On a Generalization of Prime Submodules of a Module over a Commutative Ring

Hosein Fazaeli Moghimi and Batool Zarei Jalal Abadi

ABSTRACT: Let $R$ be a commutative ring with identity, and $n \geq 1$ an integer. A proper submodule $N$ of an $R$-module $M$ is called an $n$-prime submodule if whenever $a_1 \cdots a_{n+1} m \in N$ for some non-units $a_1, \ldots, a_{n+1} \in R$ and $m \in M$, then $m \in N$ or there are $n$ of the $a_i$’s whose product is in $(N : M)$. In this paper, we study $n$-prime submodules as a generalization of prime submodules. Among other results, it is shown that if $M$ is a finitely generated faithful multiplication module over a Dedekind domain $R$, then every $n$-prime submodule of $M$ has the form $m_1 \cdots m_t M$ for some maximal ideals $m_1, \ldots, m_t$ of $R$ with $1 \leq t \leq n$.

Key Words: $n$-prime submodule, $n$-absorbing ideal, AP $n$-module.

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1. Introduction

Throughout this paper all rings are commutative with identity and all modules are unitary. Also we take $R$ as a commutative ring with identity, $U(R)$ as the set of unit elements of $R$, $M$ as an $R$-module, and $n \geq 1$ is a positive integer. A proper ideal $I$ of a ring $R$ is an $n$-absorbing ideal of $R$ if whenever $a_1 \cdots a_{n+1} \in I$ for $a_1, \ldots, a_{n+1} \in R$, then there are $n$ of the $a_i$’s whose product is in $I$. It is evident that a 1-absorbing ideal is just a prime ideal. This concept was firstly introduced for $n = 2$ by A. Badawi [3], and then it has been studied for any positive integer $n$ by D. F. Anderson and A. Badawi [1]. The authors generalized this notion to $(m, n)$-absorbing ideals with $m > n$ [11]. In fact, these ideals absorb an $n$-subproduct of every $m$-product of elements which lies in $I$. In this case, $(n + 1, n)$-absorbing ideals are just $n$-absorbing ideals. Moreover, there are several generalizations of $n$-absorbing ideals of a ring to submodules of a module (see, for example, [8,10]). In this paper, we study the notion of an $n$-prime submodule of a module as a generalization of a prime submodule.

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Let $M$ be an $R$-module. A proper submodule $N$ of $M$ is called a prime submodule if for $r \in R$, $m \in M$, $rm \in N$ implies that $r \in (N : M)$ or $m \in N$. Prime submodules have been introduced by J. Dauns in [4], and then this class of submodules has been extensively studied by several authors (see, for example, [5,7]).

**Definition 1.1.** Let $R$ be a ring, $U(R)$ the set of units of $R$, $M$ an $R$-module and $n$ a positive integer. A proper submodule $N$ of $M$ is called an $n$-prime submodule of $M$ if whenever $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M$, then $m \in N$ or there are $n$ of the $a_i$'s whose product is in $(N : M)$, where $(N : M) = \{ r \in R \mid rM \subseteq N \}$. An ideal $I$ of $R$ is called an $n$-prime ideal of $R$ if it is an $n$-prime submodule of the $R$-module $R$.

By this definition, a 1-prime submodule is just a prime submodule. Moreover, every $n$-prime ideal is an $n$-absorbing ideal, but the converse is not true in general (Example 2.6). It is shown that if $R$ is a non-local PID or a polynomial ring $S[X]$ over a domain $S$, then every $n$-prime ideal of $R$ is just a prime ideal of $R$ (Theorems 2.8 and 2.12). However, an example of an $n$-prime ideal of a ring is given which is not a prime ideal (Example 2.6).

It is shown that every $n$-prime submodule is primary. Also, if $R$ is a Bézout ring and $M$ is a faithful multiplication $R$-module, then every $n$-prime submodule contains the $n$th power of its radical (Theorem 2.4). Moreover it is proved that if $M$ is a multiplication $R$-module, then $N$ is an $n$-prime submodule of $M$ if and only if $(N : M)$ is an $n$-prime ideal of $R$ (Corollary 4.6). It is shown that if $N \times N'$ is an $n$-prime submodule of $M \times M'$, then $N$ and $N'$ are respectively an $n$-prime submodule of $M$ and $M'$. The converse is true if $(N : M) = (N' : M')$ (Theorem 3.10). Using this fact, an example of an $n$-prime submodule of a module is given which is not prime submodule (Example 3.11).

Finally, we introduce and study AP $n$-modules. Indeed, an AP $n$-module $M$ has the property that for each $n$-absorbing ideal $I$ of $R$, $IM$ is an $n$-prime submodule of $M$. If $R$ is an AP $n$-module over itself, then we call it an AP $n$-ring. For example, every Artin local ring is an AP $n$-ring for some positive integer $n$ (Theorem 4.8). Moreover, Noetherian valuation domains are AP $n$-rings for all positive integer $n$ (Theorem 4.9). It is shown that every finitely generated faithful multiplication module over an AP $n$-ring is an AP $n$-module (Corollary 4.7).

2. On $n$-prime submodules

We start with several elementary results.

**Theorem 2.1.** Let $R$ be a ring, $M$ a non-zero $R$-module and $n$ be a positive integer.

1. A proper submodule $N$ of $M$ is an $n$-prime submodule of $M$ if and only if whenever $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M$ with $t > n$, then $m \in N$ or there are $n$ of the $a_i$'s whose product is in $(N : M)$.

2. If $N$ is an $n$-prime submodule of $M$, then $N$ is a $t$-prime submodule of $M$ for all $t \geq n$. 
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2.2 There is no such prime exists, we define \( \text{rad} M \). Then the following hold:

Proof: The proof is routine, and thus it is omitted. \( \Box \)

Let \( N \) be a proper submodule of an \( R \)-module \( M \). If \( N \) is an \( n \)-prime submodule of \( M \) for some positive integer \( n \), then define \( \nu(N) = \min \{ n \mid N \text{ is an \( n \)-prime submodule of } M \} \); otherwise, set \( \nu(N) = \infty \). It is convenient to define \( \nu(M) = 0 \). Thus for any submodule \( N \) of \( M \), we have \( \nu(N) \in \mathbb{N} \cup \{ 0, \infty \} \) with \( \nu(N) = 1 \) if and only if \( N \) is a prime submodule of \( M \) and \( \nu(N) = 0 \) if and only if \( N = M \). So \( \nu(N) \) measures, in some sense, how far \( N \) is from being a prime submodule of \( M \). Clearly \( \omega(I) \leq \nu(I) \), where \( \omega(I) = \min \{ n \mid I \text{ is an } n \text{-absorbing ideal of } R \} \).

Lemma 2.2. Let \( R \) be a ring, \( M \) a non-zero \( R \)-module and \( n \) a positive integer. Then the following hold:

1. A proper submodule \( N \) of \( M \) is an \( n \)-prime submodule if and only if whenever \( a_1 \cdots a_{n+1}K \subseteq N \) for \( a_1, \ldots, a_{n+1} \in R \setminus U(R) \) and submodule \( K \) of \( M \), then \( K \subseteq N \) or there are \( n \) of the \( a_i \)'s whose product is in \( (N : M) \).

2. If a proper submodule \( N \) of \( M \) is an \( n \)-prime submodule of \( M \), then \( (N : M) \) is an \( n \)-prime ideal of \( R \) and so it is an \( n \)-absorbing ideal of \( R \). Moreover \( \omega(N : M) \leq \nu(N) \).

Proof: (1) Let \( N \) be an \( n \)-prime submodule of \( M \) and \( a_1 \cdots a_{n+1}K \subseteq N \) for \( a_1, \ldots, a_{n+1} \in R \setminus U(R) \) and for a submodule \( K \) of \( M \). Let \( K \not\subseteq N \) and \( m \in K \setminus N \). Since \( a_1 \cdots a_{n+1}m \in N \) and \( N \) is an \( n \)-prime submodule of \( M \), there are \( n \) of the \( a_i \)'s whose product is in \( (N : M) \). Conversely, if the given condition is true for a submodule \( N \) of \( M \), and \( a_1 \cdots a_{n+1} \in N \) for \( a_1, \ldots, a_{n+1} \in R \setminus U(R) \) and \( m \in M \), then it suffices to take \( K = Rm \).

(2) Let \( a_1 \cdots a_{n+1}r \in (N : M) \) for \( a_1, \ldots, a_{n+1} \in R \setminus U(R) \), \( r \in R \) and no proper subproduct of the \( a_i \)'s is in \( (N : M) \). Then \( a_1 \cdots a_{n+1}rM \subseteq N \). Thus, by (1), \( rM \subseteq N \). The “In particular” statement is clear. \( \Box \)

The converse of the Lemma 2.2(2) is not necessarily true, as the following example shows.

Example 2.3. (1) Let \( R = \mathbb{Z} \), \( M = \mathbb{Z} \oplus \mathbb{Z} \) and \( N = 4\mathbb{Z} \oplus 4\mathbb{Z} \). Then, by [1, Theorem 2.1(d)], \( (N : M) = 4\mathbb{Z} \) is a 2-absorbing ideal of \( R \), but \( N \) is not an \( n \)-prime submodule of \( M \) for any positive integer \( n \). In fact, if \( a_1 = 2 \) and \( a_2, \ldots, a_{n+1} \) are odd prime numbers, then \( a_1 \cdots a_{n+1}(2, 0) \in N \), but no proper subproduct of the \( a_i \)'s is in \( (N : M) \) and \( (2, 0) \notin N \).

(2) Let \( R = \mathbb{Z} \), \( M = \mathbb{Z}(p^\infty) \oplus \mathbb{Z}_p \) and \( N = 0 \oplus \mathbb{Z}_p \) for some prime integer \( p \). Then \( (N : M) = 0 \) is 1-prime, but by [7, Example 3.7] \( N \) is not a 1-prime submodule of \( M \).

Let \( N \) be a submodule of an \( R \)-module \( M \). By radical of \( N \), denoted \( \text{rad} N \), we mean that the intersection of all prime submodules of \( M \) containing \( N \). If there is no such prime exists, we define \( \text{rad} N = M \). For an ideal \( I \) of \( R \), we denote the radical of \( I \) by \( \sqrt{I} \).
An $R$-module $M$ is called a multiplication module, if for each submodule $N$ of $M$ there exists an ideal $I$ of $R$ such that $N = IM$. In this case, we can take $I = (N : M)$. If $N_1 = I_1M$ and $N_2 = I_2M$ are two submodules of an $R$-module $M$ for some ideals $I_1$ and $I_2$ of $R$, then $N_1N_2$ is used to denote $I_1I_2M$.

**Theorem 2.4.** Let $R$ be a ring, $M$ an $R$-module and $N$ a submodule of $M$. If $N$ is an $n$-prime submodule of $M$ for some positive integer $n$, then:

1. $N$ is a primary submodule of $M$, and so $(N : M)$ is a primary ideal of $R$ and $\sqrt{N : M}$ is a prime ideal of $R$.
2. If $(N : M)$ is a prime ideal of $R$, then $N$ is a prime submodule of $M$.
3. If $M$ is finitely generated faithful multiplication, then rad $N$ is a prime submodule of $M$.
4. If $R$ is a Bézout ring and $M$ is multiplication, then $(\text{rad} N)^n \subseteq N$. In particular, this holds if $R$ is a valuation domain.

**Proof:** (1) Let $am \in N$ for $a \in R$ and $m \in M \setminus N$. Clearly $a \in R \setminus U(R)$. Then $a^{n+1}m \in N$ implies that $a \in \sqrt{N : M}$.

(2) Since $(N : M)$ is a prime ideal of $R$, $\sqrt{N : M} = (N : M)$. Let $am \in N$ for $a \in R$ and $m \in M \setminus N$. By (1) $N$ is primary and then $a \in \sqrt{N : M} = (N : M)$. Thus $N$ is a prime submodule of $M$.

(3) Since $N$ is proper and $M$ is multiplication, by [5, Theorem 2.12], rad $N = \sqrt{N : MM}$. By (1), $\sqrt{N : M}$ is prime. Now since $M$ is finitely generated faithful, by [5, Theorem 3.1 and Lemma 2.10], rad $N \neq M$ is a prime submodule of $M$.

(4) By (1) and Lemma 2.2(2), $\sqrt{N : M}$ is a prime ideal of $R$ and $(N : M)$ is an $n$-absorbing ideal of $R$. Since $R$ is Bézout, by [1, Lemma 5.4], $(\sqrt{N : M})^n \subseteq (N : M)$. Thus by using [5, Theorem 2.12], $(\text{rad} N)^n = (\sqrt{N : M})^n M \subseteq (N : M)M = N$. □

**Theorem 2.5.** Let $(R, m)$ be a local ring, $M$ an $R$-module and $N$ a submodule of $M$ such that $m^n \subseteq (N : M)$ for some positive integer $n$. Then $N$ is an $n$-prime submodule of $M$.

**Proof:** Let $a_1 \cdots a_n+1m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M \setminus N$. Since $R \setminus U(R) = m$ and $m^n \subseteq (N : M)$, every $n$-subproduct of the $a_i$’s is in $(N : M)$. □

**Example 2.6.** Let $R = \mathbb{Z}_p^\omega$ and $m = \bar{p}R$, where $p \in \mathbb{Z}$ is a positive integer. Then $(R, m)$ is local. Every proper ideal of $R$ has the form $I_n = \bar{p}^nR$ for $n < t$. Thus by Theorem 2.5, $I_n$ is an $n$-prime ideal of $R$.

**Corollary 2.7.** Let $R$ be a Noetherian ring, $M$ an $R$-module and $N$ a $p$-primary submodule of $M$ for some prime ideal $p$ of $R$. Then $N_p$ is an $n$-prime submodule of $M_p$ for some positive integer $n$. 
Proof: Let $N$ be a $p$-primary submodule of $M$. Then $(N : M)$ is a $p$-primary ideal of $R$. Thus by [9, Theorem 5.37], $(N : M)_p$ is a $pR_p$-primary ideal of $R_p$.

Since $(R_p, pR_p)$ is Noetherian local ring, $p^nR_p \subseteq (N : M)_p \subseteq (N_p : M_p)$ for some positive integer $n$. Now by Theorem 2.5, $N_p$ is an $n$-prime submodule of $M_p$. □

**Theorem 2.8.** Let $R$ be a PID and $n > 1$ an integer.

1. If $(R, \mathfrak{m})$ is local, then every ideal of $R$ is $n$-prime for some positive integer $n$.

2. If $R$ is not local, then every $n$-prime ideal of $R$ is prime.

Proof: (1) Let $I$ be an ideal of $R$. Since every non-zero prime ideal of $R$ is maximal and $(R, \mathfrak{m})$ is local, $I$ is $\mathfrak{m}$-primary. Now since $R$ is Noetherian, $\mathfrak{m}^n \subseteq I$ for some positive integer $n$. Then by Theorem 2.5, $I$ is an $n$-prime ideal of $R$.

(2) Let $R$ be a non-local PID. Then $R$ has at least two distinct prime elements. Now if $I$ is an $n$-prime ideal of $R$, then $I$ is primary by Theorem 2.4(1). Thus $I = p^tR$ for some prime element $p$ of $R$ and positive integer $t \leq n$. Let $t \neq 1$ and $a_1 = \cdots = a_{t-1} = r = p$ and $a_t = \cdots = a_{n+1} = q$ which $q \neq p$ is a prime element of $R$. Then $a_1 \cdots a_{n+1}r \in I$. However, $r \notin I$ and no proper $n$-subproduct is in $I$, a contradiction. Therefore $t = 1$ and hence $I$ is prime. □

**Remark 2.9.** It is clear that every $n$-prime ideal of $R$ is an $n$-absorbing ideal of $R$. However, the converse need not be true in general. For example, if $R = \mathbb{Z}$ and $I = 4\mathbb{Z}$, then $I$ is a 2-absorbing ideal of $R$ which is not a 2-prime ideal of $R$ by Theorem 2.8.

**Theorem 2.10.** Let $R$ be a ring such that every proper ideal of $R$ is an $n$-prime ideal for some positive integer $n$. Then $R$ is a local ring.

Proof: Let $m_1$ and $m_2$ be two maximal ideals of $R$. Then $I = m_1 \cap m_2$ is an $n$-prime ideal for some positive integer $n$. By Theorem 2.4(1), $I$ is a primary ideal of $R$. Then $m_1 = m_2$. □

**Corollary 2.11.** Let $R$ be a ring and $n$ a positive integer such that every proper ideal of $R$ is an $n$-prime ideal of $R$. Then $R$ is local and $\dim R = 0$.

Proof: By Theorem 2.10, $R$ is local. Since every $n$-prime ideal is an $n$-absorbing ideal, by [1, Theorem 5.9], $\dim R = 0$. □

**Theorem 2.12.** Let $R = S[X]$ be a polynomial ring with coefficients in a domain $S$. Then every $n$-prime ideal of $R$ is prime.

Proof: Let $I$ be a non-prime ideal of $R = S[X]$. Then there are $f, g \in R \setminus I$ such that $fg \in I$. Since $S$ is domain, $S$ has not non-zero nilpotent element. Then by [9, Exercise 1.36], $fg+1$ is non-unit. On the other hand, $f(fg+1)^n = fg(fg+1)^n \in I$. However, $g \notin I$ and $(fg+1)^n \notin I$ and $f(fg+1)^{n-1} \notin I$. Then $I$ is not an $n$-prime ideal of $R$. □
Theorem 2.13. Let $R$ be a Dedekind domain and $M$ be a finitely generated faithful multiplication $R$-module. If $N$ is an $n$-prime submodule of $M$, then $N = N_1 \cdots N_t$ for some maximal submodules $N_1, \ldots, N_t$ of $M$ with $1 \leq t \leq n$.

Proof: Suppose that $N$ is an $n$-prime submodule of $M$. Then by Lemma 2.2(2), $(N : M)$ is an $n$-absorbing ideal of $R$. Now by [1, Theorem 5.1], $(N : M) = m_1 \cdots m_t$ for some maximal ideals $m_1, \ldots, m_t$ of $R$ with $1 \leq t \leq n$. Thus $N = (N : M)M = m_1 \cdots m_t M = m_1 M \cdots m_t M$ and $m_1 M, \ldots, m_t M$ are maximal submodules of $M$ by [5, Theorem 2.5 and Theorem 3.1].

3. Extensions of $n$-prime submodules

In this section, we investigate the stability of $n$-prime submodules in various module-theoretic constructions.

Let $N$ be a proper submodule of an $R$-module $M$. For $x \in M$, $N_x = (N : x) = \{r \in R \mid rx \in N\}$ is an ideal of $R$ and clearly $(N : M) \subseteq (N : x)$.

Proposition 3.1. Let $R$ be a ring and $M$ an $R$-module. If $N$ is an $n$-prime submodule of $M$, then $N_x$ is an $n$-prime ideal of $R$ and so is $n$-absorbing ideal of $R$ for all $x \in M \setminus N$. Moreover $\omega(N_x) \leq \nu(N)$ for all $x \in M$.

Proof: Let $N$ be an $n$-prime submodule of $M$ and $a_1 \cdots a_{n+1} r \in N_x$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $r \in R \setminus N_x$. Then $a_1 \cdots a_{n+1} rx \in N$ and $rx \notin N$. Since $N$ is an $n$-prime submodule of $M$, there are $n$ of the $a_i$’s whose product is in $(N : M) \subseteq (N : x) = N_x$. This implies that $N_x$ is an $n$-prime ideal and so is an $n$-absorbing ideal of $R$.

The “moreover” statement is clear if $x \in M \setminus N$ by above argument. If $x \in N$, then $N_x = R$ and hence $\omega(N_x) = 0 \leq \nu(N)$.

For each $r \in R$ and every submodule $N$ of $M$, we consider $N_r = (N : M r) = \{x \in M \mid rx \in N\}$.

Proposition 3.2. Let $R$ be a ring. If $N$ is an $n$-prime submodule of an $R$-module $M$, then $N_r$ is an $n$-prime submodule of $M$ for any $r \in \sqrt{N} : M \setminus (N : M)$.

Proof: Let $a_1 \cdots a_{n+1} m \in N_r$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$, $m \in M$. Then $a_1 \cdots a_{n+1} rm \in N$. Since $N$ is an $n$-prime submodule of $M$, $rm \in N$ or there are $n$ of the $a_i$’s whose product is in $(N : M)$. Thus $m \in N_r$ or there are $n$ of the $a_i$’s whose product is in $(N_r : M)$, since $(N : M) \subseteq (N_r : M)$.

Proposition 3.3. Let $R$ be a ring and $M$ an $R$-module. If $N_i$ is an $n_i$-prime submodule of $M$ such that $(N_i : M) = (N_j : M)$ for all $1 \leq i, j \leq t$, then $\cap_{i=1}^t N_i$ is an $n$-prime submodule of $M$ for $n = \max\{n_i \mid 1 \leq i \leq t\}$.

Proof: Let $t = 2$ and $n = \max\{n_1, n_2\}$. Suppose that $a_1 \cdots a_{n+1} m \in N_1 \cap N_2$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$, $m \in M$. Then $a_1 \cdots a_{n+1} m \in N_1$ and $a_1 \cdots a_{n+1} m \in N_2$. 

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Since $N_1$ and $N_2$ are respectively $n_1$-prime and $n_2$-prime, either $m \in N_1 \cap N_2$ or there are $n_1$ of the $a_i$’s whose product is in $(N_1 : M)$ or there are $n_2$ of the $a_i$’s whose product is in $(N_2 : M)$. If $m \in N_1 \cap N_2$, then we are done. In other words, there are $n$ of the $a_i$’s whose product is in $(N_1 \cap N_2 : M) = (N_1 : M) = (N_2 : M)$ for $n = \max\{n_1, n_2\}$. This implies that $N_1 \cap N_2$ is an $n$-prime submodule of $M$. The proof for $t > 2$ is follows similarly by induction on $t$. 

The following example shows that Proposition 3.3 is not true in general.

**Example 3.4.** Let $R = \mathbb{Z}$, $M = \mathbb{Z} \oplus \mathbb{Z}$, $N_1 = \mathbb{Z} \oplus \mathbb{Z}$ and $N_2 = \mathbb{Z} \oplus \mathbb{Z}$. Then $N_1$ and $N_2$ are 1-prime submodules, but $N = N_1 \cap N_2 = 6\mathbb{Z} \oplus \mathbb{Z}$ is not an $n$-prime submodule for all positive integer $n$. Since $(N : M) = 6\mathbb{Z}$ is not $n$-prime, by Theorem 2.8 and then by Lemma 2.2(2), $N$ is not an $n$-prime submodule of $M$.

**Theorem 3.5.** Let $R$ be a ring, $M$ an $R$-module and $N$ an $n$-prime submodule of $M$. Then for any submodule $K$ of $M$ either $K \subseteq N$ or $N \cap K$ is an $n$-prime submodule of $K$.

**Proof:** Let $K$ be a submodule of $M$ such that $K \nsubseteq N$. Then $N \cap K \subset K$. Now if $a_1 \cdots a_{n+1} k \in N \cap K$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $k \in K$, then $a_1 \cdots a_{n+1} k \in N$. Since $N$ is an $n$-prime submodule of $M$, $k \in N$ or there are $n$ of the $a_i$’s whose product is in $(N : M)$. Thus $k \in N \cap K$ or there are $n$ of the $a_i$’s whose product is in $(N \cap K : K)$, since $(N : M) \subseteq (N \cap K : K)$. $\square$

**Theorem 3.6.** Let $R$ be a ring and $f : M \rightarrow M'$ be a homomorphism of $R$-modules. Then the following hold:

1. If $N'$ is an $n$-prime submodule of $M'$ such that $f(M) \nsubseteq N'$, then $f^{-1}(N')$ is an $n$-prime submodule of $M$.

2. If $f$ is surjective and $N$ is an $n$-prime submodule of $M$ such that $\ker f \subseteq N$, then $f(N)$ is an $n$-prime submodule of $M'$.

**Proof:**

(1) Let $N'$ be an $n$-prime submodule of $M'$ and $a_1 \cdots a_{n+1} m \in f^{-1}(N')$ for non-unit elements $a_1, \ldots, a_{n+1} \in R$ and $m \in M$. Then $a_1 \cdots a_{n+1} f(m) = f(a_1 \cdots a_{n+1} m) \in N'$. Since $N'$ is an $n$-prime submodule of $M$, $f(m) \in N'$ or there are $n$ of the $a_i$’s whose product is in $(N' : M')$. Hence $m \in f^{-1}(N')$ or there are $n$ of the $a_i$’s whose product is in $(f^{-1}(N') : M)$, since $(N' : M') \subseteq (f^{-1}(N') : M)$.

(2) Let $N$ be an $n$-prime submodule of $M$ and $a_1 \cdots a_{n+1} m' \in f(N)$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m' \in M'$. Since $f$ is surjective, $m' = f(m)$ for some $m \in M$. Then

$$a_1 \cdots a_{n+1} m' = a_1 \cdots a_{n+1} f(m) = f(a_1 \cdots a_{n+1} m) = f(n)$$

for some $n \in N$. Thus $a_1 \cdots a_{n+1} m - n = \ker f \subseteq N$. Therefore $a_1 \cdots a_{n+1} m \in N$. Since $N$ is an $n$-prime submodule of $M$, either $m \in N$ or there are $n$ of the $a_i$’s whose product is in $(N : M)$. Hence $m' \in f(N)$ or there are $n$ of the $a_i$’s whose product is in $(f(N) : M')$ (Note that, $(N : M) \subseteq (f(N) : M')$, since $f$ is surjective). $\square$
Corollary 3.7. Let $R$ be a ring, $M$ an $R$-module and $N$, $K$ proper submodules of $M$ such that $N \subseteq K$. Then $K$ is an $n$-prime submodule of $M$ if and only if $K/N$ is an $n$-prime submodule of $M/N$.

Proof: Consider the natural projection $\pi : M \to M/N$ defined by $\pi(m) = m + N$ and use Theorem 3.6. □

Let $R$ be a ring and $M$ an $R$-module. Let $N$ be a submodule of $M$. A submodule $K$ of $M$ maximal with respect to the property that $K \cap N = 0$ is called a complement of $N$ in $M$. A submodule $K$ of $M$ will be called complement in $M$ if there exists a submodule $N$ of $M$ such that $K$ is a complement of $N$ in $M$. A submodule $N$ of $M$ will be called essential if $N \cap K \neq 0$ for every non-zero submodule $K$ of $M$. Also a submodule $N$ of $M$ will be called essential in a submodule $L$ of $M$ containing $N$, if $N$ is essential as a submodule of $L$. It is not difficult to prove that if $K$ is a complement in $M$, then $K$ is not essential in any submodule $L$ of $M$ containing $K$.

Theorem 3.8. Let $R$ be a ring, $M$ an $R$-module and $N$ an $n$-prime submodule of $M$. If $K$ is a submodule of $M$ containing $N$ such that $K/N$ is a complement in $M/N$, then $K$ is an $n$-prime submodule of $M$.

Proof: Let $a_1 \cdots a_{n+1} m \in K$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \notin K$. Then $L = K + Rm$ is a submodule of $M$ which contains $K$ properly and $a_1 a_2 \cdots a_{n+1} L \subseteq K$. $K/N$ is not essential in $L/N$, since $K/N$ is a complement in $M/N$. Thus there exists a submodule $L'$ of $L$ such that $N \subseteq L'$ and $K \cap L' = N$. Let $m' \in L' \setminus N$. Then $a_1 a_2 \cdots a_{n+1} m' \in a_1 a_2 \cdots a_{n+1} L' \subseteq (a_1 a_2 \cdots a_{n+1} L) \cap L' \subseteq K \cap L' = N$. Since $N$ is an $n$-prime submodule of $M$, there are $n$ of the $a_i$’s whose product is in $(N : M) \subseteq (K : M)$. Hence $K$ is an $n$-prime submodule of $M$. □

Let $M$ be an $R$-module. By zero divisors of $M$, denoted $Z_R(M)$, we mean that the set of elements $r \in R$ such that $rm = 0$ for some non-zero element $m \in M$.

Theorem 3.9. Let $R$ be a ring, $M$ an $R$-module and $N$ a submodule of $M$. Let $S$ be a multiplicatively closed subset of $R$ such that $S \cap Z_R(M/N) = \emptyset$. If $N$ is an $n$-prime submodule of $M$, then $S^{-1}N$ is an $n$-prime submodule of $S^{-1}M$.

Proof: Let $N$ be an $n$-prime submodule of $M$. Since $S \cap Z_R(M/N) = \emptyset$, it is easily seen that $S^{-1}N \neq S^{-1}M$. Suppose that $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} m \in S^{-1}N$ for $\frac{a_1}{s_1}, \ldots, \frac{a_{n+1}}{s_{n+1}} \in S^{-1}R \setminus U(S^{-1}R)$ and $\frac{m}{t} \in S^{-1}M$. Then $\frac{a_1}{s_1} \cdots \frac{a_{n+1}}{s_{n+1}} m \frac{t}{s} = \frac{m}{t}$ for some $n \in N$ and $t \in S$. Thus $a_1 \cdots a_{n+1} tm = s_1 \cdots s_{n+1} su \in N$ for some $u \in S$. Clearly $a_i$’s are non-unit in $R$. Thus, since $N$ is an $n$-prime submodule of $M$, there are $n$ of the $a_i$’s whose product is in $(N : M)$ or there are $n - 1$ of the $a_i$’s whose product with $tu$ is in $(N : M)$ or $m \in N$. If $m \in N$, then $\frac{m}{t} \in S^{-1}N$. If there are $n$ of the $a_i$’s whose product is in $(N : M)$, then there are $n$ of the $a_i$’s whose product is in $S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M)$. If there are $n - 1$ of the $a_i$’s
whose product with $tu$ is in $(N : M)$, for example $a_1 \cdots a_{n-1}(tu) \in (N : M)$, then $a_1 \cdots a_n(tu) \in (N : M)$. Thus
\[
\frac{a_1 \cdots a_{n-1} a_n}{s_1 \cdots s_n} = \frac{a_1 \cdots a_{n-1} a_n(tu)}{s_1 \cdots s_n(tu)} = \frac{a_1 \cdots a_n(tu)}{s_1 \cdots s_n(tu)} \in S^{-1}(N : M) \subseteq (S^{-1}N : S^{-1}M).
\]
This implies that $S^{-1}N$ is an $n$-prime submodule of $S^{-1}M$. \hfill \Box

**Theorem 3.10.** Let $R$ be a ring, $M$, $M'$ $R$-modules, $N$ a submodule of $M$, $N'$ a submodule of $M'$ and $I$, $I'$ two ideals of $R$. Then the following hold:

1. If $N \times N'$ is an $n$-prime submodule of $M \times M'$, then $N$ and $N'$ are respectively an $n$-prime submodule of $M$ and $M'$. The converse is true if $(N : M) = (N' : M')$.

2. $N$ (resp. $N'$) is an $n$-prime submodule of $M$ (resp. $M'$) if and only if $N \times M'$ (resp. $M \times N'$) is an $n$-prime submodule of $M \times M'$.

3. If $I \times I'$ is an $n$-prime submodule of the $R$-module $R \times R$, then $I$ and $I'$ are $n$-prime ideals of $R$. The converse is true if $I = I'$.

4. If $I \times I'$ is an $n$-prime ideal of $R \times R$, then $I$ and $I'$ are $n$-prime ideals of $R$. The converse is true if $I = I'$.

5. $I$ (resp. $I'$) is an $n$-prime ideal of $R$ (resp. $R'$) if and only if $I \times R'$ (resp. $R \times I'$) is an $n$-prime submodule of the $R$-module $R \times R'$.

6. $I$ (resp. $I'$) is an $n$-prime ideal of $R$ (resp. $R'$) if and only if $I \times R'$ (resp. $R \times I'$) is an $n$-prime ideal of $R \times R'$.

**Proof:**

1. Let $N \times N'$ be an $n$-prime submodule of $M \times M'$ and let $a_1 \cdots a_{n+1}m \in N$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $m \in M$. Then $a_1 \cdots a_{n+1}(m, 0) \in N \times N'$. Thus $(m, 0) \in N \times N'$ or there are $n$ of the $a_i$’s whose product is in $(N \times N' : M \times M') = (N : M) \cap (N' : M')$. Hence $N$ is an $n$-prime submodule of $M$. By a similar argument, $N'$ is an $n$-prime submodule of $M'$. Conversely let $a_1 \cdots a_{n+1}(m, m') \in N \times N'$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $(m, m') \in M \times M' \setminus N \times N'$. Then $m \notin N$ or $m' \notin N'$. Let $m \notin N$. Then $a_1 \cdots a_{n+1}m \in N$, implies that there are $n$ of the $a_i$’s whose product is in $(N : M) = (N \times N' : M \times M')$.

2. Let $N$ be an $n$-prime submodule of $M$ and $a_1 \cdots a_{n+1}(m, m') \in N \times M'$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $(m, m') \in M \times M'$. Then $a_1 \cdots a_{n+1}m \in N$. Since $N$ is an $n$-prime submodule of $M$, either $m \in N$ or there are $n$ of the $a_i$’s whose product is in $(N : M) = (N \times M' : M \times M')$. This implies that $N \times M'$ is an $n$-prime submodule of $M \times M'$. The converse is similar to (1). By a similar argument, $N'$ is an $n$-prime submodule of $M'$ if and only if $M \times N'$ is an $n$-prime submodule of $M \times M'$. 


(3) By (1).
(4) The proof is similar to the proof of (1).
(5) By (2).
(6) The proof is similar to the proof of (2).

**Example 3.11.** Let \( R = \mathbb{Z}_{p^n}, M = R \oplus R, N_n = I_n \oplus I_n, L_n = R \oplus I_n \) and \( K_n = I_n \oplus R(n < t) \). Then by Example 2.6 and Theorem 3.10(3), \( N_n, L_n \) and \( K_n \) are prime submodules of \( M \).

Let \( R \) be a ring and \( M \) an \( R \)-module. Then \( R(+)M = R \times M \) is a ring with identity \((1, 0)\) under addition defined by \((r, m) + (s, n) = (r + s, m + n)\) and multiplication defined by \((r, m)(s, n) = (rs, rn + sm)\). We view \( R \) as a subring of \( R(+)M \) via \( r \mapsto (r, 0) \).

**Theorem 3.12.** Let \( R \) be a ring, \( M \) an \( R \)-module, \( I \) an \( n_1 \)-absorbing ideal of \( R \) and \( N \) an \( n_2 \)-prime submodule of \( M \) with \( IM \subseteq N \). Then \( I(+)N \) is an \( n \)-absorbing ideal of \( R(+)M \) for \( n = n_1 + n_2 \). Conversely if \( I(+)N \) is an \( n \)-absorbing ideal of \( R(+)M \), then \( I \) is an \( n \)-absorbing ideal of \( R \).

**Proof:** Let \( n = n_1 + n_2 \). Assume that \((a_1, m_1) \cdots (a_{n_1+1}, m_{n_1+1}) \in I(+)N\) for \((a_1, m_1), \ldots, (a_{n_1+1}, m_{n_1+1}) \in R(+)M\). Without loss of generality suppose that these elements are not in \( U(R(+)M) \). Then \( a_1 \cdots a_{n_1+1} \in I \) and

\[
\sum_{i=1}^{n_1+1} a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n_1+1}m_i \in N
\]  

(3.1)

Since \( I \) is an \( n_1 \)-absorbing ideal of \( R \), there are \( n_1 \) of the \( a_i \)'s whose product is in \( I \). For example, let \( a_1 \cdots a_{n_1} \in I \). The terms of (3.1) that contain \( a_1 \cdots a_{n_1} \), are in \( IM \subseteq N \). Thus \( \sum_{i=1}^{n_1} a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n_1+1}m_i \subseteq N \), where \( a_0 \) is assumed that to be 1. But

\[
\sum_{i=1}^{n_1} a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n_1+1}m_i = a_{n_1+1} \cdots a_{n_1+1}m_1 + \sum_{i=1}^{n_1} a_1 \cdots a_{i-1}a_{i+1} \cdots a_{n_1+1}m_i \in N
\]

Since \( U(R(+)M) = U(R(+)M) \) by [2, Theorem 3.7], \( a_i \)'s (\( 1 \leq i \leq n_1 + 1 \)) are non-unit. Now, since \( N \) is an \( n_2 \)-prime submodule of \( M \), \( \sum_{i=1}^{n_1+1} a_1 \cdots a_{i-1}a_{i+1}m_i \in N \) or there are \( n_2 \) of the \( a_i \)'s (\( n_1 + 1 \leq i \leq n_1 + 1 \)) whose product is in \( (N : M) \). If \( \sum_{i=1}^{n_1+1} a_1 \cdots a_{i-1}a_{i+1}m_i \in N \), then \( (a_1, m_1) \cdots (a_{n_1+1}, m_{n_1+1}) \in I(+)N \), and if there are \( n_2 \) of the \( a_i \)'s (\( n_1 + 1 \leq i \leq n_1 + 1 \)) whose product is in \( (N : M) \), for example \( a_{n_1+1} \cdots a_n \in (N : M) \), then

\[
(a_1, m_1) \cdots (a_{n_1+1}, m_{n_1+1})(a_{n_1+1}, m_{n_1+1}) \cdots (a_n, m_n) \in I(+)N\
\]

Hence \( I(+)N \) is an \( n = n_1 + n_2 \)-absorbing ideal of \( R(+)M \).

Now let \( I(+)N \) be an \( n \)-absorbing ideal of \( R(+)M \), and let \( a_1 \cdots a_{n_1+1} \in I \) for \( a_1, \ldots, a_{n_1+1} \in R \). Then \((a_1, 0) \cdots (a_{n_1+1}, m_{n_1+1}) \in I(+)N \). Thus there are \( n \) of \((a_i, 0)\)'s whose product is in \( I(+)N \). Hence there are \( n \) of the \( a_i \)'s whose product is in \( I \) and so \( I \) is an \( n \)-absorbing ideal of \( R \).

\( \square \)
4. AP n-modules

Let $R$ be a ring, $M$ an $R$-module and $I$ a proper ideal of $R$. Let $M_n(I)$ denote a submodule of $M$ generated by the following set:
$$\{m \mid a_1 \cdots a_{n+1} m \in IM \text{ for some } a_1, \ldots, a_{n+1} \in R \setminus U(R) \text{ such that } a_1 \cdots a_{n+1} \not\in I\}.$$

Lemma 4.1. Let $R$ be a ring, $M$ an $R$-module and $I$ an $n$-absorbing ideal of $R$. If $M_n(I) \subseteq IM \neq M$, then $IM$ is an $n$-prime submodule of $M$.

Proof: Let $a_1 \cdots a_{n+1} m \in IM$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ such that no proper subproduct of the $a_i$’s is in $(IM : M)$. Since $I \subseteq (IM : M)$ and $I$ is an $n$-absorbing ideal of $R$, $a_1 \cdots a_{n+1} \not\in I$. Thus $m \in M_n(I) \subseteq IM \neq M$, and hence $IM$ is an $n$-prime submodule of $M$. \qed

The following example shows that Lemma 4.1 fails if the condition that $M_n(I) \subseteq IM$ is removed.

Example 4.2. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Then $I = 4\mathbb{Z}$ is a 2-absorbing ideal of $R$, but $IM = 4\mathbb{Z} \oplus 4\mathbb{Z}$ is not a 2-prime submodule of $M$. It is easily seen that $2\mathbb{Z} \oplus 2\mathbb{Z} \subseteq M_2(I)$, and thus $M_2(I) \not\subseteq IM$.

Definition 4.3. Let $R$ be a ring, $M$ an $R$-module and $n$ a positive integer. We say that $M$ is an AP $n$-module if $M_n(I) \subseteq IM \neq M$ for any $n$-absorbing ideal $I$ of $R$. Also, $R$ is called an AP $n$-ring if $R$ is an AP $n$-module as $R$-module.

Remark 4.4. We say that $M$ is an AP $n$-module because for any $n$-absorbing ideal $I$ of $R$, $IM$ is an $n$-Prime submodule of $M$ by Lemma 4.1.

Lemma 4.5. Let $R$ be a ring and $n$ a positive integer. Then $R$ is an AP $n$-ring if and only if every $n$-absorbing ideal of $R$ is an $n$-prime ideal of $R$.

Proof: Let $R$ be an AP $n$-ring, $I$ an $n$-absorbing ideal of $R$ and let $a_1 \cdots a_{n+1} r \in I$ for $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ and $r \in R$ such that no proper subproduct of the $a_i$’s is in $I$. Since $I$ is $n$-absorbing, $a_1 \cdots a_{n+1} \not\in I$. Thus $r \in M_n(I) \subseteq I$. Hence $I$ is $n$-prime. Conversely suppose that every $n$-absorbing ideal of $R$ is an $n$-prime ideal of $R$. Let $I$ be an $n$-absorbing ideal of $R$ and $r \in R$ be a generator of $M_n(I)$. Then $a_1 \cdots a_{n+1} r \in IR = I$ for some $a_1, \ldots, a_{n+1} \in R \setminus U(R)$ such that $a_1 \cdots a_{n+1} \not\in I$. Thus no proper subproduct of the $a_i$’s is in $I$ and hence $r \in I$, since $I$ is an $n$-prime ideal of $R$. Therefore $R$ is an AP $n$-ring. \qed

Corollary 4.6. Let $M$ be a multiplication $R$-module, $N$ a proper submodule of $M$ and $n$ a positive integer. Consider the following statements:

1. $N$ is an $n$-prime submodule of $M$.
2. $(N : M)$ is an $n$-prime ideal of $R$.
Then (1) ⇔ (2) ⇒ (3). Moreover, if \( M \) is a finitely generated faithful module, then (3) ⇒ (2).

**Proof:** (1) ⇒ (2) By Lemma 2.2(2).

(2) ⇒ (1) Let \( I = (N : M) \). We show that \((M_n(I) : M) \subseteq I\) and use Lemma 4.1. Let \( r \in (M_n(I) : M) \) and \( m \in M \setminus M_n(I) \). Then \( rm \in M_n(I) \). Thus \( rm = s_1m_1 + \cdots + s_tm_t \) for some \( s_i \in R \) and \( m_i \in M_n(I) \) (\( 1 \leq i \leq t \)) by definition of \( M_n(I) \). Since \( m_i \in M_n(I) \), there are \( a_{i_1}, \ldots, a_{i_{n+1}} \in R \setminus U(R) \) such that \( a_{i_1} \cdots a_{i_{n+1}}m_i \in IM \) and \( a_{i_1} \cdots a_{i_{n+1}} \notin I \). Then \( \prod_{i=1}^t \prod_{j=1}^{n+1} a_{i_j} rm \in IM \). Since \( m \notin M_n(I) \),

\[
\prod_{i=1}^t \prod_{j=1}^{n+1} a_{i_j} r \in I.
\]

Thus we have \( \prod_{i=2}^t \prod_{j=1}^{n+1} a_{i_j} r \in I \), since \( I \) is an \( n \)-prime and no proper subproduct of \( a_{i_j} \)'s (\( 1 \leq j \leq n+1 \)) is in \( I \). Repeating this process follows that \( r \in I \). Hence \( (M_n(I) : M) \subseteq I \). Since \( M \) is multiplication, \( M_n(I) \subseteq IM = N \).

(2) ⇒ (3) Clear.

(3) ⇒ (2) By [5, Theorem 3.1], \((N : M) = I\). \( \square \)

**Corollary 4.7.** Let \( R \) be a ring and \( M \) a finitely generated faithful multiplication \( R \)-module. Then \( R \) is an AP \( n \)-ring if and only if \( M \) is an AP \( n \)-module.

**Proof:** Let \( R \) be an AP \( n \)-ring and \( I \) an \( n \)-absorbing ideal of \( R \). Then \( I \) is an \( n \)-prime ideal of \( R \), by Lemma 4.5. Since \( M \) is a multiplication module, by the proof of Corollary 4.6(2) ⇒ (1)), \( M_n(I) \subseteq IM \). Now since \( M \) is a finitely generated faithful multiplication module, by [5, Theorem 3.1], \( IM \neq M \). Hence \( M \) is an AP \( n \)-module. Conversely suppose that \( M \) is an AP \( n \)-module and \( I \) is an \( n \)-absorbing ideal of \( R \). Then by Lemma 4.1, \( IM \) is an \( n \)-prime submodule of \( M \). Since \( M \) is a finitely generated faithful multiplication module, by Lemma 2.2(2) and [5, Theorem 3.1], \((IM : M) = I \) is an \( n \)-prime ideal of \( R \). Hence by Lemma 4.5, \( R \) is an AP \( n \)-ring. \( \square \)

**Theorem 4.8.** Let \((R,m) \) be an Artinian local ring and \( n \) a positive integer such that \( m^n = m^{n+1} = \cdots \). Then every ideal of \( R \) is an \( n \)-prime ideal. In particular, \( R \) is an AP \( n \)-ring.

**Proof:** Note that \( R \) is Noetherian and \( \text{dim} R = 0 \), by [9, Corollary 8.45]. Let \( I \) be an ideal of \( R \). Then \( I \) is \( m \)-primary. Thus \( m^n \subseteq m^t \subseteq I \) for some positive integer \( t \leq n \). Hence by Theorem 2.5, \( I \) is an \( n \)-prime ideal of \( R \). The “in particular” statement is clear. \( \square \)

**Theorem 4.9.** Let \( R \) be a Noetherian valuation domain and \( n \) a positive integer. Then \( R \) is an AP \( n \)-ring.
Proof: Note that $(R, m)$ is a local PID and then $\dim R = 1$. Let $I$ be an $n$-absorbing ideal of $R$. By [9, Theorem 15.42], $I = m^t$ for some positive integer $t$. Let $m = Rp$ for some prime element $p \in R$ and $t > n$. Then $p^n = rp^t$ for some $r \in R$, since $I$ is an $n$-absorbing ideal of $R$. Thus $rp^{t-n} = 1$ and hence $p$ is unit, which is a contradiction. Therefore $t \leq n$. Then by Theorem 2.5, $I$ is a $t$-prime ideal and so it is an $n$-prime ideal of $R$. Hence by Lemma 4.5, $R$ is an AP $n$-ring.

Corollary 4.10. Let $R$ be a DVR and $n$ a positive integer. Then $R$ is an AP $n$-ring.

Example 4.11. Let $R = \mathbb{Z}[\sqrt{-5}]$, $M = 2R + (\sqrt{-5} - 1)R$ and $n$ a positive integer. Since $R$ is a Dedekind domain, $M$ is a finitely generated faithful multiplication $R$-module. Then by Corollary 4.10, $R_p$ is an AP $n$-ring for all non-zero prime ideal $P$ of $R$. Since $M_P$ is a finitely generated faithful multiplication $R_P$-module, by Corollary 4.7, $M_P$ is an AP $n$-module.

Theorem 4.12. Let $R$ be a zero-dimensional Bézout ring and $n$ a positive integer. Then for each prime ideal $P$ of $R$, $R_P$ is an AP $n$-ring.

Proof: Without loss of generality, we may assume that $R$ is a local ring. Let $I$ be an $n$-absorbing ideal of $R$. Since $(R, m)$ is local and $\dim R = 0$, $\sqrt{I} = m$. By [1, Lemma 5.4], $m^n \subseteq I$. Thus by Theorem 2.5, $I$ is an $n$-prime ideal of $R$.

Proposition 4.13. Let $R$ be a ring, $M$ an $R$-module and $N_i$ an $n_i$-prime submodule of $M$ for all $1 \leq i \leq t$. Then $\bigcap_{i=1}^{t} N_i : M$ is an $n$-absorbing ideal of $R$ for $n = n_1 + \cdots + n_t$. Moreover if $M$ is a multiplication AP $n$-module, then $\bigcap_{i=1}^{t} N_i$ is an $n$-prime submodule of $M$.

Proof: By Theorem 4.6, $(N_i : M)$ is an $n_i$-absorbing ideal of $R$. Hence by [1, Theorem 2.1(c)], $(\bigcap_{i=1}^{t} N_i : M) = \bigcap_{i=1}^{t} (N_i : M)$ is an $n$-absorbing ideal of $R$ for $n = n_1 + \cdots + n_t$.

For “moreover” part, since $M$ is multiplication, $\bigcap_{i=1}^{t} N_i = (\bigcap_{i=1}^{t} N_i : M)M$. Therefore $\bigcap_{i=1}^{t} N_i$ is an $n$-prime submodule of $M$, by Lemma 4.1.

Let $R$ be a ring and $M$ an $R$-module. If $I$ is an $n_1$-absorbing ideal of a ring $R$ and $N$ is an $n_2$-prime submodule of an $R$-module $M$, then $IN$ is not necessarily an $n$-prime submodule of $M$ for some positive integer $n$, as the following example shows.

Example 4.14. Let $R = \mathbb{Z}$ and $M = \mathbb{Z} \oplus \mathbb{Z}$. Then $I = 4\mathbb{Z}$ is a 2-absorbing ideal of $R$ and $N = 3\mathbb{Z} \oplus \mathbb{Z}$ is a 1-prime (prime) submodule of $M$ but $IN = 12\mathbb{Z} \oplus 4\mathbb{Z}$ is not an $n$-prime submodule of $M$ for any positive integer $n$. Since $(IN : M) = 12\mathbb{Z}$ is not an $n$-prime ideal of $R$, by Theorem 2.8.
Theorem 4.15. Let $R$ be a ring, $M$ a finitely generated faithful multiplication $R$-module, $I$ an $n_1$-absorbing ideal of $R$ and $N$ an $n_2$-prime submodule of $M$. If $R$ is an AP $n$-ring for $n = n_1 + n_2$ and two ideals $I$ and $(N : M)$ are comaximal, then $IN$ is an $n$-prime submodule of $M$.

Proof: Since $M$ is a multiplication $R$-module, $(IN : M) = IN = I(N : M)M$. By [5, Theorem 3.1], hypotheses and [1, Theorem 2.1(e)], $(IN : M) = I \cap (N : M)$ is an $n$-absorbing ideal of $R$ if $n = n_1 + n_2$. Hence, $IN$ is an $n$-prime submodule of $M$, by Corollary 4.7 and Lemma 4.1.

Lemma 4.16. Let $M$ be a finitely generated faithful multiplication $R$-module. Then the ideals $I_1, \ldots, I_t$ are pairwise comaximal ideals of $R$ if and only if $N_1 = I_1M, \ldots, N_t = I_tM$ are pairwise comaximal submodules of $M$. In this case, $N_1 \cap \cdots \cap N_t = N_1 \cap \cdots \cap N_t$.

Proof: The necessity is clear. To prove the sufficiency, we observe that $(I_1 + I_2)M = I_1M + I_2M = N_1 + N_2 = M$. Since $M$ is finitely generated faithful multiplication, by [5, Theorem 3.1], $I_1 + I_2 = R$. In this case, by [5, Corollary 1.7], $N_1N_2 = I_1I_2M = (I_1 \cap I_2)M = I_1M \cap I_2M = N_1 \cap N_2$. Now the assertion follows by induction on $t$.

Theorem 4.17. Let $R$ be a ring and $M$ a finitely generated faithful multiplication $R$-module. If $R$ is an AP $n$-ring and $P_1, \ldots, P_n$ are prime submodules of $M$ that are pairwise comaximal, then $N = P_1 \cdots P_n$ is an $n$-prime submodule of $M$.

Proof: By [5, Corollary 2.11], $P_i = p_iM$ for some prime ideal $p_i$ of $R$ and by Lemma 4.16, $p_i$’s are pairwise comaximal. Then $N = P_1 \cdots P_n = p_1 \cdots p_nM$. By [1, Theorem 2.6], $p_1 \cdots p_n$ is an $n$-absorbing ideal of $R$. Therefore $N$ is an $n$-prime submodule of $M$, by Corollary 4.7 and Lemma 4.1.

Lemma 4.18. Let $R$ be a ring, $M$ a multiplication $R$-module and $N$ a maximal submodule of $M$. If $M$ is an AP $n$-module, then $N^n$ is an $n$-prime submodule of $M$. Moreover, $\nu(N^n) \leq n$, and $\nu(N^n) = n$ if $N^{n+1} \subseteq N^n$.

Proof: By [5, Theorem 2.5], $N = mM$ for some maximal ideal $m$ of $R$. Then $m^n$ is an $n$-absorbing ideal of $R$, by [1, Lemma 2.8]. Hence, $N^n = m^nM$ is an $n$-prime submodule of $M$, by Lemma 4.1. The first part of the “moreover” statement is clear. Now if $N^{n+1} \subseteq N^n$, then $m^{n+1} \subseteq m^n$. Thus by [1, Lemma 2.8], $\omega(m^n) = n$. On the other hand, $(N^n : M) = m^n$ and by Lemma 2.2(2), $\omega(N^n : M) \leq \nu(N^n)$. Hence $\nu(N^n) = n$.

Theorem 4.19. Let $R$ be a ring, $M$ a multiplication $R$-module and $N_1, \ldots, N_n$ are maximal submodules of $M$. If $M$ is an AP $n$-module, then $N = N_1 \cdots N_n$ is an $n$-prime submodule of $M$. Moreover, $\nu(N) \leq n$.
Proof: By [5, Theorem 2.5], $N_i = m_i M$ for some maximal ideal $m_i$ of $R$. Then $N = m_1 M \cdots m_n M = m_1 \cdots m_n M$ and $m_1 \cdots m_n$ is an $n$-absorbing ideal of $R$ by [1, Theorem 2.9]. Since $M$ is an AP $n$-module, $N$ is an $n$-prime submodule of $M$, by Lemma 4.1. The “moreover” statement is clear. □

We call a submodule $N$ is a minimal $n$-prime submodule of $M$ if $N$ is minimal among all $n$-prime submodules of $M$ with respect to inclusion.

Proposition 4.20. Let $R$ be a ring, $M$ a finitely generated faithful multiplication $R$-module and $n$ a positive integer. If $M$ is an AP $n$-module, then the set of minimal $n$-prime submodules of $M$ is equal to

$$\{IM \mid I \text{ is a minimal } n\text{-absorbing ideal of } R\}.$$ 

Proof: Let $I$ be a minimal $n$-absorbing ideal of $R$. Since $M$ is an AP $n$-module, $IM$ is an $n$-prime submodule of $M$. Assume that $N$ is an $n$-prime submodule of $M$ such that $N \subseteq IM$. Then by [5, Theorem 3.1],

$$(N : M) \subseteq (IM : M) = I.$$ 

Since $I$ is a minimal $n$-absorbing ideal of $R$ and $(N : M)$ is an $n$-absorbing ideal of $R$ by Lemma 2.2(2), $(N : M) = I$. Hence $N = (N : M)M = IM$ and thus $IM$ is a minimal $n$-prime submodule of $M$. Now, assume that $N$ is a minimal $n$-prime submodule of $M$. Then $(N : M)$ is an $n$-absorbing ideal of $R$ and $N = (N : M)M$. Assume that $I$ is an $n$-absorbing ideal of $R$ such that $I \subseteq (N : M)$. Then $IM \subseteq N$ and $IM$ is an $n$-prime submodule of $M$. Thus minimality of $N$ implies that $IM = N$. Therefore $I = (IM : M) = (N : M)$. Hence $(N : M)$ is a minimal $n$-absorbing ideal of $R$. □

At the end of this paper should be noted that every $n$-absorbing ideal of a ring $R$ contains a minimal $n$-absorbing ideal [6, Corollary 2.2]. Now if $M$ is a finitely generated faithful multiplication AP $n$-module, then by Proposition 4.20 every $n$-prime submodule $N$ of $M$ contains a minimal $n$-prime submodule of $M$.

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