1}^2 \right) \right]^2 \\
= \frac{1}{16} \left[ \frac{p_{1}p_{3}}{(1 + \lambda)(3 + \lambda)} - \frac{p_{2}^2}{(2 + \lambda)^2} \right. \\
- \frac{5 - 6\lambda + \lambda^2}{4(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)}p_{2}^2 \\
+ \frac{25 + 51\lambda - 9\lambda^2 + \lambda^3}{64(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)}p_{1}^4 \right].
\] (3.20)

For convenience of notation, we write \( p_{1} = p \) (0 ≤ \( p \) ≤ 2). Putting the values of \( p_{2} \) and \( p_{3} \) from Lemma 2.3 in (3.20), we obtain

\[
|a_{2}a_{4} - a_{3}^2| = \frac{1}{16} \left| \frac{p_{1}(p^2 + 2(4 - p^2)p_{2}x - (4 - p^2)p_{2}x^2 + 2(4 - p^2)(1 - |x|^2)z)}{4(1 + \lambda)(3 + \lambda)} \\
- \frac{(5 + 6\lambda - \lambda^2)p^2(4 - p^2)(1 - |x|^2)z}{8(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
- \frac{(4 - p^2)^2}{4(2 + \lambda)^2} \right. \\
+ \frac{25 + 51\lambda - 9\lambda^2 + \lambda^3}{64(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)}p_{1}^4 \right| \\
= \frac{1}{16} \left| \frac{\lambda^3 - \lambda^2 + 19\lambda + 1}{64(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)}p_{1}^4 \\
- \frac{1 + 2\lambda - \lambda^2}{8(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)}(4 - p^2)p_{2}x \\
- \frac{p^2 + 4\lambda^2 + 16\lambda + 12}{4(1 + \lambda)(2 + \lambda)^2(3 + \lambda)}(4 - p^2)x^2 \\
+ \frac{p(4 - p^2)(1 - |x|^2)z}{2(1 + \lambda)(3 + \lambda)} \right| ,
\] (3.21)

for some \( x \) (|\( x \)| ≤ 1) and for some \( z \) (|\( z \)| ≤ 1). An application of triangle inequality in (3.21) and replacing |\( x \)| by \( y \) in the resulting equation with assumption that
(p^2 + 4\lambda^2 + 16\lambda + 12 - 8p - 8\lambda p - 2\lambda^2 p) > 0, we get

\[ |a_2 a_4 - a_3^2| \leq \frac{1}{16} \left[ \frac{\lambda^3 - \lambda^2 + 19\lambda + 1}{64(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} p^4 \\
+ \frac{(1 + 2\lambda - \lambda^2)(4 - p^2)p^2 y}{8(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
+ \frac{p^2 + 4\lambda^2 + 16\lambda + 12 - 8p - 8\lambda p - 2\lambda^2 p}{4(1 + \lambda)(2 + \lambda)^2(3 + \lambda)} (4 - p^2) y^2 \\
+ \frac{(4 - p^2)p}{2(1 + \lambda)(3 + \lambda)} \right] = F(p, y; \lambda) \quad (0 \leq p \leq 2, \ 0 \leq y \leq 1)(say). \quad (3.22)

Differentiating on both sides of (3.22) with respect to \( y \), we have

\[ \frac{\partial F(p, y; \lambda)}{\partial y} = \frac{1}{16} \left[ \frac{(1 + 2\lambda - \lambda^2)(4 - p^2)p^2}{8(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
+ \frac{p^2 + 4\lambda^2 + 16\lambda + 12 - 8p - 8\lambda p - 2\lambda^2 p}{2(1 + \lambda)(2 + \lambda)^2(3 + \lambda)} (4 - p^2) y \right] \quad (3.23)

It is observed that \( \frac{\partial F(p, y; \lambda)}{\partial y} > 0 \) for \( 0 < p < 2, \ 0 < y < 1 \). Thus \( F(p, y; \lambda) \) is an increasing function of \( y \) which implies \( F(p, y; \lambda) \) cannot have maximum in the interior of the closed rectangle \([0, 2] \times [0, 1] \). Therefore, for fixed \( p \in [0, 2] \),

\[ \max_{0 \leq y \leq 1} F(p, y; \lambda) = F(p, 1, 1) = H(p; \lambda)(say), \quad (3.24) \]

where

\[ H(p; \lambda) = \frac{1}{16} \left[ \frac{\lambda^3 - \lambda^2 + 19\lambda + 1}{64(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} p^4 \\
+ \frac{(1 + 2\lambda - \lambda^2)(4 - p^2)p^2}{8(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
+ \frac{p^2 + 4\lambda^2 + 16\lambda + 12 - 8p - 8\lambda p - 2\lambda^2 p}{(1 + \lambda)(2 + \lambda)^2(3 + \lambda)} (4 - p^2) \right]. \quad (3.25)

Therefore

\[ H'(p; \lambda) = \frac{1}{16} \left[ \frac{3(\lambda^3 - \lambda^2 + 19\lambda + 1)}{16(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} p^3 + \frac{p(1 + 2\lambda - \lambda^2)(2 - p^2)}{2(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
- \frac{3p^2}{2(1 + \lambda)(3 + \lambda)} \right], \quad (3.26)

and

\[ H''(p; \lambda) = \frac{1}{16} \left[ \frac{3(\lambda^3 - \lambda^2 + 19\lambda + 1)}{16(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} p^2 + \frac{(1 + 2\lambda - \lambda^2)(2 - 3p^2)}{2(1 + \lambda)^2(2 + \lambda)^2(3 + \lambda)} \\
- \frac{3p^2}{(1 + \lambda)(3 + \lambda)} \right]. \quad (3.27)\]
By elementary calculus we have $H''(p; \lambda) < 0$ for $0 \leq p \leq 2$ and $H(p; \lambda)$ has real critical point at $p = 0$. This shows that the maximum of $H(p; \lambda)$ occurs at $p = 0$. Thus, the upper bound in (3.22) corresponds to $p = 0$ and $y = 1$ from which we get the required estimate (3.19).

The result is sharp for the functions

$$\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z} = \sqrt{1 + z},$$

or

$$\frac{zf''(z)}{(1 - \lambda)f(z) + \lambda z} = \sqrt{1 + z^2}.$$  

The proof of Theorem 3.8 is thus completed.

Remark 3.9. Putting $\lambda = 0$ and $\lambda = 1$ in Theorem 3.8, we get the result of Raza and Malik (see [27]) and Sahoo and Patel (see [28]).

The sharp upper bound for the fourth coefficient of the function $f \in ML^*_\lambda$ is given by the following theorem.

Theorem 3.10. Let the function $f$ given by (1.1) be in the class $ML^*_L$. Then

$$|a_4| \leq \frac{1}{2(3 + \lambda)} \quad (0 \leq \lambda \leq 1).$$

(3.26)

Proof. Proceeding similarly as in the proof of Theorem 3.8 and making use of Lemma 2.2 in (3.15) assuming that $(1 - 4\lambda - 3\lambda^2) > 0$, it follows that

$$|a_4| \leq \frac{1}{16(3 + \lambda)} \left[ \frac{1 + 5\lambda^2}{8(1 + \lambda)(2 + \lambda)}p^3 + \frac{(1 - 4\lambda - 3\lambda^2)}{2(1 + \lambda)(2 + \lambda)}(4 - p^2)py 
+ (4 - p^2)py^2 + 2(4 - p^2)(1 - y^2) \right]$$

$$= T(p, y; \lambda) \text{ (say)}.$$  

(3.27)

Now we maximize the function $T(p, y; \lambda)$ on the closed rectangle $[0, 2] \times [0, 1]$. Suppose that the maximum of $T$ occurs at the interior point of $[0, 2] \times [0, 1]$. Differentiating (3.27) with respect to $y$, we obtain

$$\frac{\partial T}{\partial y} = \frac{1}{16(3 + \lambda)} \left[ \frac{(1 - 4\lambda - 3\lambda^2)}{2(1 + \lambda)(2 + \lambda)}p + 2(p - 2)y \right] (4 - p^2).$$

(3.28)

It is clear that $\frac{\partial T}{\partial y} < 0$ for $0 < p < 2$ and $0 < y < 1$. Thus, $T(p, y, \lambda)$ is an decreasing function of $y$, contradicting our assumption. Therefore,

$$\max_{0 \leq y \leq 1} T(p, y; \lambda) = T(p, 0, \lambda) = \frac{1}{16(3 + \lambda)} \left[ \frac{1 + 5\lambda^2}{8(1 + \lambda)(2 + \lambda)}p^3 + 2(4 - p^2) \right]$$

$$= S(p) \text{ (say)}. \quad (3.29)$$
From (3.29), we have
\[
S'(p) = \frac{1}{16(3 + \lambda)} \left( \frac{3(1 + 5\lambda^2)p^2}{8(1 + \lambda)(2 + \lambda)} - 4p \right),
\]
and
\[
S''(p) = \frac{1}{16(3 + \lambda)} \left( \frac{3(1 + 5\lambda^2)p}{4(1 + \lambda)(2 + \lambda)} - 4 \right).
\]
For extreme values of $S(p)$, consider $S'(p) = 0$. From (3.30), we have
\[
\frac{3(1 + 5\lambda^2)p^2}{8(1 + \lambda)(2 + \lambda)} - 4p = 0
\]
\[
\Rightarrow p \left[ \frac{3(1 + 5\lambda^2)p}{8(1 + \lambda)(2 + \lambda)} - 4 \right] = 0
\]
We now discuss the following cases.

**Cases 1:** If $p = 0$, then
\[
S''(p) = -\frac{1}{4(3 + \lambda)} < 0.
\]
By the second derivative test, $S(p)$ has maximum value at $p = 0$.

**Cases 2:** If $p \neq 0$, then (3.33) gives
\[
p = \frac{32(1 + \lambda)(2 + \lambda)}{3(1 + 5\lambda^2)}.
\]
Using the value of $p$ given in (3.33) in (3.31), we get
\[
S''(p) = \frac{1}{4(3 + \lambda)} > 0 \quad (0 \leq \lambda \leq 1).
\]
Hence by second derivative test, $S(p)$ has minimum value at $p$, where $p$ is given by (3.33).

From the above discussion, it is clear that $S(p)$ attains its maximum at $p = 0$. Thus, the upper bound in (3.27) corresponds to $p = 0$ and $y = 0$ from which we get the required estimate (3.26).

The estimate in (3.26) is sharp for the function $f \in A$ given by
\[
\frac{zf'(z)}{(1 - \lambda)f(z) + \lambda z} = \sqrt{1 + z^3} \quad (z \in \mathbb{U}).
\]
This complete the prove of Theorem 3.10.

**Remark 3.11.** Taking $\lambda = 0$ and $\lambda = 1$ in Theorem 3.10, we get the upper bounds for $|a_4|$ for the class of $SL^*$ and $R$ respectively studied by Raza and Malik [27] and Sahoo and Patel [28].

**Acknowledgement:** The authors would like to express their gratitude to the reviewers for careful reading of the manuscript and making valuable suggestions which leads to better presentation of the paper.
References


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