Triple Almost \((\lambda_m, \mu_n, \gamma_k)\) Lacunary Riesz \(\chi_{R_{\lambda_m, \mu_n, \gamma_k}}^3\) Sequence Spaces Defined by Orlicz Function

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ABSTRACT: In this paper we introduce a new concept for generalized almost \((\lambda_m, \mu_n, \gamma_k)\) convergence in \(\chi_{R_{\lambda_m, \mu_n, \gamma_k}}^3\) –Riesz spaces strong \(P\)– convergent to zero with respect to an Orlicz function and examine some properties of the resulting sequence spaces. We also introduce and study statistical convergence of generalized almost \((\lambda_m, \mu_n, \gamma_k)\) convergence in \(\chi_{R_{\lambda_m, \mu_n, \gamma_k}}^3\) –Riesz space and also some inclusion theorems are discussed.

Key Words: Analytic sequence, Orlicz function, Double sequences, Riesz space, Riesz convergence, Pringsheim convergence.

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1. Introduction

Throughout \(w, \chi\) and \(\Lambda\) denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write \(w^3\) for the set of all complex triple sequences \((x_{m,n,k})\), where \(m, n, k \in \mathbb{N}\), the set of positive integers. Then, \(w^3\) is a linear space under the coordinate wise addition and scalar multiplication.

We can represent triple sequences by matrix. In case of double sequences we write in the form of a square. In the case of a triple sequence it will be in the form of a box in three dimensional case.

Some initial work on double series is found in Apostol [1] and double sequence spaces is found in Hardy [7], Subramanian et al. [8], Deepmala et al. [9,9] and many others. Later on investigated by some initial work on triple sequence spaces is found in Sahiner et al. [11], Esi et al. [2,3,4,5], Savas et al. [6], Subramanian et al. [12], Prakash et al. [13,14] and many others.

Let \((x_{m,n,k})\) be a triple sequence of real or complex numbers. Then the series \(\sum_{m,n,k=1}^\infty x_{m,n,k}\) is called a triple series. The triple series \(\sum_{m,n,k=1}^\infty x_{m,n,k}\) give one
space is said to be convergent if and only if the triple sequence \( (S_{mnk}) \) is convergent, where

\[
S_{mnk} = \sum_{i,j,q=1}^{m,n,k} x_{ijq}(m, n, k = 1, 2, 3, \ldots).
\]

A sequence \( x = (x_{mnk}) \) is said to be triple analytic if

\[
\sup_{m,n,k} |x_{mnk}|^{\frac{1}{m+n+k}} < \infty.
\]

The vector space of all triple analytic sequences are usually denoted by \( \Lambda^3 \). A sequence \( x = (x_{mnk}) \) is called triple entire sequence if

\[
|x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.
\]

The vector space of all triple entire sequences are usually denoted by \( \Gamma^3 \).

Let the set of sequences with this property be denoted by \( \Lambda^3 \) and \( \Gamma^3 \) is a metric space with the metric

\[
d(x, y) = \sup_{m,n,k} \left\{ |x_{mnk} - y_{mnk}|^{\frac{1}{m+n+k}} : m, n, k \in \mathbb{N} \right\},
\]

for all \( x = \{x_{mnk}\} \) and \( y = \{y_{mnk}\} \) in \( \Gamma^3 \). Let \( \phi = \{\text{finite sequences}\} \).

Consider a triple sequence \( x = (x_{mnk}) \). The \((m, n, k)\)th section \( x^{[m,n,k]} \) of the sequence is defined by

\[
x^{[m,n,k]} = \sum_{i,j,q=0}^{m,n,k} x_{ijq}\delta_{ijq} \quad \text{for all } m, n, k \in \mathbb{N},
\]

with 1 in the \((m, n, k)\)th position and zero otherwise.

A sequence \( x = (x_{mnk}) \) is called triple gai sequence if

\[
((m + n + k)! |x_{mnk}|^{\frac{1}{m+n+k}} \to 0,
\]
as \( m, n, k \to \infty \). The triple gai sequences will be denoted by \( \chi^3 \).

2. Definitions and Preliminaries

A triple sequence \( x = (x_{mnk}) \) has limit 0 (denoted by \( P - \lim x = 0 \)) (i.e)

\[
((m + n + k)! |x_{mnk}|^{\frac{1}{m+n+k}} \to 0 \text{ as } m, n, k \to \infty.
\]

We shall write more briefly as \( P - \text{convergent to } 0 \).
Definition 2.1. A function $M : [0, \infty) \to [0, \infty)$ is said to be an Orlicz function if it satisfies the following conditions

(i) $M$ is continuous, convex and non-decreasing;
(ii) $M(0) = 0$, $M(x) > 0$ and $M(x) \to \infty$ as $x \to \infty$.

Remark 2.2. If the convexity of an Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$, then this function is called modulus function.

Definition 2.3. Let $(q^r_{rst}), \overline{Q^r_{rst}}, \overline{Q^r_{st}}$ be sequences of positive numbers and

\[ Q_r = \begin{bmatrix}
q_{11} & q_{12} & \ldots & q_{1s} & 0 & \ldots \\
q_{21} & q_{22} & \ldots & q_{2s} & 0 & \ldots \\
\vdots & \vdots & & \vdots & \vdots & \ddots \\
q_{r1} & q_{r2} & \ldots & q_{rs} & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots
\end{bmatrix} \]

where $q_{11} + q_{12} + \ldots + q_{rs} \neq 0$,

\[ \overline{Q}_s = \begin{bmatrix}
\overline{q}_{11} & \overline{q}_{12} & \ldots & \overline{q}_{1s} & 0 & \ldots \\
\overline{q}_{21} & \overline{q}_{22} & \ldots & \overline{q}_{2s} & 0 & \ldots \\
\vdots & \vdots & & \vdots & \vdots & \ddots \\
\overline{q}_{r1} & \overline{q}_{r2} & \ldots & \overline{q}_{rs} & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots
\end{bmatrix} \]

where $\overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0$,

\[ \overline{Q}_t = \begin{bmatrix}
\overline{q}_{11} & \overline{q}_{12} & \ldots & \overline{q}_{1s} & 0 & \ldots \\
\overline{q}_{21} & \overline{q}_{22} & \ldots & \overline{q}_{2s} & 0 & \ldots \\
\vdots & \vdots & & \vdots & \vdots & \ddots \\
\overline{q}_{r1} & \overline{q}_{r2} & \ldots & \overline{q}_{rs} & 0 & \ldots \\
0 & 0 & \ldots & 0 & 0 & \ldots
\end{bmatrix} \]

where $\overline{q}_{11} + \overline{q}_{12} + \ldots + \overline{q}_{rs} \neq 0$. Then the transformation is given by

\[ T_{rst} = \frac{1}{\lambda_i \mu_r \gamma_j} \frac{1}{Q_q \overline{Q}_s \overline{Q}_t} \sum_{m=1}^{r} \sum_{n=1}^{s} \sum_{k=1}^{t} \overline{q}_{mn} \overline{q}_{nk} \Gamma((m + n + k)! |x_{mnk}|)^{1/m+n+k} \]

is called the Riesz mean of triple sequence $x = (x_{mnk})$. If $P - \lim_{r\to \infty} T_{rst} (x) = 0$, $0 \in \mathbb{R}$, then the sequence $x = (x_{mnk})$ is said to be Riesz convergent to 0. If $x = (x_{mnk})$ is Riesz convergent to 0, then we write $P_R - \lim x = 0$. 
Definition 2.4. Let \( \lambda = (\lambda_{m_i}) \), \( \mu = (\mu_{n_i}) \) and \( \gamma = (\gamma_{k_j}) \) be three non-decreasing sequences of positive real numbers such that each tending to \( \infty \) and \( \lambda_{m_i+1} \leq \lambda_{m_i} + 1 \), \( \mu_{n_i+1} \leq \mu_{n_i} + 1 \), \( \mu_1 = 1 \), \( \mu_{n_i} \) is tending to \( \infty \). Let \( I_{m_i} = [m_i - \lambda_{m_i}, +1, m_i] \), \( I_{n_i} = [n_i - \mu_{n_i}, +1, n_i] \) and \( I_{k_j} = [k_j - \gamma_{k_j}, +1, k_j] \). For any set \( K \subseteq \mathbb{N} \times \mathbb{N} \times \mathbb{N} \), the number

\[
\delta_{\lambda, \mu, \gamma}(K) = \lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_i}\gamma_{k_j}} \left| \{(i,j) : i \in I_{m_i}, j \in I_{n_i}, k \in I_{k_j}, (i,j) \in K \} \right|
\]

is called the \((\lambda, \mu, \gamma)\)-density of the set \( K \) provided the limit exists.

Definition 2.5. A triple sequence \( x = (x_{mnk}) \) of numbers is said to be \((\lambda, \mu, \gamma)\)-statistical convergent to a number \( \xi \) provided that for each \( \epsilon > 0 \),

\[
\lim_{m,n,k\to\infty} \frac{1}{\lambda_{m_i}\mu_{n_i}\gamma_{k_j}} \frac{1}{Q_i Q_j Q_k} \left| \{(i,j) : i \in I_{m_i}, j \in I_{n_i}, k \in I_{k_j}, x_{mnk} - \xi \geq \epsilon \} \right| = 0,
\]

(i.e) the set

\[
K(\epsilon) = \frac{1}{\lambda_{m_i}\mu_{n_i}\gamma_{k_j}} \frac{1}{Q_i Q_j Q_k} \left| \{(i,j) : i \in I_{m_i}, j \in I_{n_i}, k \in I_{k_j}, x_{mnk} - \xi \geq \epsilon \} \right|
\]

has \((\lambda, \mu, \gamma)\)-density zero. In this case the number \( \xi \) is called the \((\lambda, \mu, \gamma)\)-statistical limit of the sequence \( x = (x_{mnk}) \) and we write \( \text{St}_{(\lambda, \mu, \gamma)}(x_{mnk}) = \xi \).

Definition 2.6. The triple sequence \( \theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} \) is called triple lacunary if there exist three increasing sequences of integers such that

\[
m_0 = 0, h_i = m_i - m_{i-1} \to \infty \text{ as } i \to \infty,
\]

\[
n_0 = 0, h_j = n_\ell - n_{\ell-1} \to \infty \text{ as } \ell \to \infty
\]

and

\[
k_0 = 0, k_j = k_j - k_{j-1} \to \infty \text{ as } j \to \infty.
\]

Let \( m_{i,\ell,j} = m_i n_\ell k_j \), \( h_{i,\ell,j} = h_i h_j \), and \( \theta_{i,\ell,j} \) is determine by

\[
I_{i,\ell,j} = \{(m_i, n_\ell, k_j) : m_i - 1 < m_i \text{ and } n_{\ell-1} < n \leq n_\ell \text{ and } k_{j-1} < k \leq k_j\},
\]

\[
q_k = \frac{m_k}{m_{k-1}}, q_j = \frac{n_\ell}{n_{\ell-1}}, \bar{q}_j = \frac{k_j}{k_{j-1}}.
\]

Using the notations of lacunary sequence and Riesz mean for triple sequences, \( \theta_{i,\ell,j} = \{(m_i, n_\ell, k_j)\} \) be a triple lacunary sequence and \( q_m q_n q_k \) be sequences of positive real numbers such that

\[
Q_m = \sum_{m \in [0,m_i]} p_m, Q_n = \sum_{n \in [0,n_\ell]} p_n, Q_k = \sum_{k \in [0,k_j]} p_k
\]
and

\[ H_i = \sum_{m \in (0, m_i]} p_{m_i}, \overline{H} = \sum_{n \in (0, n_i]} p_{n_i}, \overline{H} = \sum_{k \in (0, k_j]} p_{k_j}. \]

Clearly, \( H_i = Q_{m_i} - Q_{m_{i-1}} \), \( \overline{H_\ell} = Q_{n_\ell} - Q_{n_{\ell-1}} \), and \( \overline{H_j} = Q_{k_j} - Q_{k_{j-1}} \). If the Riesz transformation of triple sequences is RH-regular, and \( H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \) as \( i \to \infty \), \( \overline{H_\ell} = \sum_{n \in (0, n_\ell]} p_{n_\ell} \to \infty \) as \( \ell \to \infty \), and \( \overline{H_j} = \sum_{k \in (0, k_j]} p_{k_j} \to \infty \) as \( j \to \infty \), then \( \theta_{i, \ell, j} = \{(m_i, n_\ell, k_j)\} \) is a triple lacunary sequence. If the assumptions \( Q_r \to \infty \) as \( r \to \infty \), \( \overline{Q_s} \to \infty \) as \( s \to \infty \), and \( \overline{Q_t} \to \infty \) as \( t \to \infty \) may not be enough to obtain the conditions \( H_i \to \infty \) as \( i \to \infty \), \( \overline{H_\ell} \to \infty \) as \( \ell \to \infty \), and \( \overline{H_j} \to \infty \) as \( j \to \infty \) respectively. For any lacunary sequences \( (m_i), (n_\ell) \) and \( (k_j) \) are integers. Throughout the paper, we assume that

\[
\begin{align*}
Q_r &= q_{11} + q_{12} + \ldots + q_{r s} \to \infty \quad (r \to \infty), \\
Q_s &= q_{11} + q_{12} + \ldots + q_{r s} \to \infty \quad (s \to \infty), \\
\overline{Q_t} &= q_{11} + q_{12} + \ldots + q_{r s} \to \infty \quad (t \to \infty),
\end{align*}
\]

such that \( H_i = Q_{m_i} - Q_{m_{i-1}} \to \infty \) as \( i \to \infty \), \( \overline{H_\ell} = Q_{n_\ell} - Q_{n_{\ell-1}} \to \infty \) as \( \ell \to \infty \), and \( \overline{H_j} = Q_{k_j} - Q_{k_{j-1}} \to \infty \) as \( j \to \infty \). Let \( Q_m, n_\ell, k_j = Q_{m_i}, Q_{n_\ell}, Q_{k_j} \), \( H_{i, \ell} = H_i \overline{H_\ell} \overline{H_j} \), \( \lambda_{i, \ell, j} = \{(m_i, n_\ell, k_j)\} \),

\[ I'_{i, \ell, j} = \{(m, n, k) : Q_{m_i} - Q_{m_{i-1}} < m < Q_{m_i}, Q_{n_\ell} - Q_{n_{\ell-1}} < n < Q_{n_\ell}, \text{and } Q_{k_{j-1}} < k < Q_{k_j}\}, \]

\[ V_i = \frac{Q_{m_i}}{Q_{m_{i-1}}} \quad \text{and} \quad \overline{V_\ell} = \frac{Q_{n_\ell}}{Q_{n_{\ell-1}}} \quad \text{and} \quad \overline{V_j} = \frac{Q_{k_j}}{Q_{k_{j-1}}} \cdot \]

If we take \( q_m = 1, \overline{q_n} = 1 \) and \( \overline{q_k} = 1 \) for all \( m, n, k \) then \( H_{i, \ell}, Q_{i, \ell, j}, V_{i, \ell, j} \) and \( I'_{i, \ell, j} \) reduce to \( h_{i, \ell, j}, q_{i, \ell, j}, v_{i, \ell, j} \) and \( I_{i, \ell, j} \).

Let \( f \) be an Orlicz function and \( p = (p_{mnk}) \) be any factorable triple sequence of strictly positive real numbers, we define the following sequence spaces:

\[
\begin{align*}
\left[ \lambda_{R, m_i, n_\ell, k_j}^3, \theta_{i, \ell, j}, q, f, p \right] = & \quad \left\{ P - \lim_{i, \ell, j \to \infty} \frac{1}{H_{i, \ell, j}} \right. \\
& \left. \sum_{i \in I_{i, \ell, j}} \sum_{\ell \in I_{i, \ell, j}} \sum_{j \in I_{i, \ell, j}} q_{m_n^*k} f \left((m + n + k)! \left|x_{m+i, n+\ell, k+j}\right|^{p_{mnk}}\right) \right\} = 0,
\end{align*}
\]
uniformly in \( i, \ell \) and \( j \).

\[
\left[ \Lambda_{R}^{3} \mu_{n_{k_{j}}} \gamma_{k_{j}}, \theta_{ij}, q, f, p \right] = \left\{ x = (x_{m_{nk}}) : P \sup_{i,\ell,j} \frac{1}{\lambda_{m_{i}}n_{i_{k_{j}}}} \frac{1}{H_{i_{j}}} \sum_{i \in I_{i_{j}}} \sum_{\ell \in I_{i_{j}}} \sum_{j \in I_{i_{j}}} q_{m_{i}}q_{n_{j}}q_{k_{j}} \left| f \left| x_{m_{i}+n_{i}+\ell_{j}+k_{j}} \right|^{p_{mnk}} \right| < \infty \right\},
\]

uniformly in \( i, \ell \) and \( j \).

Let \( f \) be an Orlicz function, \( p = p_{mnk} \) be any factorable double sequence of strictly positive real numbers and \( Q_{r} = q_{11} \cdots q_{r_{s}} \), \( \overline{Q}_{r} = q_{11} \cdots q_{r_{s}} \) and \( \overline{Q}_{r} = q_{11} \cdots q_{r_{s}} \). If we choose \( q_{m} = 1, q_{n} = 1 \) and \( \overline{q}_{k} = 1 \) for all \( m, n \) and \( k \), then we obtain the following sequence spaces.

\[
\left[ \chi_{R}^{3} \mu_{n_{k_{j}}} \gamma_{k_{j}}, q, f, p \right] = \left\{ P \lim_{i,\ell,j \to \infty} m \frac{1}{\lambda_{i_{j}}n_{j}} \frac{1}{Q_{i_{j}}} \sum_{i = 1}^{m} \sum_{\ell = 1}^{n} \sum_{j = 1}^{k} q_{m_{i}}q_{n_{j}}q_{k_{j}} \left| f \left| (m + n + k)! \right| x_{m_{i}+n_{i}+\ell_{j}+k_{j}} \right|^{p_{mnk}} \right| = 0 \right\},
\]

uniformly in \( i, \ell \) and \( j \).

\[
\left[ \Lambda_{R}^{3} \mu_{n_{k_{j}}} \gamma_{k_{j}}, q, f, p \right] = \left\{ P \sup_{i,\ell,j} \frac{1}{\lambda_{i_{j}}n_{j}} \frac{1}{Q_{i_{j}}} \sum_{m = 1}^{i} \sum_{n = 1}^{\ell} \sum_{k = 1}^{j} q_{m_{i}}q_{n_{j}}q_{k_{j}} \left| f \left| (m + n + k)! \right| x_{m_{i}+n_{i}+\ell_{j}+k_{j}} \right|^{p_{mnk}} \right| < \infty \right\},
\]

uniformly in \( i, \ell \) and \( j \).

3. Main Results

**Theorem 3.1.** If \( f \) be any Orlicz function and a bounded factorable positive triple number sequence \( p_{mnk} \) then \( \left[ \chi_{R}^{3} \mu_{n_{k_{j}}} \gamma_{k_{j}}, \theta_{ij}, q, f, p \right] \) is linear space.

**Proof:** The proof is easy. Therefore omit the proof. \( \square \)
Theorem 3.2. For any Orlicz function \( f \), we have
\[
\left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right] \subseteq \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, p \right].
\]

Proof: Let \( x \in \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right] \) so that for each \( i, \ell \) and \( j \)
\[
\left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right] = \left\{ P - \lim inf_{i,\ell,j \to \infty} \frac{1}{\lambda_{i,\ell,j}} \frac{1}{H_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_{mnk} \chi_{\lambda_{i,n,\ell,k}} \left((m + n + k)! |x_{m+i,n+\ell,k+j}|^{p_{mnk}} \right) = 0 \right\},
\]
uniformly in \( i, \ell \) and \( j \). Since \( f \) is continuous at zero, for \( \epsilon > 0 \) and choose \( \delta \) with \( 0 < \delta < 1 \) such that \( f(t) < \epsilon \) for every \( t \) with \( 0 \leq t \leq \delta \). We obtain the following,
\[
\frac{1}{\lambda_{i,\ell,j}} \frac{1}{h_{itj}} (h_{itj} \epsilon) + \frac{1}{\lambda_{i,\ell,j}} \frac{1}{h_{itj}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} f \left((m + n + k)! |x_{m+i,n+\ell,k+j}|^{1/(m+n+k)} \right)
\]
\[
= \frac{1}{h_{itj}} (h_{itj} \epsilon) + \frac{1}{\lambda_{i,\ell,j}} \frac{1}{h_{itj}} K \delta^{-1} f (2) h_{itj} \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right].
\]
Hence \( i, \ell \) and \( j \) goes to infinity, we are granted \( x \in \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right]. \)

Theorem 3.3. Let \( \theta_{i,\ell,j} = \{m_{i,n_{\ell},k_{j}}\} \) be a triple lacunary sequence and \( q, \pi, \pi_{ij} \)
with \( \lim inf_{i} V_{i} > 1 \), \( \lim inf_{\ell} V_{\ell} > 1 \) and \( \lim inf_{j} V_{j} > 1 \) then for any Orlicz function \( f \),
\[
\left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, f, q, p \right] \subseteq \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, \theta_{itj}, q, f, p \right].
\]

Proof: Suppose \( \lim inf_{i} V_{i} > 1 \), \( \lim inf_{\ell} V_{\ell} > 1 \) and \( \lim inf_{j} V_{j} > 1 \) then there exists \( \delta > 0 \) such that \( V_{i} > 1 + \delta \), \( V_{\ell} > 1 + \delta \) and \( V_{j} > 1 + \delta \). This implies \( \frac{H_{i,j}}{q_{mn}} \geq \frac{\delta}{1 + \delta} \), \( \frac{\pi_{ij}}{q_{ij}} \geq \frac{\delta}{1 + \delta} \) and \( \frac{\pi_{ij}}{q_{ij}} \geq \frac{\delta}{1 + \delta} \). Then for \( x \in \left[ \chi_{\lambda_{m_{\ell},\nu_{\ell}}}, f, q, p \right] \), we can write for each \( i, \ell \) and \( j \)
\[
A_{i,\ell,j} = \frac{1}{\lambda_{i,\ell,j}} \frac{1}{H_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_{mnk} \chi_{\lambda_{i,n,\ell,k}} \left((m + n + k)! |x_{m+i,n+\ell,k+j}|^{p_{mnk}} \right)
\]
\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\left( \frac{1}{Q_{m_i,Q_{n_{\ell-1}},Q_{k_{j-1}}} H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}} \right)
\]

\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\left( \frac{1}{Q_{m_i,Q_{n_{\ell-1}},Q_{k_{j-1}}} H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}} \right)
\]

\[
\frac{1}{\lambda_i \mu \gamma_j} \frac{1}{H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}}
\]

\[
\left( \frac{1}{Q_{m_i,Q_{n_{\ell-1}},Q_{k_{j-1}}} H_{i\ell j}} \sum_{m=1}^{m_i} \sum_{n=1}^{n_{\ell-1}} \sum_{k=1}^{k_j} q_{m,n,k} \prod_{k=1}^{1/m+n+k} f ((m + n + k)! | x_{m+i,n+\ell,k+j})^{1/m+n+k})^{\mu_{n,k}} \right)
\]

Since \( x \in \chi_{R_{m_i,\mu_{n_{\ell-1}},Q_{k_{j-1}}}}, f, q, p \), the last three terms tend to zero uniformly in
m, n, k in the sense, thus, for each i, \ell and j

\[ A_{i, \ell, j} = \frac{1}{\lambda_i \mu_{i,\ell, j}} \frac{Q_m, Q_n, Q_{k_j}}{H_{itj}} \]

\[ + O(1). \]

Since \( \frac{1}{\lambda_i \mu_{i,\ell, j}} H_{itj} = \frac{1}{\lambda_i \mu_{i,\ell, j}} Q_m, Q_n, Q_{k_j} - \frac{1}{\lambda_i \mu_{i,\ell, j}} Q_{m-1}, Q_{n-1}, Q_{k_{j-1}} \) we are granted for each i, \ell and j the following

\[ \frac{1}{\lambda_i \mu_{i,\ell, j}} Q_m, Q_n, Q_{k_j} \leq 1 + \frac{\delta}{\delta} \text{ and } \frac{1}{\lambda_i \mu_{i,\ell, j}} Q_{m-1}, Q_{n-1}, Q_{k_{j-1}} \leq 1 - \frac{\delta}{\delta}. \]

The terms

\[ \left( \frac{1}{\lambda_i \mu_{i,\ell, j}} \frac{1}{Q_m, Q_n, Q_{k_j}} \right) \sum_{m=1}^{n} \sum_{n=1}^{k_j} \sum_{k=1}^{m} q_m q_n q_k \left[ f\left( (m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right) \right]^{\frac{1}{m+1-n+k}} \]

and

\[ \left( \frac{1}{\lambda_i \mu_{i,\ell, j}} \frac{1}{Q_{m-1}, Q_{n-1}, Q_{k_{j-1}}} \right) \sum_{m=1}^{n} \sum_{n=1}^{k_{j-1}} \sum_{k=1}^{m} q_m q_n q_k \left[ f\left( (m+n+k)! \left| x_{m+i, n+\ell, k+j} \right| \right) \right]^{\frac{1}{m+1-n+k}} \]

are both gai sequences for all r, s and u. Thus \( A_{itj} \) is a gai sequence for each i, \ell and j. Hence \( x \in \left[ \chi_{R_{m, n, k_j}}^{3}, \theta_{itj}, q, p \right] \).

**Theorem 3.4.** Let \( \theta_{i, \ell, j} = \{ m_i, n_i, k_j \} \) be a triple lacunary sequence and \( q_m q_n q_k \) with \( \lambda \text{lim sup}_{V_i} V_i < \infty, \lambda \text{lim sup}_{V_\ell} V_\ell < \infty \) and \( \lambda \text{lim sup}_{V_j} V_j < \infty \) then for any Orlicz function \( f, \)

\[ \left[ \chi_{R_{m, n, k_j}}^{3}, \theta_{itj}, q, f, p \right] \subseteq \left[ \chi_{R_{m, n, k_j}}^{3}, q, f, p \right]. \]
Proof: Since \( \lim \sup_i V_i < \infty \), \( \lim \sup_{\ell} \overline{V}_{\ell} < \infty \) and \( \lim \sup_j \overline{V}_j < \infty \) there exists \( H > 0 \) such that \( V_i < H \), \( \overline{V}_{\ell} < H \) and \( \overline{V}_j < H \) for all \( i, \ell \) and \( j \). Let \( x \in [1, H], \theta_{i, \ell, j}, q, f, p \) and \( \epsilon > 0 \). Then there exist \( i_0 > 0, \ell_0 > 0 \) and \( j_0 > 0 \) such that for every \( a \geq i_0 \), \( b \geq \ell_0 \) and \( c \geq j_0 \) and for all \( i, \ell \) and \( j \).

\[
A_{abc}^\prime = \frac{1}{\lambda_i \mu_{\ell j} H_{abc}} \sum_{m \in I_{a,b,c}} \sum_{n \in I_{a,b,c}} \sum_{k \in I_{a,b,c}} q_{m} \eta_{n} \eta_{k}^{t} \cdot \left[ f \left( (m + n + k)! \left| x_{m + i, n + \ell, k + j} \right| \right)^{1/(m + n + k)} \right]^{P_{mnk}} \rightarrow 0 \text{ as } m, n, k \rightarrow \infty.
\]

Let \( G' = \max \left\{ A'_{a,b,c} : 1 \leq a \leq i_0, \ 1 \leq b \leq \ell_0 \text{ and } 1 \leq c \leq j_0 \right\} \) and \( p, r \) and \( t \) be such that \( m_{i-1} < p \leq m_i \), \( n_{\ell-1} < r \leq n_{\ell} \) and \( k_{j-1} < t \leq k_j \). Thus we obtain the following:

\[
\frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{a = 1}^{m_{i}} \sum_{b = 1}^{n_{\ell}} \sum_{c = 1}^{k_{j}} \left[ f \left( (m + n + k)! \left| x_{m + i, n + \ell, k + j} \right| \right)^{1/(m + n + k)} \right]^{P_{mnk}} \\
\leq \frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{i = 1}^{m_i} \sum_{j = 1}^{n_{\ell}} \sum_{j = 1}^{k_j} \left[ f \left( (m + n + k)! \left| x_{m + i, n + \ell, k + j} \right| \right)^{1/(m + n + k)} \right]^{P_{mnk}} \\
= \frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{a = 1}^{m_{i}} \sum_{b = 1}^{n_{\ell}} \sum_{c = 1}^{k_{j}} H_{a,b,c} A_{a,b,c}^{\prime} + \\
\frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{i_0 < a \leq i} \cup \sum_{n_0 < b \leq n} \cup \sum_{j_0 < c \leq j} H_{a,b,c} A_{a,b,c}^{\prime}
\]

\[
\leq \frac{G' Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} + \frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{i_0 < a \leq i} \cup \sum_{n_0 < b \leq n} \cup \sum_{j_0 < c \leq j} H_{a,b,c} A_{a,b,c}^{\prime}
\]

\[
\leq \frac{G' Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} + \frac{1}{\lambda_i \mu_{\ell j} Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}} \sum_{i_0 < a \leq i} \cup \sum_{n_0 < b \leq n} \cup \sum_{j_0 < c \leq j} H_{a,b,c} A_{a,b,c}^{\prime}
\]

\[
G' Q_{m_{i-1}, Q_{n_{\ell-1}}, Q_{k_{j-1}}}^{t}
\]
Triple Almost Lacunary Riesz $\lambda_{s,m,n}^{3}$ defined by Orlicz Func.139

$$
\frac{1}{\lambda_i \mu_i \gamma_j Q_{m_i-1} Q_{n_i-1} Q_{k_j-1}} + \\
\left( \sup_{a \geq i, b \geq j, c \geq j} A'_{a,b,c} \right) \frac{1}{\lambda_i \mu_i \gamma_j Q_{m_i-1} Q_{n_i-1} Q_{k_j-1}} \sum H_{a,b,c} \\
\leq \frac{G' Q_{m_i} Q_{n_i} Q_{k_j}}{\lambda_i \mu_i \gamma_j Q_{m_i-1} Q_{n_i-1} Q_{k_j-1}} + \epsilon \sum H_{a,b,c} \\
= \frac{G' Q_{m_i} Q_{n_i} Q_{k_j}}{\lambda_i \mu_i \gamma_j Q_{m_i-1} Q_{n_i-1} Q_{k_j-1}} + V_i V_j \epsilon \\
\leq \frac{G' Q_{m_i} Q_{n_i} Q_{k_j}}{\lambda_i \mu_i \gamma_j Q_{m_i-1} Q_{n_i-1} Q_{k_j-1}} + \epsilon H^3.
$$

Since $Q_{m_i} Q_{n_i} Q_{k_j} \rightarrow \infty$ as $i, \ell, j \rightarrow \infty$ approaches infinity, it follows that

$$
\frac{1}{\lambda_i \mu_i \gamma_j Q_{m_i} Q_{n_i} Q_{k_j}} \sum_{m=1}^{p} \sum_{n=1}^{q} \sum_{k=1}^{t} q_i q_n q_k \left[ f\left( (m+n+k)! |x_{m+n+k}| \right) \right]^{1/m+n+k} = 0,
$$

uniformly in $i, \ell$ and $j$. Hence $x \in \left[ \lambda_{s,m,n}^{3}, q, f, p, \right]$.  

**Corollary 3.5.** Let $\theta_{i,j,k} = \{m_i, n_i, k_j\}$ be a triple lacunary sequence and $q_{m_i} q_{n_i} q_k$ be sequences of positive numbers. If $1 < \lim_{\delta \to 0} V_{\theta_{ij}} \leq \lim_{\delta \to 0} \sup_{i,j} < \infty$, then for any Orlicz function $f$,

$$
\left[ \lambda_{s,m,n}^{3}, \theta_{i,j}, q, f, p \right] = \left[ \lambda_{s,m,n}^{3}, q, f, p \right].
$$

**Definition 3.6.** Let $\theta_{i,j,k} = \{m_i, n_i, k_j\}$ be a triple lacunary sequence. The triple number sequence $x$ is said to be $S$-convergent to 0 provided that for every $\epsilon > 0$,

$$
P - \lim_{i \to J} \frac{1}{\lambda_i \mu_i \gamma_j H_{itj}} \sup_{i,tj} \left\{ \left( m, n, k \right) \in I_{itj} : q_i q_n q_k \left[ (m+n+k)! |x_{m+n+k}| \right]^{1/m+n+k} \right\} \geq \epsilon \right\} = 0.
$$

In this case we write $S\left[ \lambda_{s,m,n}^{3}, \theta_{i,j} \right] - P \rightarrow x = 0$. 

Theorem 3.7. Let \( \theta_{i,\ell,j} = \{m_i, n_\ell, k_j\} \) be a triple lacunary sequence. If \( I'_{i,\ell,j} \subseteq I_{i,\ell,j} \), then the inclusion

\[
\left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j}, q \right] \subseteq S \left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j} \right]
\]

is strict and

\[
\left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j}, q \right] - P - \lim_{x} = S \left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j} \right] - P - \lim_{x} = 0.
\]

Proof: Let

\[
K_Q_{i,\ell,j}(\epsilon) = \left\{ (m,n,k) \in I'_{i,\ell,j} : q_m \overline{q}_n \overline{q}_k \left[ \left( (m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/(m+n+k)} , 0 \right] \right\} \geq \epsilon
\]

(3.1)

Suppose that \( x \in \left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j}, q \right] \). Then for each \( i, \ell \) and \( j \)

\[
P - \lim_{x} \lambda_{i,\ell,j} H_{i,\ell,j} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_m \overline{q}_n \overline{q}_k \left[ \left( (m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/(m+n+k)} , 0 \right] = 0.
\]

Since

\[
\frac{1}{\lambda_{i,\ell,j} H_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_m \overline{q}_n \overline{q}_k \left[ \left( (m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/(m+n+k)} , 0 \right] \geq \frac{1}{\lambda_{i,\ell,j} H_{i,\ell,j}} \sum_{m \in I_{i,\ell,j}} \sum_{n \in I_{i,\ell,j}} \sum_{k \in I_{i,\ell,j}} q_m \overline{q}_n \overline{q}_k \left[ \left( (m+n+k)! \left| x_{m+i,n+\ell,k+j} \right| \right)^{1/(m+n+k)} , 0 \right] = \frac{K_{i,\ell,j}(\epsilon)}{\lambda_{i,\ell,j} H_{i,\ell,j}}
\]

for all \( i, \ell \) and \( j \), we get \( P - \lim_{x} \frac{K_{i,\ell,j}(\epsilon)}{\lambda_{i,\ell,j} H_{i,\ell,j}} = 0 \) for each \( i, \ell \) and \( j \). This implies that \( x \in S \left[ \chi_{R^{\lambda_{m_i},\mu_{n_\ell},\gamma_{k_j}}}^{3} \theta_{i,\ell,j} \right] \).
We show that this inclusion is strict, let $x = (x_{mnk})$ be defined as
\[
(x_{mnk}) = \begin{bmatrix}
1 & 2 & 3 & \ldots \\
1 & 2 & 3 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
\lambda_{\mu_i\gamma_j}(\sqrt{H_{i,\ell,j}})^{m+n+k} & 0 & \ldots \\
\lambda_{\mu_i\gamma_j}(\sqrt{H_{i,\ell,j}})^{m+n+k} & \lambda_{\mu_i\gamma_j}(\sqrt{H_{i,\ell,j}})^{m+n+k} & \lambda_{\mu_i\gamma_j}(\sqrt{H_{i,\ell,j}})^{m+n+k} & \ldots \\
\lambda_{\mu_i\gamma_j}(\sqrt{H_{i,\ell,j}})^{m+n+k} & 0 & \ldots \\
\vdots & \vdots & \vdots & \ddots \\
0 & 0 & 0 & \ldots \\
0 & 0 & 0 & \ldots \\
\end{bmatrix}
\]
and $q_m = 1$, $q_n = 1$, and $q_k = 1$ for all $m, n, k$. Clearly, $x$ is unbounded sequence. For $\epsilon > 0$ and for all $i, \ell$ and $j$ we have
\[
\left| \left\{ (m, n, k) \in I_{ij} : q_m \tilde{q}_n \tilde{q}_k \left( (m+n+k)! \left| x_{mn+i,n+\ell,k+j} \right| \right)^{1/(m+n+k), 0} \right\} \right| \geq \epsilon \\
= P - \lim_{I_{ij}} \left( \frac{\lambda_{\mu_i\gamma_j}(m+n+k)!}{\sqrt{H_{i,\ell,j}}^{m+n+k}} \left( \frac{\sqrt{H_{i,\ell,j}}^{m+n+k}}{\sqrt{H_{i,\ell,j}}^{m+n+k}} \right) \right)^{1/(m+n+k)} \frac{1}{m+n+k} \\
= 0.
\]
Therefore $x \in S^{\tilde{\lambda}_{\lambda_{\mu_i\gamma_j}, \theta_{I_{ij}}}}$ with the $P - \lim = 0$. Also note that
\[
P - \lim_{I_{ij}} \frac{1}{\lambda_{\mu_i\gamma_j}(m+n+k)!} \sum_{m \in I_{ij}} \sum_{n \in I_{ij}} \sum_{k \in I_{ij}} q_m \tilde{q}_n \tilde{q}_k \left( (m+n+k)! \left| x_{mn+i,n+\ell,k+j} \right| \right)^{1/(m+n+k), 0} \\
= P \frac{1}{2} \left( \lim_{I_{ij}} \lambda_{\mu_i\gamma_j}(m+n+k)! \left( \frac{\sqrt{H_{i,\ell,j}}^{m+n+k}}{\sqrt{H_{i,\ell,j}}^{m+n+k}} \right)^{1/(m+n+k)} + 1 \right) \\
= \frac{1}{2}.
\]
Hence $x \notin S^{\tilde{\lambda}_{\lambda_{\mu_i\gamma_j}, \theta_{I_{ij}}}, q}$. \qed

**Theorem 3.8.** Let $I_{ij} \subseteq I_{I_{ij}}$. If the following conditions hold, then
\[
S^{\tilde{\lambda}_{\lambda_{\mu_i\gamma_j}, \theta_{I_{ij}}}, q} \subset S^{\tilde{\lambda}_{\lambda_{\mu_i\gamma_j}, \theta_{I_{ij}}}, q_{I_{ij}}}
\]
and
\[
\chi^3_{\lambda_{\mu_{\nu_{\gamma_j}}, \theta_{\delta_j}, q}} - P - \lim x = S_{\chi^3_{\lambda_{\mu_{\nu_{\gamma_j}}, \theta_{\delta_j}}}} - P - \lim x = 0.
\]

(1) \(0 < \mu < 1\) and 0 \(\leq \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right] < 1\).

(2) \(1 < \mu < \infty\) and 1 \(\leq \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right] < \infty\).

**Proof:** Let \(x = (x_{m,n})\) be strongly \(\chi^3_{\lambda_{\mu_{\nu_{\gamma_j}}, \theta_{\delta_j}, q}} - \) almost \(P\) convergent to the limit 0. Since

\[
q_m \overline{P}_n \bar{Q}_k \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right]^\mu \\
\geq q_m \overline{P}_n \bar{Q}_k \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right]
\]

for (1) and (2), for all \(i, \ell\) and \(j\), we have

\[
\frac{1}{\lambda_i \mu_{\gamma_j} H_{\delta_j}} \sum_{m \in I_{\ell,j}} \sum_{n \in I_{\ell,j}} \sum_{k \in I_{\ell,j}} q_m \overline{P}_n \bar{Q}_k \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right]^\mu \\
\geq \frac{1}{\lambda_i \mu_{\gamma_j} H_{\delta_j}} \sum_{m \in I_{\ell,j}} \sum_{n \in I_{\ell,j}} \sum_{k \in I_{\ell,j}} q_m \overline{P}_n \bar{Q}_k \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right]
\]

where \(K_{\gamma_{\ell,j}} (\epsilon)\) is as in (3.1). Taking limit \(i, \ell, j \to \infty\) in both sides of the above inequality, we conclude that \(S_{\chi^3_{\lambda_{\mu_{\nu_{\gamma_j}}, \theta_{\delta_j}}}} - P - \lim x = 0\). \(\Diamond\)

**Definition 3.9.** A triple sequence \(x = (x_{m,n})\) is said to be Riesz lacunary of \(\chi\) almost \(P\) convergent 0 if \(P - \lim_{i,\ell,j} w^{i_{\ell,j}}_{m,n} (x) = 0\), uniformly in \(i, \ell\) and \(j\), where

\[
w^{i_{\ell,j}}_{m,n} (x) = w^{i_{\ell,j}}_{m,n} = \frac{1}{\lambda_i \mu_{\gamma_j} H_{\delta_j}} \sum_{m \in I_{\ell,j}} \sum_{n \in I_{\ell,j}} \sum_{k \in I_{\ell,j}} q_m \overline{P}_n \bar{Q}_k \left[\left[(m + n + k)! |x_{m+i,n+\ell,k,j}| \right]^{1/(m+n+k)}, 0\right].
\]

**Definition 3.10.** A triple sequence \((x_{m,n})\) is said to be Riesz lacunary \(\chi\) almost statistically summable to 0 if for every \(\epsilon > 0\) the set

\[K_{\epsilon} = \{(i, \ell, j) \in \mathbb{N} \times \mathbb{N} \times \mathbb{N} : \left|w^{i_{\ell,j}}_{m,n} (x, 0)\right| \geq \epsilon\}\]
Let $\text{Theorem 3.11.}$.

That is, for every $\epsilon > 0$, $P - \lim_{rst \to \epsilon} \left| \left\{ i \leq r, \ell \leq s, j \leq t : \left| w_{mnk}^{ij} \right| \geq \epsilon \right\} \right| = 0$, uniformly in $i, \ell$ and $j$.

**Theorem 3.11.** Let $I'_{ij} \subseteq I_{ij}$ and $q_m \gamma_n \bar{\gamma}_k \left( (m+n+k)! |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \leq M$ for all $m, n, k \in \mathbb{N}$ and for each $i, \ell$ and $j$. Let $x = (x_{mnk})$ be

$$S_{\chi_{R^{\lambda m + \mu n + \gamma k}}^{\delta} \theta_{ij}} - P - \lim_{x \to \epsilon} = 0.$$

Let

$$K_{Q_{ij}}(e) = \left\{ (m,n,k) \in I'_{ij} : q_m \gamma_n \bar{\gamma}_k \left( (m+n+k)! |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \right\} \left( (m+n+k)! |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \geq \epsilon \right\}.$$

Then

$$w_{mnk}^{ij} \left( \frac{1}{\lambda_{ij} \mu_{i,j} \gamma_{j}} H_{ij} \right) \sum_{m \in I'_{ij}} \sum_{n \in I'_{ij}} \sum_{k \in I'_{ij}} q_m \gamma_n \bar{\gamma}_k \left( (m+n+k)! |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \left( (m+n+k)! |x_{m+i,n+\ell,k+j}| \right)^{1/m+n+k} \geq \epsilon \right\}.$$

for each $i, \ell$ and $j$, which implies that $P - \lim_{i, \ell, j} w_{mnk}^{ij} (x) = 0$, uniformly in $i, \ell$ and $j$. Hence $S_{t \to \epsilon} - P - \lim_{x \to \epsilon} = 0$ uniformly in $i, \ell, j$. Hence

$$\left[ \chi_{R^{\lambda m + \mu n + \gamma k}}^{\delta} \theta_{ij} \right]_{st_{2}} - P - \lim_{x \to \epsilon} = 0.$$

To see that the converse is not true, consider the triple lacunary sequence $\theta_{ij} \{ (2^{i-1}3^{j-1}4^{i-1}) \}$, $q_m = 1, \gamma_n = 1, \bar{\gamma}_k = 1$ for all $m, n$ and $k$, and the triple sequence $x = (x_{mnk})$ defined by $x_{mnk} = (-1)^{m+n+k}$ for all $m, n$ and $k$.

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