On almost $b$-continuous functions in a bitopological space

by

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Abstract: The aim of this paper is to investigate some properties of almost $b$-continuous function in a bitopological space. Relationships with some other types of functions are also investigated.

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1. Introduction

The notion of a bitopological space $(X, \tau_1, \tau_2)$, where $X$ is a non-empty set and $\tau_1, \tau_2$ are topologies on $X$, was introduced by Kelly [7]. In 1996, Andrijevic [2] introduced the concept of $b$-open set in a topological space. Later Al-Hawary and Al-Omari [1] defined the notion $b$-open set and $b$-continuity in a bitopological space and established several fundamental properties. Sengul [11] defined the notion of almost $b$-continuous function in a topological space and established relationships between several properties of this notion with other known results. In addition to this, Duszynski et al. [6] introduced the concept of almost $b$-continuous function in a bitopological space. In the light of the above results, the purpose of this paper is to study almost $b$-continuity in a bitopological space and to obtain several characterizations and properties of this concept.

Bitopological space and its properties have many useful applications in real world. In 2010, Salama [10] used lower and upper approximations of Pawlak’s rough sets by using a class of generalized closed set of bitopological space for data reduction of rheumatic fever data sets. Fuzzy topology integrated support vector machine (FTSVM)-classification method for remotely sensed images based on standard support vector machine (SVM)
were introduced by using fuzzy topology by Zhang et al. [16]. For some of recent applications of generalized forms of topological or bitopological space as fuzzy, rough version etc one may refer to [10, 14, 16]. Acharjee and Tripathy [20] used concept of \((\gamma, \delta)-BSC\) set of bitopology to determine poverty patterns and equilibria in mixed budget. Recently, Acharjee and Tripathy [21] investigated some fundamental results in soft bitopology, a new area of research created in 2014.

2. Preliminaries

Throughout this paper, bitopological spaces \((X, \tau_1, \tau_2)\) and \((Y, \sigma_1, \sigma_2)\) are represented by \(X\) and \(Y\); on which no separation axiom is assumed and \((i, j)\) means the topologies \(\tau_i\) and \(\tau_j\); where \(i, j \in \{1, 2\}, i \neq j\). For a subset \(A\) of \((X, \tau_1, \tau_2)\), \(i\)-int \((A)\) (respectively, \(i\)-cl \((A)\)) denotes interior (resp. closure) of \(A\) with respect to the topology \(\tau_i\), where \(i \in \{1, 2\}\).

Now, we list some definitions and results those will be used throughout this article.

**Definition 2.1.** Let \((X, \tau_1, \tau_2)\) be a bitopological space, then a subset \(A\) of \(X\) is said to be

(a) \((i, j)\)-b-open ([1]) if \(A \subseteq i\)-int \((j\)-cl \((A)) \cup j\)-cl \((i\)-int \((A))\).

(b) \((i, j)\)-regular open ([3]) if \(A = i\)-int \((j\)-cl \((A))\).

(c) \((i, j)\)-regular closed ([4]) if \(A = i\)-cl \((j\)-int \((A))\).

The complement of \((i, j)\)-b-open set is said to be \((i, j)\)-b-closed set.

**Definition 2.2.** ([1]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subseteq X\). Then

(a) \((i, j)\)-b-closure of \(A\); denoted by \((i, j)\)-bcl \((A)\), is defined as the intersection of all \((i, j)\)-b-closed sets containing \(A\).

(b) \((i, j)\)-b-interior of \(A\); denoted by \((i, j)\)-bint \((A)\), is defined as the union of all \((i, j)\)-b-open sets contained in \(A\).

**Lemma 2.1.** ([1]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subseteq X\). Then

(a) \((i, j)\)-bint \((A)\) is \((i, j)\)-b-open.

(b) \((i, j)\)-bcl \((A)\) is \((i, j)\)-b-closed.

(c) \(A\) is \((i, j)\)-b-open if and only if \(A = (i, j)\)-bint \((A)\).
A is \((i,j)\)-b-closed if and only if \(A = (i,j)\)-\text{bcl}(A).

**Lemma 2.2.** ([9]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subseteq X\). Then

(a) \(X \setminus (i,j)\)-\text{bcl}(A) = (i,j)-\text{bint}(X \setminus A)

(b) \(X \setminus (i,j)\)-\text{bint}(A) = (i,j)-\text{bcl}(X \setminus A)

**Lemma 2.3.** ([1]) Let \((X, \tau_1, \tau_2)\) be a bitopological space and \(A \subseteq X\). Then \(x \in (i,j)-\text{bcl}(A)\), if and only if for every \((i,j)\)-b-open set \(U\) containing \(x\) such that \(U \cap A \neq \emptyset\).

**Definition 2.3.** A function \(f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)\) is said to be

(a) \((i,j)\)-b-continuous ([1]) if \(f^{-1}(A)\) is \((i,j)\)-b-open in \(X\), for each \(\sigma_i\)-open set \(A\) of \(Y\).

(b) \((i,j)\)-weakly b-continuous ([13]) if for each \(x \in X\) and each \(\sigma_i\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i,j)\)-b-open set \(U\) containing \(x\) such that \(f(U) \subseteq j\)-\text{cl}(V).

**Definition 2.4.** ([8]) Let \((X, \tau_1, \tau_2)\) be a bitopological space. A point \(x \in X\) is said to be an \((i,j)\)-\(\delta\)-cluster point of \(A\) if \(A \cap U \neq \emptyset\); for every \((i,j)\)-regular open set \(U\) containing \(x\). The set of all \((i,j)\)-\(\delta\)-cluster points of \(A\) is called \((i,j)\)-\(\delta\)-closure of \(A\) and it is denoted by \((i,j)\)-\text{cl\_}\((\delta)\)(A). A subset \(A\) of \(X\) is said to be \((i,j)\)-\(\delta\)-closed if the set of all \((i,j)\)-\(\delta\)-cluster points of \(A\) is a subset of \(A\). The complement of an \((i,j)\)-\(\delta\)-closed set is an \((i,j)\)-\(\delta\)-open. So, a subset of \(X\) is \((i,j)\)-\(\delta\)-open; if it is expressible as union of \((i,j)\)-regular open sets.

3. \((i,j)\)-almost b-continuous functions

**Definition 3.1.** ([6]) A function \(f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)\) is said to be \((i,j)\)-almost b-continuous at a point \(x \in X\); if for each \(\sigma_i\)-open set \(V\) of \(Y\) containing \(f(x)\), there exists an \((i,j)\)-b-open set \(U\) of \(X\) containing \(x\) such that \(f(U) \subseteq j\)-\text{int}(j\text{-cl}(V))\).

If \(f\) is \((i,j)\)-almost b-continuous at every point \(x\) of \(X\), then it is called \((i,j)\)-almost b-continuous.

**Theorem 3.1.** The following statements are equivalent for a function \(f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)\).

(a) \(f\) is \((i,j)\)-almost b-continuous.

(b) \((i,j)\)-\text{bcl}(f^{-1}(i\text{-cl}(j\text{-int}(i\text{-cl}(B)))))) \subseteq f^{-1}(i\text{-cl}(B))\), for every subset \(B\) of \(Y\).
and by Lemma 2.1, we have
\[ \text{int}(f^{-1}(i-cl(j-int(G)))) \subseteq f^{-1}(G), \]
for every \((i,j)\)-regular closed set \(G\) of \(Y\).

(d) \((i,j)\)-bcl\((f^{-1}(i-cl(V))) \subseteq f^{-1}(i-cl(V))\), for every \(\sigma_j\)-open set \(V\) of \(Y\).

(e) \(f^{-1}(V) \subseteq (i,j)\)-bint\((f^{-1}(i-int(j-cl(V))))\), for every \(\sigma_i\)-open set \(V\) of \(Y\).

**Proof.** (a) \(\Rightarrow\)(b) Let, \(x \in X\) and \(B\) is any subset of \(Y\). We assume that \(x \in X \setminus f^{-1}(i-cl(B))\) and so, \(f(x) \in Y \setminus i-cl(B)\). Then, there exists a \(\sigma_i\)-open set \(C\) of \(Y\) containing \(f(x)\) such that \(C \cap B = \emptyset\). Therefore \(C \cap i-cl(j-int(i-cl(B))) = \emptyset\) and hence \(i-int(j-cl(C)) \cap i-cl(j-int(i-cl(B))) = \emptyset\). By the given hypothesis, there exists an \((i,j)\)-b-open set \(D\) such that \(f(D) \subseteq i-int(j-cl(C))\). So, we have \(D \cap f^{-1}(i-cl(j-int(i-cl(B))) = \emptyset\). Therefore by Lemma 2.3, we have \(x \in X \setminus (i,j)\)-bcl\((f^{-1}(i-cl(j-int(i-cl(B))))\)). Hence, \((i,j)\)-bcl\((f^{-1}(i-cl(j-int(i-cl(B))))\)) \subseteq f^{-1}(i-cl(B)).

(b) \(\Rightarrow\)(c) Let, \(G\) be an \((i,j)\)-regular closed set in \(Y\). Therefore, \(G = i-cl(j-int(G))\). Now, \((i,j)\)-bcl\((f^{-1}(i-cl(j-int(G)))) = (i,j)\)-bcl\((f^{-1}(i-cl(j-int(i-cl(j-int(G)))))) \subseteq f^{-1}(i-cl(j-int(G))) = f^{-1}(G)\).

(c) \(\Rightarrow\)(d) Let, \(V\) be \(\sigma_j\)-open in \(Y\). Therefore, \(i-cl(V)\) is \((i,j)\)-regular closed in \(Y\). Hence by (c) we have, \((i,j)\)-bcl\((f^{-1}(i-cl(V)))) \subseteq (i,j)\)-bcl\((f^{-1}(i-cl(j-int(i-cl(V)))) \subseteq f^{-1}(i-cl(V))\).

(d) \(\Rightarrow\)(e) Let, \(V\) be \(\sigma_i\)-open in \(Y\) and so, \(Y \setminus j-cl(V)\) is \(\sigma_j\)-open in \(Y\).
Hence by (d) we have, \((i,j)\)-bcl\((f^{-1}(i-cl(V)))) \subseteq f^{-1}(i-cl(Y \setminus j-cl(V))).

\[
\Rightarrow (i,j)\)-bcl\((f^{-1}(Y \setminus i-int(j-cl(V)))) \subseteq f^{-1}(Y \setminus i-int(j-cl(V)))
\]
\[
\Rightarrow (i,j)\)-bcl\((X \setminus f^{-1}(i-int(j-cl(V)))) \subseteq X \setminus f^{-1}(i-int(j-cl(V)))
\]
\[
\Rightarrow X \setminus (i,j)\)-bint\((f^{-1}(i-int(j-cl(V)))) \subseteq X \setminus f^{-1}(i-int(j-cl(V))) \subseteq X \setminus f^{-1}(V)
\]
Hence \(f^{-1}(V) \subseteq (i,j)\)-bint\((f^{-1}(i-int(j-cl(V))))\).

(e) \(\Rightarrow\)(a) Let, \(x \in X\) and \(V\) be a \(\sigma_i\)-open set in \(Y\) containing \(f(x)\). Then, \(x \in f^{-1}(V) \subseteq (i,j)\)-bint\((f^{-1}(i-int(j-cl(V))))\). Putting \(U = (i,j)\)-bint\((f^{-1}(i-int(j-cl(V))))\) and by Lemma 2.1, we have \(U\) is \((i,j)\)-b-open and \(U \subseteq f^{-1}(i-int(j-cl(V)))\). So \(f(U) \subseteq i-int(j-cl(V))\). Hence, \(f\) is \((i,j)\)-almost \(b\)-continuous.

**Theorem 3.2.** The following statements are equivalent for a function \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\).

(a) \(f\) is \((i,j)\)-almost \(b\)-continuous.

(b) \(f((i,j)\)-bcl\((A)) \subseteq (i,j)\)-cl\(_\delta\)(\(f(A)\)), for every subset \(A\) of \(X\).
(c) \((i, j)\)-bcl\((f^{-1}(B))\) \(\subseteq f^{-1}((i, j)\text{-}\overline{\text{cl}}(B))\), for every subset \(B\) of \(Y\).

(d) \(f^{-1}(C)\) is \((i, j)\)-b-closed in \(X\) for every \((i, j)\)-\(\delta\)-closed subset \(C\) of \(Y\).

(e) \(f^{-1}(D)\) is \((i, j)\)-b-open in \(X\) for every \((i, j)\)-\(\delta\)-open subset \(D\) of \(Y\).

**Proof.** (a)⇒(b) Let, \(A\) be a subset of \(X\) containing \(x\) and \(V\) be a \(\sigma\text{-}\overline{\text{open}}\) set of \(Y\) containing \(f(x)\). Since, \(f\) is \((i, j)\)-almost \(b\)-continuous, there exists an \((i, j)\)-b-open set \(U\) containing \(x\) such that, \(f(U) \subseteq i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(V))\). Let, \(x \in (i, j)\text{-}\text{bcl}(A)\), then by Lemma 2.3, we have \(U \cap A \neq \emptyset\); hence \(\emptyset \neq f(U) \cap f(A) \subseteq i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(V)) \cap f(A)\). Since, \(V\) is \(\sigma\text{-}\overline{\text{open}}\) in \(Y\), \(V \subseteq i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(V))\) and \(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(V))\) is \((i, j)\)-regular open in \(Y\). Hence, \(f(x) \in (i, j)\text{-}\text{bcl}(A)\). Consequently, \((i, j)\text{-}\text{bcl}(A) \subseteq f^{-1}((i, j)\text{-}\overline{\text{cl}}(f(A)))\). This implies that \(f((i, j)\text{-}\text{bcl}(A)) \subseteq (i, j)\text{-}\overline{\text{cl}}(f(A))\).

(b)⇒(c) Suppose, \(B\) is any subset of \(Y\). Then by (b), \(f((i, j)\text{-}\text{bcl}(f^{-1}(B))) \subseteq (i, j)\text{-}\overline{\text{cl}}\delta(f(f^{-1}(B))) \subseteq (i, j)\text{-}\overline{\text{cl}}\delta(B)\). This implies \((i, j)\text{-}\text{bcl}(f^{-1}(B)) \subseteq f^{-1}((i, j)\text{-}\overline{\text{cl}}\delta(B))\).

(c)⇒(d) Let, \(C\) be an \((i, j)\)-\(\delta\)-closed subset of \(Y\). Then by (c), \((i, j)\text{-}\text{bcl}(f^{-1}(C)) \subseteq f^{-1}(C)\) and so, \(f^{-1}(C)\) is \((i, j)\)-b-closed in \(X\).

(d)⇒(e) Let, \(D\) be an \((i, j)\)-\(\delta\)-open subset of \(Y\). Then, \(Y \setminus D\) is \((i, j)\)-\(\delta\)-closed in \(Y\). By (d), \(f^{-1}(Y \setminus D) = X \setminus f^{-1}(D)\) is \((i, j)\)-b-closed in \(X\). Hence, \(f^{-1}(D)\) is \((i, j)\)-b-open in \(X\).

(e)⇒(a) Let, \(A\) be a \(\sigma\text{-}\overline{\text{open}}\) subset of \(Y\) containing \(f(x)\). Then, \(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A))\) is \((i, j)\)-regular open in \(Y\) containing \(f(x)\). Since, \(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A))\) is \((i, j)\)-\(\delta\)-open in \(Y\), thus by (e), \(f^{-1}(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A)))\) is \((i, j)\)-b-open in \(X\). Now, \(A \subseteq i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A))\). This implies that, \(f^{-1}(A) \subseteq f^{-1}(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A))) = (i, j)\text{-}\text{bint}(f^{-1}(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(A))))\). Hence, by theorem 3.1, \(f\) is \((i, j)\)-b-continuous.

**Definition 3.2.**([15]) A function \(f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)\) is said to have \((i, j)\)-b interiority condition, if \((i, j)\)-bint\((f^{-1}(j\text{-}\overline{\text{cl}}(V))) \subseteq f^{-1}(V)\), for every \(\sigma_1\text{-}\overline{\text{open}}\) subset \(V\) of \(Y\).

**Theorem 3.3.** Let, \(f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2)\) be a function. If \(f\) is \((i, j)\)-almost \(b\)-continuous and satisfies \((i, j)\)-b interiority condition, then \(f\) is \((i, j)\)-b-continuous.

**Proof.** Let, \(U\) be a \(\sigma\text{-}\overline{\text{open}}\) subset of \(Y\). By hypothesis, \(f\) is \((i, j)\)-almost \(b\)-continuous. Therefore by theorem 3.1, we have \(f^{-1}(U) \subseteq (i, j)\text{-}\text{bint}(f^{-1}(i\text{-}\text{int}(j\text{-}\overline{\text{cl}}(U)))) \subseteq (i, j)\text{-}\text{bint}(f^{-1}(j\text{-}\overline{\text{cl}}(U)))\). Again by the \((i, j)\)-b interiority condition of \(f\), we get \((i, j)\text{-}\text{bint}(f^{-1}(j\text{-}\overline{\text{cl}}(U))) \subseteq f^{-1}(U)\). Thus we get \(f^{-1}(U) = (i, j)\text{-}\text{bint}(f^{-1}(j\text{-}\overline{\text{cl}}(U)))\) and so \(f^{-1}(U)\) is \((i, j)\)-b-open, by Lemma 2.1. Hence \(f\) is \((i, j)\)-b-continuous.
Definition 3.3. ([7]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise Hausdorff or pairwise \(T_2\), if for each pair of distinct points \(x\) and \(y\) of \(X\), there exist a \(\tau_i\)-open set \(U\) containing \(x\) and a \(\tau_j\)-open set \(V\) containing \(y\) such that \(U \cap V = \emptyset\).

Definition 3.4. ([15]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise \(b-T_2\), if for each pair of distinct points \(x\) and \(y\) of \(X\), there exist a \((i, j)\)-open set \(U\) containing \(x\) and a \((j, i)\)-open set \(V\) containing \(y\) such that \(U \cap V = \emptyset\).

Theorem 3.4. Let \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function such that, \(Y\) is pairwise \(T_2\). If for any two distinct points \(x\) and \(y\) of \(X\), following conditions are hold
(a) \(f(x) \neq f(y)\)
(b) \(f\) is \((i, j)\)-weakly \(b\)-continuous at \(x\),
(c) \(f\) is \((j, i)\)-almost \(b\)-continuous at \(y\),
then \(X\) is a pairwise \(b-T_2\) space.

Proof. Let \(x, y \in X\) such that \(x \neq y\). Suppose, \(Y\) is pairwise \(T_2\). Therefore, there exist a \(\sigma_1\)-open set \(U\) and a \(\sigma_2\)-open set \(V\) such that \(f(x) \in U\), \(f(y) \in V\) and \(U \cap V = \emptyset\).
Since \(U \cap V = \emptyset\), so we have \(j-\text{cl}(U) \cap (j-\text{int}(i-\text{cl}(V))) = \emptyset\). Again since \(f\) is \((i, j)\)-weakly \(b\)-continuous at \(x\) and \((j, i)\)-almost \(b\)-continuous at \(y\), therefore there exists an \((i, j)\)-open set \(F\) in \(X\) such that \(x \in F\), \(f(F) \subseteq j-\text{cl}(U)\) and there exists a \((j, i)\)-open set \(G\) in \(X\) such that \(y \in G\), \(f(G) \subseteq j-\text{int}(i-\text{cl}(V))\). Thus, \(F \cap G = \emptyset\). Hence, \(X\) is a pairwise \(b-T_2\) space.

Definition 3.5. ([4]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be pairwise Urysohn, if for each pair of distinct points \(x\) and \(y\) of \(X\), there exist a \(\tau_i\)-open set \(U\) containing \(x\) and a \(\tau_j\)-open set \(V\) containing \(y\) such that \(j-\text{cl}(U) \cap i-\text{cl}(V) = \emptyset\).

Theorem 3.5. Let \(f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) be a function, such that \(Y\) is a pairwise Urysohn space. If \(f\) is pairwise almost \(b\)-continuous, then \(X\) is pairwise \(b-T_2\) space.

Proof. Let \(x, y \in X\) such that \(x \neq y\). Therefore, \(f(x) \neq f(y)\). Since, \(Y\) is pairwise Urysohn,
therefore there exist a \(\sigma_1\)-open set \(U\) containing \(f(x)\) and a \(\sigma_2\)-open set \(V\) containing \(f(y)\) such that \(j-\text{cl}(U) \cap i-\text{cl}(V) = \emptyset\).
This implies \(i-\text{int}(j-\text{cl}(U)) \cap j-\text{int}(i-\text{cl}(V)) = \emptyset\). Hence, \(f^{-1}(i-\text{int}(j-\text{cl}(U))) \cap f^{-1}(j-\text{int}(i-\text{cl}(V))) = \emptyset\) and so, \((i, j)\)-\(b\text{-int}(f^{-1}(i-\text{int}(j-\text{cl}(U)))) \cap (j, i)\)-\(b\text{-int}(f^{-1}(j-\text{int}(i-\text{cl}(V)))) = \emptyset\). Again, since \(f\) is pairwise almost \(b\)-continuous, therefore by theorem 3.1, we have \(x \in f^{-1}(U) \subseteq (i, j)\)-\(b\text{-int}(f^{-1}(i-\text{int}(j-\text{cl}(U))))\) and \(y \in f^{-1}(V) \subseteq (j, i)\)-\(b\text{-int}(f^{-1}(j-\text{int}(i-\text{cl}(V))))\). Hence, \(X\) is pairwise \(b-T_2\) space.

Theorem 3.6. Let \(f : (X_1, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((i, j)\)-weakly \(b\)-continuous, \(g : (X_2, \psi_1, \psi_2) \rightarrow (Y, \sigma_1, \sigma_2)\) is \((i, j)\)-almost \(b\)-continuous and \(Y\) is pairwise Hausdorff, then the set \(\{(x, y) \in X_1 \times X_2 : f(x) = g(y)\}\) is \((i, j)\)-\(b\)-closed in \(X_1 \times X_2\).
Proof. Let, \( G = \{(x, y) \in X_1 \times X_2 : f(x) = g(y)\} \) and \((x, y) \in (X_1 \times X_2) \setminus G\). Thus, we get \(f(x) \neq f(y)\). Since \( Y \) is pairwise Hausdorff, therefore there exist a \(\sigma_i\)-open set \( U_1 \) and a \(\sigma_j\)-open set \( U_2 \) of \( Y \) such that \( f(x) \in U_1, g(y) \in U_2 \) and \( U_1 \cap U_2 = \emptyset \). Since, \( U_1 \) and \( U_2 \) are disjoint, hence \( j-cl(U_1) \cap (i-int(j-cl(U_2))) = \emptyset \). Also, \( f \) is \((i, j)\)-weakly \( b \)-continuous, thus, there exists an \((i, j)\)-\( b \)-open set \( V_1 \) containing \( x \) such that \( f(V_1) \subseteq j-cl(U_1) \). Again \( g \) is \((i, j)\)-almost \( b \)-continuous, thus, there exists an \((i, j)\)-\( b \)-open set \( V_2 \) containing \( y \) such that \( g(V_2) \subseteq i-int(j-cl(U_2)) \). Thus, we obtain \( (x, y) \in V_1 \times V_2 \subseteq (X_1 \times X_2) \setminus G \) and \( V_1 \times V_2 \) is \((i, j)\)-\( b \)-open in \( X_1 \times X_2 \). It implies \( G \) is \((i, j)\)-\( b \)-closed in \( X_1 \times X_2 \).

Definition 3.6. ([13]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-almost regular, if for every \( x \in X \) and for every \( \tau_i\)-open set \( V \) of \( X \), there exists a \( \tau_i\)-open set \( U \) containing \( x \) such that \( x \in U \subseteq j-cl(U) \subseteq i-int(j-cl(V)) \).

Lemma 3.1. ([6]) For a function \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \), the following statements are equivalent:

(a) \( f \) is \((i, j)\)-almost \( b \)-continuous.

(b) \( f^{-1}(i-int(j-cl(V))) \) is \((i, j)\)-\( b \)-open in \( X \), for each \( \sigma_i \)-open set \( V \) in \( Y \).

(c) \( f^{-1}(i-cl(j-int(F))) \) is \((i, j)\)-\( b \)-closed in \( X \), for each \( \sigma_i \)-closed set \( F \) in \( Y \).

(d) \( f^{-1}(F) \) is \((i, j)\)-\( b \)-closed in \( X \), for each \((i, j)\)-regular closed set \( F \) of \( Y \).

(e) \( f^{-1}(V) \) is \((i, j)\)-\( b \)-open in \( X \), for each \((i, j)\)-regular open set \( V \) of \( Y \).

Theorem 3.7. Let \( f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2) \) be a function, such that \( Y \) is \((i, j)\)-almost regular. Then, \( f \) is \((i, j)\)-almost \( b \)-continuous if and only if \( f \) is \((i, j)\)-weakly \( b \)-continuous.

Proof. Necessity: It is obvious that \((i, j)\)-almost \( b \)-continuity implies \((i, j)\)-weakly \( b \)-continuity.

Sufficiency: Assume that \( f \) is \((i, j)\)-weakly \( b \)-continuous. Let, \( U \) be an \((i, j)\)-\( b \)-open set in \( Y \) such that, \( x \in f^{-1}(U) \). This implies \( f(x) \in U \). Since \( Y \) is \((i, j)\)-almost regular, therefore there exists a \((i, j)\)-\( b \)-regular open set \( V \) in \( Y \) such that \( f(x) \in V \subseteq j-cl(V) \subseteq U \). Again since \( f \) is \((i, j)\)-weakly \( b \)-continuous, therefore there exists an \((i, j)\)-\( b \)-open set \( W \) in \( X \) containing \( x \) such that \( f(W) \subseteq j-cl(V) \subseteq U \). Thus, we get \( W \subseteq f^{-1}(U) \). Thus, \( x \in W = (i, j)-bint(W) \subseteq (i, j)-bint(f^{-1}(U)) \). Hence \( f^{-1}(U) \subseteq (i, j)-bint(f^{-1}(U)) \). Consequently, \( f^{-1}(U) = (i, j)-bint(f^{-1}(U)) \) and so, \( f^{-1}(U) \) is \((i, j)\)-\( b \)-open. By Lemma 3.1, \( f \) is \((i, j)\)-almost \( b \)-continuous.

Definition 3.7. ([12]) A bitopological space \((X, \tau_1, \tau_2)\) is said to be \((i, j)\)-semi regular,
if for every $x \in X$ and for every $\tau_i$-open set $V$ of $X$, there exists a $\tau_i$-open set $U$ containing $x$ such that $x \in U \subseteq i\text{-}int(j\text{-}cl(U)) \subseteq V$.

**Theorem 3.8.** Let, $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function, such that $Y$ is $(i, j)$-semi regular. If $f$ is $(i, j)$-almost $b$-continuous, then $f$ is $(i, j)$-$b$-continuous.

**Proof.** Let, $U$ be a $\sigma_i$-open set of $Y$ containing $f(x)$. Therefore, $x \in f^{-1}(U)$. Since, $Y$ is $(i, j)$-semi regular, thus there exists a $\sigma_i$-open set $V$ such that $f(x) \in V \subseteq i\text{-}int(j\text{-}cl(V)) \subseteq U$. Again, $f$ is $(i, j)$-almost $b$-continuous, so; there exists an $(i, j)$-$b$-open set $W$ in $X$ containing $x$ such that $f(W) \subseteq i\text{-}int(j\text{-}cl(V)) \subseteq U$. So, $x \in W = (i, j)$-cl$(W) \subseteq (i, j)$-cl$(f^{-1}(U))$ and hence $f^{-1}(U) \subseteq (i, j)$-cl$(f^{-1}(U))$. Hence, $f^{-1}(U) = (i, j)$-cl$(f^{-1}(U))$. Now by Lemma 2.1, $f^{-1}(U)$ is $(i, j)$-$b$-open in $X$. Consequently, $f$ is $(i, j)$-$b$-continuous.

**Definition 3.8.** A function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is said to be $(i, j)$-almost $b$-open if $f(U) \subseteq i\text{-}int(j\text{-}cl(f(U)))$, for every $(i, j)$-$b$-open set $U$ of $X$.

**Theorem 3.9.** If a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ is $(i, j)$-almost $b$-open and $(i, j)$-weakly $b$-continuous, then $f$ is $(i, j)$-almost $b$-continuous.

**Proof.** Let $V$ be a $\sigma_i$-open set of $Y$ containing $f(x)$. Since, $f$ is $(i, j)$-weakly $b$-continuous, thus there exists an $(i, j)$-$b$-open set $U$ in $X$ containing $x$ such that $f(U) \subseteq j$-cl$(V)$. Also, $f$ is $(i, j)$-almost $b$-open, therefore $f(U) \subseteq i\text{-}int(j\text{-}cl(f(U))) \subseteq i\text{-}int(j\text{-}cl(V))$. Hence, $f$ is $(i, j)$-almost $b$-continuous.

**Lemma 3.2.** ([6]) For a function $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$, the following statements are equivalent:

(a) $f$ is $(i, j)$-almost $b$-continuous.

(b) For each $x \in X$ and each $(i, j)$-regular open set $V$ of $Y$ containing $f(x)$, there exists an $(i, j)$-$b$-open $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.

(c) For each $x \in X$ and each $(i, j)$-$\delta$-open set $V$ of $Y$ containing $f(x)$, there exists an $(i, j)$-$b$-open $U$ in $X$ containing $x$ such that $f(U) \subseteq V$.

**Theorem 3.10.** If $f : (X, \tau_1, \tau_2) \rightarrow (Y, \sigma_1, \sigma_2)$ be a function and $g : X \rightarrow X \times Y$ be the function defined by $g(x) = (x, f(x))$, for every $x \in X$, then $g$ is $(i, j)$-almost $b$-continuous if and only if $f$ is $(i, j)$-almost $b$-continuous.

**Proof.** Let, $x \in X$ and $V$ be an $(i, j)$-regular open set of $Y$ such that $f(x) \in V$. Then $g(x) = (x, f(x)) \in X \times V$ is $(i, j)$-regular open in $X \times Y$. Since, $g$ is $(i, j)$-almost $b$-continuous, thus there exists an $(i, j)$-$b$-open set $U$ containing $x$ such that $g(U) \subseteq X \times Y$.
Thus we get \( f(U) \subseteq V \). Hence by Lemma 3.2, we have \( f \) is \((i, j)\)-almost \( b\)-continuous.

Conversely, let, \( x \in X \) and \( W \) be an \((i, j)\)-regular open set of \( X \times Y \) such that \( g(x) = (x, f(x)) \in X \times Y \). Then, there exists an \((i, j)\)-regular open set \( V \) in \( Y \) such that \( U \times V \subseteq W \). Since, \( f \) is \((i, j)\)-almost \( b\)-continuous, hence there exists an \((i, j)\)-\( b \)-open set \( A \) containing \( x \) such that \( f(A) \subseteq V \). Let, \( B = U \cap A \), then \( B \) is an \((i, j)\)-\( b \)-open set containing \( x \) and so; \( g(B) \subseteq U \times V \subseteq W \). Hence, \( g \) is \((i, j)\)-almost \( b \)-continuous.

**Theorem 3.11.** If \( g : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-almost \( b \)-continuous and \( A \) is \((i, j)\)-\( \delta \)-closed set in \( X \times Y \), then \( P_X(A \cap G(g)) \) is \((i, j)\)-\( b \)-closed in \( X \), where \( P_X \) denotes the projection of \( X \times Y \) onto \( X \) and \( G(g) \) denotes the graph of \( g \).

**Proof.** Let, \( A \) be \((i, j)\)-\( \delta \)-closed set in \( X \times Y \). Consider \( x \in (i, j)\)-\( bcl(P_X(A \cap G(g))) \). Again, let \( U \) be a \( \tau \)-open set of \( X \) containing \( x \) and \( V \) be a \( \sigma \)-open set of \( Y \) containing \( g(x) \). Since, \( g \) is \((i, j)\)-almost \( b \)-continuous, therefore by theorem 3.1, \( x \in g^{-1}(V) \subseteq (i, j)\)-\( bint(g^{-1}(i-int(j-cl(V)))) \) and \( U \cap (i, j)\)-\( bint(g^{-1}(i-int(j-cl(V)))) \) is \((i, j)\)-\( b \)-open in \( X \) containing \( x \). Since, \( x \in (i, j)\)-\( bcl(P_X(A \cap G(g))) \), therefore \( [U \cap (i, j)\)-\( bint(g^{-1}(i-int(j-cl(V)))) \] \( \cap \) \( P_X(A \cap G(g)) \) containing some point \( y \) of \( X \), which implies \( (y, g(y)) \in A \) and \( g(y) \in i-int(j-cl(V)) \). Then, \( \emptyset \neq U \times (i-int(j-cl(V))) \cap A \subseteq i-int(j-cl(U \times V)) \cap A \) and hence, \( (x, g(x)) \in (i, j)\)-\( \delta \)-closed. Since, \( A \) is \((i, j)\)-\( \delta \)-closed, \( (x, g(x)) \in A \cap G(g) \) and \( x \in P_X(A \cap G(g)) \). Therefore, \((i, j)\)-\( bcl(P_X(A \cap G(g))) \subseteq P_X(A \cap G(g)) \). Hence, \( P_X(A \cap G(g)) \) is \((i, j)\)-\( b \)-closed.

**Definition 3.9.**([3]) Let, \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \subseteq X \), then \( A \) is said to be \((i, j)\)-\( \delta \)\-quasi \( H \)\-closed relative to \( X \); if for each cover \( \{B_\alpha : \alpha \in \Delta \} \) of \( A \) by \( \tau \)-open subsets of \( X \), there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( A \subseteq \bigcup \{j-cl(B_\alpha) : \alpha \in \Delta_0 \} \), where \( \Delta \) is an index set.

**Definition 3.10.** Let \((X, \tau_1, \tau_2)\) be a bitopological space and \( A \subseteq X \), then \( A \) is said to be \((i, j)\)-\( b \)-compact relative to \( X \), if every cover of \( A \) by \((i, j)\)-\( b \)-open sets of \( X \) has a finite subcover.

**Theorem 3.12.** If a function \( f : (X, \tau_1, \tau_2) \longrightarrow (Y, \sigma_1, \sigma_2) \) is \((i, j)\)-almost \( b \)-continuous and \( A \) is \((i, j)\)-\( b \)-compact relative to \( X \), then \( f(A) \) is \((i, j)\)-\( \delta \)-quasi \( H \)-closed relative to \( Y \).

**Proof.** Let, \( A \) be \((i, j)\)-\( b \)-compact relative to \( X \) and \( \{B_\alpha : \alpha \in \Delta \} \) be any cover of \( f(A) \) by \( \sigma \)-open sets of \( Y \). Therefore, \( f(A) \subseteq \bigcup \{B_\alpha : \alpha \in \Delta \} \) and so; \( A \subseteq \bigcup \{f^{-1}(B_\alpha) : \alpha \in \Delta \} \). Since, \( f \) is \((i, j)\)-almost \( b \)-continuous, therefore by theorem 3.1, we have \( f^{-1}(B_\alpha) \subseteq (i, j)\)-\( bint(f^{-1}(i-int(j-cl(B_\alpha)))) \subseteq (i, j)\)-\( bint(f^{-1}(j-cl(B_\alpha)))) \subseteq \Delta \). Also, \( A \subseteq \bigcup \{(i, j)\)-\( bint(f^{-1}(j-cl(B_\alpha)))) : \alpha \in \Delta \} \). Also, \( A \) is \((i, j)\)-\( b \)-compact relative to \( X \) and \((i, j)\)-\( bint(f^{-1}(j-cl(B_\alpha)))) \) is \((i, j)\)-\( b \)-open for each \( \alpha \in \Delta \), therefore there exists a finite subset \( \Delta_0 \) of \( \Delta \) such that \( A \subseteq \bigcup \{(i, j)\)-\( bint(f^{-1}(j-cl(B_\alpha)))) : \alpha \in \Delta_0 \} \). This implies \( f(A) \subseteq \bigcup \{f((i, j)\)-\( bint(f^{-1}(j-cl(B_\alpha)))) : \alpha \in \Delta_0 \} \subseteq \bigcup \{j-cl(B_\alpha) : \alpha \in \Delta_0 \} \). Hence,
$f(A)$ is $(i,j)$-quasi $H$-closed relative to $Y$.

**Conflict of interest:** Authors declare that there is no conflict of interest.

**References**


