On Fully-Convex Harmonic Functions and their Extension

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Abstract: Uniformly convex univalent functions that introduced by Goodman, maps every circular arc contained in the open unit disk with center in it into a convex curve. On the other hand, a fully-convex harmonic function, maps each subdisk $|z| = r < 1$ onto a convex curve. Here we synthesis these two ideas and introduce a family of univalent harmonic functions which are fully-convex and uniformly convex also. In the following we will mention some examples of this subclass and obtain a necessary and sufficient conditions and finally a coefficient condition is given as an application of some convolution results.

Key Words: Uniformly convex function, Fully-Convex function, Harmonic function, Convolution.

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1. Introduction and Preliminaries

Let $D = \{ z \in \mathbb{C} : |z| < 1 \}$ be the open unit disk in complex plane. Let $A$ be the familier class of all analytic functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n$$

(1.1)

in the open unit disk $D$. Let $S$ denotes the family of all functions $f(z)$ of the form (1.1) that are univalent in $D$ and normalized with $f(0) = 0$ and $f'(0) = 1$.

A conformal function $f(z)$ is said to be starlike if every point of its range can be connected to the origin by a radial line that lies entirely in that region. The class of all starlike functions in $S$ is shown by $S^*$ [9] and $f(z) \in S^*$ if and only if $\Re \{ z \frac{f'(z)}{f(z)} \} > 0$. Starlikeness is a hereditary property for conformal mappings, so if $f(z) \in S$, and if $f$ maps $D$ onto a domain that is starlike with respect to the origin, then the image of every subdisk $|z| < r < 1$ is also starlike with respect to the origin.

2010 Mathematics Subject Classification: Primary 30C45; Secondary 31C05, 31A05.

Submitted December 29, 2016. Published October 04, 2017
An analytic function $f(z)$ is said to be convex if its range $f(D)$ is a convex set. It has shown that every convex function $f$ in $S$ satisfy following analytic property

$$\Re \left\{ 1 + z \frac{f''(z)}{f'(z)} \right\} > 0$$

The class of all convex functions in $S$ is denoted by $\mathcal{K}$ [9].

The subclass of uniformly starlike functions, $\mathcal{UST}$ introduced by Goodman [6] and studied in analytic and geometric view.

**Definition 1.1.** [6] A function $f(z) \in S^*$ is said to be uniformly starlike in $D$ if it has the property that for every circular arc $\gamma$ contained in $D$, with center $\zeta \in D$, the arc $f(\gamma)$ be starlike with respect to $f(\zeta)$. We denote the family of all uniformly starlike functions by $\mathcal{UST}$ and we have,

$$\mathcal{UST} = \left\{ f(z) \in S : \Re \left( \frac{(z - \zeta)f'(z)}{f(z) - f(\zeta)} \right) > 0, \ (z, \zeta) \in \mathbb{D}^2 \right\} \quad (1.2)$$

It’s clear that $\mathcal{UST} \subset S^*$ and every function in $\mathcal{UST}$ maps each subdisk $\{ |z - \zeta| < \rho \} \subset \mathbb{D}$ onto a domain starlike with respect to $f(\zeta)$. Goodman [5] also defined the subclass of convex functions with this property that map each disk $\{ |z - \zeta| < \rho \} \subset \mathbb{D}$ onto a convex domain and called it uniformly convex function and denoted the set of all these functions by $\mathcal{UCE}$.

**Definition 1.2.** [5] A function $f(z) \in \mathcal{K}$ is said to be uniformly convex in $D$ if it has the property that for every circular arc $\gamma$ contained in $D$, with center $\zeta \in D$, the arc $f(\gamma)$ be a convex arc. We have,

$$\mathcal{UCE} = \left\{ f(z) \in S : \Re \left( 1 + (z - \zeta) \frac{f''(z)}{f'(z)} \right) \geq 0, \ (z, \zeta) \in \mathbb{D}^2 \right\} \quad (1.3)$$

A summary of early works on uniformly starlike and uniformly convex functions can be found in [10].

The complex-valued function $f(x, y) = u(x, y) + iv(x, y)$ is complex-valued harmonic function in $D$ if $f$ is continuous and $u$ and $v$ are real harmonic in $D$. We denote $H$ the family of continuous complex-valued functions which are harmonic in the open unit disk $\mathbb{D}$. In simply-connected domain $\mathbb{D}$, $f \in H$ has a canonical representation $f = h + g$, where $h$ and $g$ are analytic in $D$ [3,4]. Then, $g$ and $h$ have expansions in Taylor series as $h(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$, so we may represent $f$ by a power series of the form

$$f(z) = h(z) + g(z) = \sum_{n=0}^{\infty} a_n z^n + \sum_{n=0}^{\infty} b_n z^n \quad (1.4)$$

The Jacobian of a function $f = u + iv$ is $J_f(z) = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = |h'(z)|^2 - |g'(z)|^2$, and $f(z) = h(z) + g(z)$ is sense-preserving if $J_f(z) > 0$. In 1984, Clunie and Shiel-Small
investigated the class $S_H$, consisting of sense-preserving univalent harmonic functions $f(z) = h(z) + g(z)$ in simply-connected domain $D$ which normalized by $f(0) = 0$ and $f_z(0) = 1$ with the form,

$$f(z) = h(z) + g(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n$$ \hspace{1cm} (1.5)

The subclass $S_0^H$ of $S_H$ includes all functions $f \in S_H$ with $f_z(0) = 0$, so $S \subset S_0^H \subset S_H$. Clunie and Sheil-Small also considered convex functions in $S_H$, denoted by $K_H$. The hereditary property of convexity for conformal maps does not generalize to univalent harmonic mappings. If $f$ is a univalent harmonic map of $D$ onto a convex domain, then the image of the disk $|z| < r$ is convex for each radius $r \leq \sqrt{2} - 1$, but not necessarily for any radius in the interval $\sqrt{2} - 1 < r < 1$. In fact, the function

$$f(z) = \text{Re} \frac{z}{1 - z} + i\text{Im} \frac{z}{(1 - z)^2}$$ \hspace{1cm} (1.6)

is a harmonic mapping of the disk $D$ onto the half-plane $\text{Re} w > -\frac{1}{2}$, but the image of the disk $|z| \leq r$ fails to be convex for every $r$ in the interval $\sqrt{2} - 1 < r < 1$ [4]. Thus we need a property to explain convexity of a map in a hereditary form in whole disk. We have following definition.

**Definition 1.3.** [2] A harmonic mapping $f$ with $f(0) = 0$ of the unit disk is said to be fully-convex if it maps every circle $|z| = r < 1$ in a one-to-one manner onto a convex curve.

For $f \in S_H$, the family of fully-convex harmonic functions denotes by $F \mathcal{K}_H$. In 1980 Mocanu gave a relation between fully-starlikeness and a differential operator of a non-analytic function [7]. Let

$$Df = zf_z - \overline{zf_z}$$ \hspace{1cm} (1.7)

be the differential operator and

$$D^2f = D(Df) = zzf_{zz} + \overline{zf_{zz}} + zf_z + \overline{zf_z}$$ \hspace{1cm} (1.8)

**Lemma 1.4.** [7] Let $f \in C^1(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $\text{Re} \frac{Df(z)}{f(z)} > 0$ then $f$ is univalent and fully-starlike in $\mathbb{D}$.

**Lemma 1.5.** Let $f \in C^2(\mathbb{D})$ is a complex-valued function such that $f(0) = 0$, $f(z) \neq 0$ for all $z \in \mathbb{D} \setminus \{0\}$, and $J_f(z) > 0$ in $\mathbb{D}$ and $\text{Re} \frac{D^2f(z)}{Df(z)} > 0$ then $f$ is univalent and fully-convex in $\mathbb{D}$.
Since for a sense-preserving complex-valued function \( f(z) \), \( Df \neq 0 \), if \( f(z) \in S_H \) and satisfies condition such as \( \text{Re} \left( \frac{Df(z)}{f(z)} \right) > 0 \) or \( \text{Re} \left( \frac{D^2 f(z)}{Df(z)} \right) > 0 \) for all \( z \in \mathbb{D} - \{0\} \), then \( f \) maps every circle \( 0 < |z| = r < 1 \) onto a simple closed curve \([7]\). However, a fully-starlike mapping need not be univalent \([2]\), we restrict our discussion to \( S_H \).

2. Definition and Examples

For a harmonic function \( f(z) = h(z) + g(z) \in S_H \), and \( \zeta \in \mathbb{D} \) we define the operator

\[
Df(z, \zeta) = \frac{(z - \zeta)f_z(z) - (\bar{z} - \bar{\zeta})f_{\bar{z}}(z)}{(z - \zeta)h'(z) - (\bar{z} - \bar{\zeta})g'(z)}
\]

is harmonic also. For \( \zeta = 0 \) the operator \( Df(z, 0) = zf_z - \bar{z}f_{\bar{z}} = Df(z) \) is previous operator \((1.7)\). Differentiating of the operator \( Df(z, \zeta) \) gives us

\[
D^2 f(z, \zeta) = D(Df(z, \zeta))
= D((z - \zeta)h'(z) - (\bar{z} - \bar{\zeta})g'(z))
= (z - \zeta)^2 h''(z) + (z - \zeta)h'(z)
+ (\bar{z} - \bar{\zeta})g'(z)
\]

For \( \zeta = 0 \) the operator \( D^2 f(z, 0) = z^2 h''(z) + \bar{z}^2 g''(z) + zh'(z) + \bar{z}g'(z) = D^2 f(z) \) has described by Al-Amiri and Mocanu \([1]\). Similar to definition \((1.1)\) we say that for an arbitrary function:

**Definition 2.1.** A function \( f \in S_H \) is said to be uniformly fully-convex harmonic function in \( \mathbb{D} \) if it has the property that for every circular arc \( \gamma \) contained in \( \mathbb{D} \), with center \( \zeta \in \mathbb{D} \), the arc \( f(\gamma) \) is convex in \( f(\mathbb{D}) \).

We denote the set of all uniformly fully-convex harmonic functions in \( \mathbb{D} \) by \( UFKH \). The following theorem gives analytic equivalency for above definition:

**Theorem 2.2.** Let \( f \in S_H \). \( f \in UFKH \) if and only if

\[
\text{Re} \left( \frac{D^2 f(z, \zeta)}{Df(z, \zeta)} \right) > 0 , \ (z, \zeta) \in \mathbb{D}^2
\]

**Proof:** Let \( \gamma : \zeta + re^{i\theta} \) with \( \theta_1 \leq \theta \leq \theta_2 \) be a circular arc centered at \( \zeta \) and contained in \( \mathbb{D} \), then the image of \( \gamma \) under \( f \) is convex if the argument of the tangent to the image be a non-decreasing function of \( \theta \), that is,

\[
\frac{\partial}{\partial \theta} \left( \arg \left( \frac{f(z) - f(\zeta)}{Df(z, \zeta)} \right) \right) \geq 0
\]

Hence

\[
\text{Im} \left( \frac{\partial}{\partial \theta} \left( \log \left( \frac{f(z) - f(\zeta)}{Df(z, \zeta)} \right) \right) \right) \geq 0
\]
But for a circular arc $\gamma$, set $z = \zeta + re^{i\theta}$, then $\frac{\partial}{\partial\theta}z = i(z-\zeta)$ and a brief computation will give us

$$\frac{\partial}{\partial\theta}\{f(z) - f(\zeta)\} = i\left\{(z-\zeta)f_z(z) - (z-\zeta)f_\zeta(z)\right\} = i\mathbf{D}f(z, \zeta)$$

then

$$\frac{\partial}{\partial\theta}\log i\mathbf{D}f(z, \zeta) = \frac{\partial}{\partial\theta}\log i\left\{(z-\zeta)h'(z) - (z-\zeta)g'(z)\right\} = \frac{i[h'(z) + (z-\zeta)h''(z)]}{i\mathbf{D}f(z, \zeta)i(z-\zeta)} = i\mathbf{D}^2f(z, \zeta)$$

Therefore, we must have

$$\text{Im} \frac{\partial}{\partial\theta}\log i\mathbf{D}f(z, \zeta) = \text{Re} \frac{\mathbf{D}^2f(z, \zeta)}{\mathbf{D}f(z, \zeta)} \geq 0$$
as we want. \(\square\)

It should be noted that $\frac{\mathbf{D}^2f(z, \zeta)}{\mathbf{D}f(z, \zeta)}(0,0) = 1$, and

$$\mathcal{UFKH} = \left\{f(z) \in \mathcal{SH} : \text{Re} \frac{\mathbf{D}^2f(z, \zeta)}{\mathbf{D}f(z, \zeta)} > 0, (z, \zeta) \in \mathbb{D}^2\right\} \quad (2.4)$$

It’s simple that one checks the rotations, $e^{-i\alpha}f(e^{i\alpha}z)$ for some real $\alpha$, are preserve the class $\mathcal{UFKH}$ and the transformation $\frac{1}{t}f(tz)$ preserves this class also, where $0 < t \leq 1$. On the other hand, the class $\mathcal{UFKH}$ includes all fully-convex functions and uniformly convex functions. With $g = 0$ in (2.3), the analytic function $f(z) \in \mathcal{UFKH}$ by (2.1) and (2.2) satisfies condition

$$\text{Re} \frac{\mathbf{D}^2f(z, \zeta)}{\mathbf{D}f(z, \zeta)} = \text{Re} \frac{(z-\zeta)^2h''(z) + (z-\zeta)h'(z)}{(z-\zeta)h'(z)} = \text{Re} \left(1 + (z-\zeta)\frac{h''(z)}{h'(z)}\right) \geq 0$$

where $(z, \zeta) \in \mathbb{D}^2$. Then

**Corollary 2.3.** If $f \in \mathcal{UCV}$ be an analytic function, then $f \in \mathcal{UFKH}$. So, $\mathcal{UCV} \subset \mathcal{UFKH} \subset \mathcal{K}_H$. Goodman [5] shows the analytic function $f(z) = \frac{z}{1-Az}$ \in $\mathcal{UCV}$ if and only if $|A| \leq \frac{1}{3}$, thus the convex function $f(z) = \frac{z}{1-z} \notin \mathcal{UFKH}$.
Example 2.1. For $|\beta| < 1$ the affine mappings $f(z) = z + \bar{\beta}z \in \mathcal{UFK}_H$, since

$$\Re \left( \frac{(z - \zeta) + (z - \zeta)\beta}{(z - \zeta) - (z - \zeta)\beta} \right) \geq 0$$

is equivalent to

$$\Re \left( \frac{(z - \zeta) + (z - \zeta)\beta}{(z - \zeta) - (z - \zeta)\beta} \right) \geq 0$$

that is $(1 - |\beta|^2)|z - \zeta|^2 \geq 0$.

Corollary 2.4. For $\zeta = 0$ in (2.3), the harmonic function $f \in \mathcal{UFK}_H$ will be univalent and fully-convex in $\Delta$ by Lemma 1.5. Thus it’s clear any non fully-convex harmonic function is not in $\mathcal{UFK}_H$. The harmonic function $f(z) = \Re \frac{z}{1 - z} + i \Im \frac{z}{(1 - z)^2}$ isn’t fully-convex ([4], p.46), then $f \notin \mathcal{UFK}_H$.

In the following we will give a necessary and sufficient condition for that $f \in \mathcal{UFK}_H$. This condition is a generalization form of a theorem about fully-convex functions mentioned by Chuaqui et al. in [2], p.139.

Theorem 2.5. Let $f(z) \in \mathcal{S}_H$, $f \in \mathcal{UFK}_H$ if and only if

$$|(z - \zeta)h'(z)|^2 \Re Q_h \geq \left| (z - \zeta)g'(z) \right|^2 \Re Q_g + \Re \left\{ (z - \zeta)^3 (h''(z)g'(z) - h'(z)g''(z)) \right\}$$

where $Q_h = 1 + (z - \zeta)\frac{h''(z)}{h'(z)}$ and $Q_g = 1 + (z - \zeta)\frac{g''(z)}{g'(z)}$ for $(z, \zeta)$ in polydisk $\mathbb{D}^2$.

Proof: According to the definition, $f \in \mathcal{UFK}_H$ if and only if $\Re \frac{D^2f(z, \zeta)}{Df(z, \zeta)} > 0$ for $(z, \zeta) \in \mathbb{D}^2$, if and only if $\Re \left\{ D^2f(z, \zeta) \bar{Df(z, \zeta)} \right\} > 0$ for $(z, \zeta) \in \mathbb{D}^2$; then a simple calculation gives us (2.5).

Lemma 2.6. $f = h + \bar{\beta}h \in \mathcal{UFK}_H$ if and only if $h \in \mathcal{UCV}$, where $|\beta| < 1$.

Proof: Let $f = h + \bar{g} \in \mathcal{S}_H$ and $g = \beta h$ with $|\beta| < 1$, then $f \in \mathcal{UFK}_H$ if and only if (2.5) holds. Since in this case, $h$ and $g$ satisfy equality $Q_h = Q_g$ so (2.5) holds if and only if $|(z - \zeta)h'(z)|^2 \Re Q_h (1 - |\beta|^2) \geq 0$, or $\Re Q_h \geq 0$ that shows $h \in \mathcal{UCV}$.

Example 2.2. The analytic function $h = z + Az^2$ is in $\mathcal{UCV}$ if and only if $|A| \leq \frac{1}{6}$ [5]. By Lemma 2.6 we get $f(z) = z + Az^2 + \beta z + \beta Az^2 \in \mathcal{UFK}_H$ with $|\beta| < 1$ and $|A| \leq \frac{1}{6}$. For example, let $A = \frac{1}{6}, \beta = \frac{i}{2}$ then $f = z + \frac{1}{6}z^2 - \frac{i}{2}z - \frac{i}{12}z^2 \in \mathcal{UFK}_H$.

In Figure 1, the disk $|z - 0.7| < 0.3$ is mapped under this uniformly fully-convex harmonic function to a convex elliptical shape with center $f(\zeta) = (0.78, 0.39)$. 
3. Convolution and a sufficient condition

The convolution or Hadamard product of two harmonic functions \( f(z) \) and \( F(z) \) with canonical representations

\[
\begin{align*}
    f(z) &= h(z) + g(z) = z + \sum_{n=2}^{\infty} a_n z^n + \sum_{n=1}^{\infty} b_n z^n \\
    F(z) &= H(z) + G(z) = z + \sum_{n=2}^{\infty} A_n z^n + \sum_{n=1}^{\infty} B_n z^n
\end{align*}
\]
(3.1)

and

\[
\begin{align*}
    (f * F)(z) &= (h * H)(z) + g * G(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n z^n
\end{align*}
\]
(3.3)

is defined as

\[
\begin{align*}
    (f * F)(z) &= (h * H)(z) + g * G(z) = z + \sum_{n=2}^{\infty} a_n A_n z^n + \sum_{n=1}^{\infty} b_n B_n z^n
\end{align*}
\]

The right half-plane mapping \( \ell(z) = \frac{z}{1-z} \) acts as the convolution identity and the Koebe map \( k(z) = \frac{z}{(1-z)^2} \) acts as derivative operation over functions convolution. We have some properties for convolution over analytic functions \( f \) and \( g \):

\[
\begin{align*}
    f * g &= g * f \\
    f * \ell &= f \\
    z f'(z) &= f * k(z)
\end{align*}
\]

where \( \alpha \in \mathbb{C} \). For a given subset \( \mathcal{V} \subset \mathcal{A} \), its dual set \( \mathcal{V}^* \) is defined by

\[
\mathcal{V}^* = \left\{ g \in \mathcal{A} : \frac{f * g(z)}{z} \neq 0, \ \forall f \in \mathcal{V}, \ \forall z \in \mathbb{D} \right\}
\]
(3.4)
Nezhmetdinov (1997) proved that class $UCV$ is dual set for certain family of functions from $A$. He proved ([8], Theorem 2, p.43) that the class $UCV$ is the dual set of a subset of $A$ consisting of functions $\varphi : \mathbb{D} \to \mathbb{C}$ given by
\[
\varphi(z) = \frac{z}{(1-z)^3} \left[ 1 - z - \frac{4z}{(\alpha+i)^2} \right] \tag{3.5}
\]
where $\alpha \in \mathbb{R}$. He determined the uniform estimate $|a_n(\varphi)| \leq n(2n-1)$ for the $n$-th Taylor coefficient of $\varphi(z)$:

**Lemma 3.1.** [8] Let $G$ is all function $\varphi \in A$ of the form (3.5), then $UCV = G^*$ and $|a_n(\varphi)| \leq n(2n-1)$ for all $n \geq 2$.

For obtaining a sufficient condition in class $\mathcal{UFK}_H$, we define the dual set of a harmonic function. Let $A_H$ be the class of complex-valued harmonic functions $f(z) = h(z) + g(z)$ in simply connected domain $D$ of the form (1.5) which are not necessarily sense-preserving univalent on $D$. We define the dual set of a subset of $A_H$:

**Definition 3.2.** For a given subset $V_H \subset A_H$, the dual set $V_H^*$ is
\[
V_H = \left\{ F = H + \overline{G} \in A_H : \frac{h+G}{z} \neq 0, \forall f = h + \overline{G} \in V_H, \forall z \in \mathbb{D} \right\} \tag{3.6}
\]

**Theorem 3.3.** Let $\alpha \in \mathbb{R}$, $|w| = 1$ and
\[
G_H = \left\{ \varphi - \sigma \overline{\varphi} : \varphi(z) = \frac{z}{(1-z)^3} \left[ 1 - \frac{w - i\alpha}{2 - w - i\alpha} z \right], \sigma = \frac{(1-w)(2-w-i\alpha)}{(1-w)(2-w+i\alpha)} z \in \mathbb{D} \right\}
\]
then $\mathcal{UFK}_H = G_H$. Furthermore If $\sum_{n=2}^{\infty} n(2n-1)|a_n| + n(2n-1)|b_n| < 1 - |b_1|$ then $f \in \mathcal{UFK}_H$.

It’s clear that the analytic function $\varphi$ is the same (3.5), but $\sigma$ with $|\sigma| = 1$ isn’t an arbitrary number and depend on both $w$ and $\alpha$ in $\varphi$.

**Proof:** Let $f = h + \overline{g} \in \mathcal{UFK}_H$, that is
\[
\text{Re} \frac{(z - \zeta)h''(z) + (z - \zeta)^2g''(z) + (z - \zeta)h'(z) + (z - \zeta)g'(z)}{(z - \zeta)h'(z) - (z - \zeta)g'(z)} > 0 \tag{3.7}
\]
$(z, \zeta) \in D^2$. For $\zeta = 0$ and then $z = 0$ we have $D^2f(z, \zeta) = 1$, hence the condition (3.7) may be write as
\[
\text{Re} \frac{i\alpha ((z - \zeta)h'(z) - (z - \zeta)g'(z))}{(z - \zeta)h''(z) + (z - \zeta)^2g''(z)} \neq (z - \zeta)h''(z) + (z - \zeta)^2g''(z)
\]
where $\alpha \in \mathbb{R}$. By the minimum principle for harmonic functions, it is sufficient to verify this condition for $|z| = |\zeta|$ and so, we may assume that $\zeta = wz$ with $|w| = 1$, then from the definition of the dual set for harmonic functions (3.6), with straightforward calculation we conclude that $\frac{h * \varphi}{z} + \sigma \frac{g * \varphi}{z} \neq 0$, so the first assertion follows.

For obtaining coefficients condition, let $f(z) = h(z) + g(z)$ is of the form (3.1), and $\varphi(z) = z + \sum_{n=2}^{\infty} \phi_n z^n$ be the series expansion of analytic function $\varphi(z)$, then $|\phi_n| \leq n(2n - 1)$ for all $n \geq 2$, by Lemma 3.1. From previous part we see that

$$| \frac{h * \varphi}{z} + \sigma \frac{g * \varphi}{z} | = \left| 1 + \sum_{n=2}^{\infty} a_n \phi_n z^{n-1} + \sigma \left( b_1 + \sum_{n=2}^{\infty} b_n \phi_n z^{n-1} \right) \right|$$

$$\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} |a_n| |\phi_n||z|^{n-1} - |\sigma| \sum_{n=2}^{\infty} |b_n| |\phi_n||z|^{n-1}$$

$$\geq |1 + \sigma b_1| - \sum_{n=2}^{\infty} n(2n - 1)|a_n| - \sum_{n=2}^{\infty} n(2n - 1)|b_n|$$

$$> 0$$

when $\sum_{n=2}^{\infty} n(2n - 1)|a_n| + n(2n - 1)|b_n| < 1 - |b_1|$.

\[\square\]

References

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