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Nehari Manifold and Multiplicity Result for Elliptic Equation Involving p-Laplacian Operator

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ABSTRACT: In this paper an elliptic problem involving p-Laplacian operator is considered. Existence and multiplicity of solutions are investigated. The method is based on Nehari manifold and variational method.

Key Words: Multiple solutions, Variational method, Nehari manifold.

Contents

1. Introduction and main result

In the last few years, p-Laplacian equations have received increasing attention. This theory has been developed very quickly and attracted a considerable interest from researches (See [1-12]). Since the p-Laplacian operator and fractional calculus arises from many applied fields, such as turbulent filtration in porous media, blood flow problems, rheology, modeling of viscoplasticity, material science, it is worth studying the fractional p-Laplacian equations.
Let Ω be a bounded domain of $\mathbb{R}^n$ $(n \geq 3)$ with smooth boundary $\partial\Omega$ and $1 < q < p < n$. In this paper, we consider the p-Laplacian problem of the following form

$$(\mathbf{P}_\lambda) \quad \begin{cases} -\Delta_p u = \frac{1}{p^*}\frac{\partial F(x,u)}{\partial u} + \lambda a(x)|u|^{q-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on} & \partial\Omega, \end{cases}$$

where $\lambda \in \mathbb{R}$, $p^* = \frac{np}{n-p}$ and the sign changing weight function a satisfies the following condition

$$\textbf{(A)} \quad a \in C(\Omega) \ \text{ with } \ \|a\|_\infty = 1 \ \text{ and } \ a^\pm := max(\pm a, 0) \not\equiv 0.$$

The study of p-laplacian equations using the Nehari manifold method has been the subject of several works. More precisely, Wu [15] considered the following elliptic equation

$$\textbf{(2)} \begin{cases} -\triangle_p u = \lambda f(x)|u|^{q-2}u + g(x)|u|^{r-2}u, & \text{in } \Omega, \\ u = 0 \quad \text{on } \partial\Omega, \end{cases}$$

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where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geq 3)$, $1 < q < p < r < p^*$. The author has proved that there exists $\lambda_0 > 0$ such that for $0 < \lambda < \lambda_0$, equation **(2)** has at least two positive solutions. In addition, If the weight functions $f \equiv g \equiv 1$, Ambrosetti et al. [1] have investigated equation **(2)**, they prove that there exists $\lambda_0 > 0$ such that equation **(2)** admits two positive solutions for $\lambda \in (0, \lambda_0)$, one positive solution for $\lambda = \lambda_0$ and no positive solution for $\lambda > \lambda_0$.
The starting point on the study of the system $(\mathbf{P}_\lambda)$ is its scalar version

$$\textbf{(3)} \quad \begin{cases} -\triangle_p u = |u|^{p^*-2}u + \lambda|u|^{q-2}u & \text{in } \Omega, \\ u = 0 \quad \text{on} & \partial\Omega, \end{cases}$$

with $2 \leq p \leq q < p^*$. Note that many excellent results have been worked out on the existence of solutions for problem **(3)** (See [7], [13]).
In this work, motivated by the above works, we give a very simple variational method to prove the existence of at least two nontrivial solutions of problem $(\mathbf{P}_\lambda)$.
Before stating our main result, we need the following assumptions:

(H$_1$) $F : \overline{\Omega} \times \mathbb{R} \longrightarrow \mathbb{R}$ is a C^1 function such that

$$F(x, tu) = t^{p^*} F(x, u)(t > 0) \text{ for all } x \in \overline{\Omega},\ u \in \mathbb{R}.$$

(H$_2$) $F(x, 0) = \frac{\partial F}{\partial u}(x, 0) = 0$ and $F^{\pm}(x, u) = max(\pm F(x, u), 0) \not\equiv 0$ for all $u \neq 0$.

We remark, that Using assumption **(H$_1$)**, we have the so-called Euler identity

$$u\frac{\partial F(x, u)}{\partial u} = p^* F(x, u), \quad \text{and } |F(x, u)| \leq K|u|^{p^*} \quad \text{for some constant } K > 0. \tag{1.1}$$

Our main result is the following

Theorem 1.1. *Under the assumptions **(A)**, **(H$_1$)** and **(H$_2$)**, there exists $\lambda_0 > 0$ such that for all $0 < |\lambda| < \lambda_0$, problem $(\boldsymbol{P}_\lambda)$ has at least two positive solutions.*

2. Notations and preliminaries

Throughout this paper, for $1 < l \leq p^*$, we note by S_l the best Sobolev embedding for the operator $W_0^{1,p}(\Omega) \hookrightarrow L^l(\Omega)$ which is given by

$$S_l = \inf_{u \in W_0^{1,p}(\Omega)\setminus\{0\}} \frac{\int_\Omega |\nabla u|^p}{\left(\int_\Omega |u|^l\right)^{\frac{p}{l}}}.$$

In particular, we have

$$\int_\Omega |u|^l \leq S_l^{-\frac{l}{p}} \|u\|^l \quad \text{for all } u \in W_0^{1,p}(\Omega), \tag{2.1}$$

where $\|.\|$ is the standard norm defined as

$$\|u\| = \left(\int_\Omega |\nabla u|^p dx\right)^{\frac{1}{p}}.$$

Problem $(\mathbf{P}_\lambda)$ is posed in the framework of the Sobolev space $E = W_0^{1,p}(\Omega)$. Moreover, a function u in E is said to be a weak solution of problem $(\mathbf{P}_\lambda)$ if

$$\int_\Omega |\nabla u|^{p-2}\nabla u \nabla\varphi dx - \frac{1}{p^*}\int_\Omega \frac{\partial F(x,u)}{\partial u}\varphi dx - \lambda\int_\Omega a|u|^{q-2}u\varphi dx = 0, \quad \text{for all } \varphi \in E.$$

Associated with the problem $(\mathbf{P}_\lambda)$ we define the functional $J_\lambda : E \to \mathbb{R}$ given by

$$J_\lambda(u) = \frac{1}{p}\|u\|^p - \frac{1}{p^*}\int_\Omega F(x,u)dx - \frac{\lambda}{q}\int_\Omega a(x)|u|^q dx.$$

In order to verify that $J_\lambda \in C^1(E,\mathbb{R})$, we need the following lemmas.

Lemma 2.1. *Assume that $F \in C^1(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ is positively homogenuous of degree p^*, then $\frac{\partial F}{\partial u} \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ is positively homogenuous of degree p^*-1.*

Proof. The proof is the same as that in Chu and Tang [8], so we omit it here. □

It is easily seen that using Lemma 2.1, there exists a positive constant M such that for all $u \in \mathbb{R}$, we have

$$|\frac{\partial F(x,u)}{\partial u}| \le M|u|^{p^*-1}. \tag{2.2}$$

Lemma 2.2. *(See Proposition 1 in [12]) Suppose that $\frac{\partial F(x,u)}{\partial u} \in C(\overline{\Omega}\times\mathbb{R},\mathbb{R})$ verifies condition (2.2). Then, the functional J_λ belongs to $C^1(E,\mathbb{R})$, and*

$$\langle J'_\lambda(u), u\rangle = \|u\|^p - \int_\Omega F(x,u)dx - \lambda\int_\Omega a(x)|u|^q dx. \tag{2.3}$$

As the energy functional J_λ is not bounded below on E, it is useful to consider the functional on the Nehari manifold

$$N_\lambda = \{u \in E\backslash\{0\} : \langle J'_\lambda(u), u\rangle = 0\},$$

where $\langle .,. \rangle$ denotes the usual duality between E and $E^* = W^{-1,p'}(\Omega)$ (the dual space of the Sobolev space E).
Thus, $u \in N_\lambda$ if and only if

$$\|u\|^p - \int_\Omega F(x,u)dx - \lambda\int_\Omega a(x)|u|^q dx = 0. \tag{2.4}$$

Moreover, we have the following result.

Lemma 2.3. *The energy functional J_λ is coercive and bounded below on N_λ.*

Proof. If $u \in N_\lambda$, then by (2.4) and condition **(A)** we obtain

$$\begin{aligned} J_\lambda(u) &= \frac{p^*-p}{p^*p}\|u\|^p - \lambda\frac{p^*-q}{p^*q}\int_\Omega a(x)|u|^q dx \\ &\ge \frac{p^*-p}{p^*p}\|u\|^p - |\lambda|\frac{p^*-q}{p^*q}\int_\Omega |u|^q dx. \end{aligned}$$

So, it follows from (2.1) that

$$J_\lambda(u) \geq \frac{p^*-p}{p^*p}\|u\|^p - \frac{|\lambda|}{S_q^{\frac{q}{p}}}\frac{p^*-q}{p^*q}\|u\|^q. \tag{2.5}$$

Thus, J_λ is coercive and bounded below on N_λ. □

Define

$$\phi_\lambda(u) = \langle J'_\lambda(u), u\rangle.$$

Then, by (2.4) it is easy to see that for all $u \in N_\lambda$, one has

$$\begin{aligned}
\langle \phi'_\lambda(u), u\rangle &= p\|u\|^p - p^*\int_\Omega F(x,u)dx - \lambda q\int_\Omega a(x)|u|^q dx &(2.6)\\
&= \lambda(p-q)\int_\Omega a(x)|u|^q dx - (p^*-p)\int_\Omega F(x,u)dx &(2.7)\\
&= \lambda(p^*-q)\int_\Omega a(x)|u|^q dx - (p^*-p)\|u\|^p &(2.8)\\
&= (p-q)\|u\|^p - (p^*-q)\int_\Omega F(x,u)dx. &(2.9)
\end{aligned}$$

Now, we split N_λ into three parts

$$N_\lambda^+ = \{u \in N_\lambda : \langle \phi'_\lambda(u), u\rangle > 0\},$$

$$N_\lambda^0 = \{u \in N_\lambda : \langle \phi'_\lambda(u), u\rangle = 0\},$$

$$N_\lambda^- = \{u \in N_\lambda : \langle \phi'_\lambda(u), u\rangle < 0\}.$$

Lemma 2.4. *Assume that u_0 is a local minimizer for J_λ on N_λ such that $u_0 \notin N_\lambda^0$. Then, $J'_\lambda(u_0) = 0$ in E^*.*

Proof. The proof is the same as that in Brown and Zhang [[6], Theorem 2.3], so we omit it here. . □

Lemma 2.5. *We have*
(i) If $u \in N_\lambda^+$, then $\lambda\int_\Omega a(x)|u|^q dx > 0$.
(ii) If $u \in N_\lambda^0$, then $\lambda\int_\Omega a(x)|u|^q dx > 0$ and $\int_\Omega F(x,u)dx > 0$.
(iii) If $u \in N_\lambda^-$, then $\int_\Omega F(x,u)dx > 0$.

Proof. The proofs are immediate from (2.7), (2.8) and (2.9). □

Let

$$\lambda_0 = \frac{q(p^*-p)}{p(p^*-q)}S_q^{\frac{q}{p}}\left(\frac{p-q}{K(p^*-q)}S_{p^*}^{\frac{p^*}{p}}\right)^{\frac{p-q}{p^*-p}}, \tag{2.10}$$

then we have the following lemma.

Lemma 2.6. *If* $0 < |\lambda| < \lambda_0$, *then* $N_\lambda^0 = \varnothing$.

Proof. Suppose otherwise, that $0 < |\lambda| < \lambda_0$ such that $N_\lambda^0 \neq \varnothing$. Then for $u \in N_\lambda^0$ with $u \not\equiv 0$, we have

$$0 = \langle \phi'_\lambda(u), u\rangle = \lambda(p^* - q)\int_\Omega a(x)|u|^q dx - (p^* - p)\|u\|^p \tag{2.11}$$

$$= (p - q)\|u\|^p - (p^* - q)\int_\Omega F(x,u)dx. \tag{2.12}$$

From the Hölder inequality, (1.1) and (2.1), it follows that

$$\int_\Omega F(x,u)dx \leq \int_\Omega |F(x,u)|dx \leq K\int_\Omega |u|^{p^*}dx \leq KS_{p^*}^{-\frac{p^*}{p}}\|u\|^{p^*}.$$

Hence, from (2.12), we obtain

$$\begin{aligned}\|u\|^p &= \frac{p^* - q}{p - q}\int_\Omega F(x,u)dx \\ &\leq \frac{p^* - q}{p - q}KS_{p^*}^{-\frac{p^*}{p}}\|u\|^{p^*}.\end{aligned}$$

Which yields to

$$\|u\| \geq \left(\frac{p - q}{K(p^* - q)}S_{p^*}^{\frac{p^*}{p}}\right)^{\frac{1}{p^* - p}}. \tag{2.13}$$

On the other hand, from condition (**A**) ,(2.1) and (2.11) we have

$$\begin{aligned}\|u\|^p &= \lambda\frac{p^* - q}{p^* - p}\int_\Omega a(x)|u|^q dx \\ &\leq |\lambda|\frac{p^* - q}{p^* - p}KS_q^{-\frac{q}{p}}\|u\|^q,\end{aligned}$$

so,

$$\|u\| \leq \left(|\lambda|\frac{p^* - q}{p^* - p}S_q^{-\frac{q}{p}}\right)^{\frac{1}{p - q}}. \tag{2.14}$$

Combining (2.13) and (2.14), we obtain $\lambda_0 \leq |\lambda|$, which is a contradiction. □

By lemma 2.6, for $0 < |\lambda| < \lambda_0$, we have $N_\lambda = N_\lambda^+ \cup N_\lambda^-$.
Let

$$\theta_\lambda = \inf_{u \in N_\lambda} J_\lambda(u),\ \theta_\lambda^+ = \inf_{u \in N_\lambda^+} J_\lambda(u),\ \theta_\lambda^- = \inf_{u \in N_\lambda^-} J_\lambda(u).$$

Then, we have the following.

Lemma 2.7. *If* $0 < |\lambda| < \lambda_0$, *then*

$$\theta_\lambda \le \theta_\lambda^+ < 0 \ \ \text{and} \ \ \theta_\lambda^- > d_0$$

for some $d_0 > 0$ *depending on* p, q, p^*, K, λ, S_q *and* S_{p^*}.

Proof. Let $u \in N_\lambda^+$. Then, from (2.9) we have

$$\frac{p-q}{p^*-q}\|u\|^p > \int_\Omega F(x,u)dx.$$

So

$$\begin{aligned} J_\lambda(u) &= \frac{q-p}{pq}\|u\|^p + \frac{p^*-q}{p^*q}\int_\Omega F(x,u)dx \\ &< \left(\frac{q-p}{pq} + \frac{p^*-q}{p^*q}\frac{p-q}{p^*-q}\right)\|u\|^p \\ &= -\frac{(p-q)(p^*-p)}{pqp^*}\|u\|^p < 0. \end{aligned}$$

Thus, from the definition of θ_λ and θ_λ^+, we can deduce that $\theta_\lambda \le \theta_\lambda^+ < 0$. Now, let $u \in N_\lambda^-$. Then, using (1.1) and (2.1) we obtain

$$\frac{p-q}{p^*-q}\|u\|^p < \int_\Omega F(x,u)dx \le K S_{p^*}^{-\frac{p^*}{p}}\|u\|^{p^*}.$$

This implies that

$$\|u\| > \left(\frac{p-q}{p^*-q}\frac{S_{p^*}^{\frac{p^*}{p}}}{K}\right)^{\frac{1}{p^*-p}}, \quad \forall u \in N_\lambda^-. \tag{2.15}$$

In addition, by (2.5) and (2.15)

$$\begin{aligned} J_\lambda(u) &\ge \frac{p^*-p}{pp^*}\|u\|^p - |\lambda| S_q^{-\frac{q}{p}}\frac{p^*-p}{p^*q}\|u\|^q \\ &\ge \|u\|^q\left[\frac{p^*-p}{pp^*}\|u\|^{p-q} - |\lambda| S_q^{-\frac{q}{p}}\frac{p^*-q}{p^*q}\right] \\ &> \left(\frac{p-q}{p^*-q}\frac{S_{p^*}^{\frac{p^*}{p}}}{K}\right)^{\frac{q}{p^*-p}}\left(\frac{p^*-p}{pp^*}\left(\frac{p-q}{p^*-q}\frac{S_{p^*}^{\frac{p^*}{p}}}{K}\right)^{\frac{p-q}{p^*-p}} - |\lambda| S_q^{-\frac{q}{p}}\frac{p^*-q}{p^*q}\right). \end{aligned}$$

Thus, since $0 < |\lambda| < \lambda_0$, we conclude that $J_\lambda > d_0$ for some $d_0 > 0$. This completes the proof of lemma 2.7. □

For $u \in E$ with $\int_\Omega F(x,u)dx > 0$, let

$$T = \left(\frac{(p-q)\|u\|^p}{(p^*-q)\int_\Omega F(x,u)dx} \right)^{\frac{1}{p^*-p}} > 0.$$

Then, the following lemma hold.

Lemma 2.8. *For each $u \in E$ with $\int_\Omega F(x,u)dx > 0$, we have*
(i) If $\lambda \int_\Omega a(x)|u|^q dx \le 0$, then there exists unique $t^- > T$ such that $t^- u \in N_\lambda^-$ and

$$J_\lambda(t^- u) = \sup_{t \ge 0} J_\lambda(tu).$$

(ii) If $\lambda \int_\Omega a(x)|u|^q dx > 0$, then there are unique $0 < t^+ < T < t^-$ such that $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$ and

$$J_\lambda(t^- u) = \sup_{t \ge 0} J_\lambda(tu); \quad J_\lambda(t^+ u) = \inf_{0 \le t \le T} J_\lambda(tu).$$

Proof. We fix $u \in E$ with $\int_\Omega F(x,u)dx > 0$ and we define the map m on $[0,\infty)$ as follows:

$$m(t) = t^{p-q}\|u\|^p - t^{p^*-q} \int_\Omega F(x,u)dx \quad \text{for} \ \ t \ge 0.$$

Then, it is easy to check that $m(t)$ achieves its maximum at T. Moreover,

$$\begin{aligned} m(T) &= \|u\|^q \left[\left(\frac{p-q}{p^*-q}\right)^{\frac{p-q}{p^*-q}} - \left(\frac{p-q}{p^*-q}\right)^{\frac{p^*-q}{p^*-p}} \right] \left(\frac{\|u\|^{p^*}}{\int_\Omega F(x,u)dx} \right)^{\frac{p-q}{p^*-p}} \\ &\ge \|u\|^q \left(\frac{p^*-p}{p^*-q}\right) \left(\frac{(p^*-q)S_{p^*}^{\frac{p^*}{p}}}{K(p-q)} \right)^{\frac{p-q}{p^*-p}}. \end{aligned}$$

(i) We suppose that $\lambda \int_\Omega a(x)|u|^q dx \le 0$. Since $m(0) = 0$, $m(t) \to -\infty$ as $t \to \infty$, $m'(t) > 0$ for $t < T$ and $m'(t) < 0$ for $t > T$. There is a unique $t^- > T$ such that $m(t^-) = \lambda \int_\Omega a(x)|u|^q dx \le 0$.
Now, it follows from (2.3) and (2.12) that

$$\phi'_\lambda(t^- u)t^- u = (t^-)^{1+q} m'(t^-) < 0$$

and

$$J'_\lambda(t^- u)t^- u = (t^-)^q \left(m(t^-) - \lambda \int_\Omega a(x)|u|^q dx \right) = 0.$$

Hence, $t^- u \in N_\lambda^-$. On the other hand, it is easy to see that

$$\frac{d^2}{dt^2} J_\lambda(tu) < 0 \ \text{ for } \ t > T \ \text{ and } \ \frac{d}{dt} J_\lambda(tu) = 0 \ \text{ for } \ t = t^-.$$

Thus, $J_\lambda(t^- u) = \sup_{t\geq 0} J_\lambda(tu)$.
(ii) Assume $\lambda\int_\Omega a(x)|u|^q dx > 0$. Then, by **(A)** and (2.1) and the fact that $|\lambda| < \lambda_0$ we obtain

$$m(0) = 0 < \lambda \int_\Omega a(x)|u|^q dx \leq |\lambda| S_q^{-\frac{q}{p}} \|u\|^q < m(T).$$

Then, there are unique t^+ and t^- such that $0 < t^+ < T < t^-$, $m(t^+) = \lambda\int_\Omega a(x)|u|^q dx = m(t^-)$ and $m'(t^-) < 0 < m'(t^+)$. We have $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$, and

$$\begin{cases} J_\lambda(t^+u) \leq J_\lambda(tu) \leq J_\lambda(t^-u) \quad \forall\, t \in [t^+, t^-], \\ J_\lambda(t^+u) \leq J_\lambda(tu) \ \ \forall\, 0 \leq t \leq t^+. \end{cases}$$

Thus,

$$J_\lambda(t^- u) = \sup_{t\geq 0} J_\lambda(tu) \ \text{ and } \ J_\lambda(t^+ u) = \inf_{0\leq t\leq T} J_\lambda(tu).$$

This completes the proof of lemma 2.8. □

For each $u \in E$ with $\lambda\int_\Omega a(x)|u|^q dx > 0$, put

$$\widetilde{T} = \left(\frac{\lambda(p^* - q)\int_\Omega a(x)|u|^q dx}{(p^* - p)\|u\|^p}\right)^{\frac{1}{p-q}} > 0.$$

Then we have the following lemma.

Lemma 2.9. *For each $u \in E$ with $\lambda\int_\Omega a(x)|u|^q dx > 0$, we have*
(i) If $\int_\Omega F(x,u)dx \leq 0$, then there exists a unique $0 < t_+ < \widetilde{T}$ such that $t^+ \in N_\lambda^+$ and

$$J_\lambda(t^+ u) = \inf_{t\geq 0} J_\lambda(tu).$$

(ii) If $\int_\Omega F(x,u)dx > 0$, then there are unique $0 < t^+ < \widetilde{T} < t^-$ such that $(t^- u, t^+ u) \in N_\lambda^- \times N_\lambda^+$ and

$$J_\lambda(t^- u) = \sup_{t\geq 0} J_\lambda(tu); \quad J_\lambda(t^+ u) = \inf_{0\leq t\leq \widetilde{T}} J_\lambda(tu).$$

Proof. For $u \in E$ with $\lambda\int_\Omega a(x)|u|^q dx > 0$, we can take

$$\widetilde{m}(t) = t^{p-p^*}\|u\|^p - \lambda t^{q-p^*}\int_\Omega a(x)|u|^q dx \quad \text{for } \ t > 0,$$

and similarly to the argument given in lemma 2.8, we can obtain the results of lemma 2.9. □

Proposition 2.10. *There exist minimizing sequences $\{u_n^\pm\}$ in $N_\lambda^\pm$ such that*

$$J_\lambda(u_n^\pm) = \theta_\lambda^\pm + \circ(1) \ \ \textit{and} \ \ J_\lambda'(u_n^\pm) = \circ(1) \ \ \textit{in} \ \ E^*.$$

Proof. The proof is almost the same as that in Wu[[14], Proposition 9] and is omit here. □

3. Proof of our main result

Throughout this section, the L^s norm is denoted by $\|.\|_s$ for $1 \leq s \leq \infty$, $\rightarrow$ (respectively $\rightharpoonup$) denotes strong (respectively weak) convergence and we assume that the parameter λ satisfies $0 < |\lambda| < \lambda_0$. Then we have the following results.

Theorem 3.1. *If $0 < |\lambda| < \lambda_0$, then, problem $(\boldsymbol{P}_\lambda)$ has a positive solution u_λ in N_λ^+ such that*

$$J_\lambda(u_\lambda) = \theta_\lambda = \theta_\lambda^+.$$

Proof. By Proposition 2.10, there exists a minimizing sequence $\{u_n^+\}$ for J_λ on N_λ^+ such that

$$J_\lambda(u_n^+) = \theta_\lambda^+ + o(1) \text{ and } J'_\lambda(u_n^+) = o(1) \quad \text{in} \quad E^*. \tag{3.1}$$

Then by Lemma 2.3, there exists a subsequence $\{u_n\}$ and u_λ in E such that

$$\begin{cases} u_n \rightharpoonup u_\lambda \text{ weakly in } E, \\ u_n \rightarrow u_\lambda \text{ strongly in } L^q(\Omega) \text{ and in } L^{p^*}(\Omega). \end{cases} \tag{3.2}$$

This implies that $\int_\Omega a(x)|u_n|^q dx \rightarrow \int_\Omega a(x)|u_\lambda|^q dx$ as $n \rightarrow \infty$.
Next, we will show that

$$\int_\Omega F(x,u_n)dx \rightarrow \int_\Omega F(x,u_\lambda)dx \text{ as } n \rightarrow \infty.$$

By lemma 2.1, we have

$$\frac{\partial F(x,u_n)}{\partial u} \in L^p(\Omega) \text{ and } \frac{\partial F(x,u_n)}{\partial u} \rightarrow \frac{\partial F(x,u_\lambda)}{\partial u} \text{ in } L^p(\Omega).$$

On the other hand, it follows from the Hölder inequality, that

$$\begin{aligned} \int_\Omega \left| u_n \frac{\partial F(x,u_n)}{\partial u} - u_\lambda \frac{\partial F(x,u_\lambda)}{\partial u} \right| dx &\leq \int_\Omega |(u_n - u_\lambda)\frac{\partial F(x,u_n)}{\partial u}| dx \\ &\quad + \int_\Omega |u_\lambda| \left| \frac{\partial F(x,u_n)}{\partial u} - \frac{\partial F(x,u_\lambda)}{\partial u} \right| dx \\ &\leq \|u_n - u_\lambda\|_{p^*} \| \frac{\partial F(x,u_n)}{\partial u} \|_p \\ &\quad + \|u_\lambda\|_{p^*} \| \frac{\partial F(x,u_n)}{\partial u} - \frac{\partial F(x,u_\lambda)}{\partial u} \|_p \\ &\rightarrow 0 \text{ as } n \rightarrow \infty. \end{aligned}$$

Hence, $\int_\Omega F(x,u_n)dx \rightarrow \int_\Omega F(x,u_\lambda)dx$ as $n \rightarrow \infty$.
Since

$$J_\lambda(u_n) = \frac{p^* - p}{pp^*}\|u\|^p - \lambda \frac{p^* - q}{qp^*} \int_\Omega a(x)|u_n|^q dx \geq -\lambda \frac{p^* - q}{qp^*} \int_\Omega a(x)|u_n|^q dx.$$

By (3.1) and lemma 2.7 , $J_\lambda(u_n) \to \theta_\lambda < 0$ as $n \to \infty$.
Letting $n \to \infty$, we see that

$$\lambda \int_\Omega a(x)|u_\lambda|^q dx > 0. \tag{3.3}$$

Now, we aim to prove that $u_n \to u_\lambda$ strongly in E and $J_\lambda(u_\lambda) = \theta_\lambda$.
using the fact that $u_\lambda \in N_\lambda$ and by Fatou's lemma, we get

$$\begin{aligned} \theta_\lambda &\leq J_\lambda(u_\lambda) = \frac{1}{p}\|u_\lambda\|^p - \frac{1}{p^*}\int_\Omega F(x,u_\lambda)dx - \frac{\lambda}{q}\int_\Omega a(x)|u_\lambda|^q dx \\ &\leq \liminf_{n\to\infty}\left(\frac{1}{p}\|u_n\|^p - \frac{1}{p^*}\int_\Omega F(x,u_n)dx - \frac{\lambda}{q}\int_\Omega a(x)|u_n|^q dx\right) \\ &\leq \liminf_{n\to\infty} J_\lambda(u_n) = \theta_\lambda \end{aligned}$$

This implies that

$$J_\lambda(u_\lambda) = \theta_\lambda \text{ and } \lim_{n\to\infty}\|u_n\|^p = \|u_\lambda\|^p.$$

Let $\widetilde{u}_n = u_n - u_\lambda$, then by Brézis-Lieb lemma [5] we obtain

$$\|\widetilde{u}_n\|^p = \|u_n\|^p - \|u_\lambda\|^p.$$

Therefore, $u_n \to u_\lambda$ strongly in E.
Moreover, we have $u_\lambda \in N_\lambda^+$. In fact, if $u_\lambda \in N_\lambda^-$ then, there exist t_0^+, t_0^- such that $t_0^- u_\lambda \in N_\lambda^-$ and $t_0^+ u_\lambda \in N_\lambda^+$. In particular we have $t_0^+ < t_0^- = 1$. Since

$$\frac{d^2}{dt^2}J_\lambda(t_0^+ u_\lambda) > 0 \text{ and } \frac{d}{dt}J_\lambda(t_0^+ u_\lambda) = 0 \text{ for } t = t^-.$$

there exists $t_0^+ < \widetilde{t} < t_0^-$ such that $J_\lambda(t_0^+ u_\lambda) < J_\lambda(\widetilde{t} u_\lambda)$. By Lemma 2.9, we have

$$J_\lambda(t_0^+ u_\lambda) < J_\lambda(\widetilde{t} u_\lambda) \leq J_\lambda(t_0^- u_\lambda) = J_\lambda(u_\lambda)$$

which is a contraduction.
Fin ally, by (3.1) and (3.2) it is easy to see that u_λ is a weak solution of $(\mathbf{P}_\lambda)$.
Moreover from (3.3), u_λ is nontrivial. □

Theorem 3.2. *If $0 < |\lambda| < \lambda_0$, then, problem $(\boldsymbol{P}_\lambda)$ has a nontrivial solution v_λ in N_λ^- such that*

$$J_\lambda(v_\lambda) = \theta_\lambda^-.$$

Proof. By Proposition 2.1, there exists a minimizing sequence $\{u_n\}$ for J_λ on N_λ^- such that

$$J_\lambda(u_n) = \theta_\lambda^- + o(1) \text{ and } J'_\lambda(u_n) = o(1) \text{ in } E^* \tag{3.4}$$

and

$$\begin{cases} u_n \rightharpoonup v_\lambda \text{ weakly in } E, \\ u_n \to v_\lambda \text{ strongly in } L^q(\Omega) \text{ and in } L^{p^*}(\Omega). \end{cases} \tag{3.5}$$

Moreover, by (2.9) we obtain

$$\int_\Omega F(x,u_n)dx > \frac{p-q}{p^*-q}\|u_n\|^p. \tag{3.6}$$

So, by (2.15) and (3.6) there exists a positive constant $\widetilde{C}$ such that

$$\int_\Omega F(x,u_n) > \widetilde{C}.$$

This implies that

$$\int_\Omega F(x,v_\lambda) \geq \widetilde{C}. \tag{3.7}$$

Now, we prove that $u_n \to v_\lambda$ strongly in E. Supposing otherwise, then

$$\|v_\lambda\| < \liminf_{n\to\infty} \|u_n\|.$$

By lemma 2.8, there is a unique t_0^- such that $t_0^- v_\lambda \in N_\lambda^-$. Since $u_n \in N_\lambda^-$, $J_\lambda(u_n) \geq J_\lambda(tu_n)$ for all $t \geq 0$, we have

$$J_\lambda(t_0^- v_\lambda) < \lim_{n\to\infty} J_\lambda(t_0^- u_n) \leq \lim_{n\to\infty} J_\lambda(u_n) = \theta_\lambda^-.$$

Which is a contradiction. Hence $u_n \to v_\lambda$ strongly in E.
This imply that

$$J_\lambda(u_n) \to J_\lambda(v_\lambda) = \theta_\lambda^- \text{ as } n \to \infty.$$

Finally, from (3.4) and (3.5), we obtain clearly that v_λ is a weak solution of $(\mathbf{P}_\lambda)$. Moreover, from (3.7) v_λ is nontrivial.
□

Now, Let us proof Theorem 1.1: By Theorems 3.1, we obtain a nontrivial solution of problem $(\mathbf{P}_\lambda)$ which is in N_λ^+. And by Theorem 3.1, we have a nontrivial solution of problem $(\mathbf{P}_\lambda)$ which is in N_λ^-. Since $N_\lambda^- \cap N_\lambda^+ = \emptyset$, this implies that v_λ and u_λ are distinct. The proof is complete

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