W1,N Versus C1 Local Minimizer For a Singular Functional with Neumann Boundary Condition

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Abstract: Let Ω ⊂ R^N, be a bounded domain with smooth boundary. Let g : R^+ → R^+ be a continuous function on (0, +∞) non-increasing and satisfying

\[ c_1 = \liminf_{t \to 0^+} g(t) t^\delta \leq \limsup_{t \to 0^+} t^\delta \leq c_2, \]

for some c_1, c_2 > 0 and 0 < δ < 1. Let f(x, s) = h(x, s) e^{bs N - 1}, b > 0 is a constant. Consider the singular functional I : W^{1,N}(Ω) → R defined as

\[ I(u) \overset{\text{def}}{=} \frac{1}{N} \|u\|_{W^{1,N}(Ω)}^N - \int_{Ω} G(u^+) \, dx - \int_{Ω} F(x, u^+) \, dx \]

\[ - \frac{1}{q+1} \|u\|_{L^{q+1}(\partial Ω)}^{q+1} \]

where F(x, u) = \int_0^u f(x, s) \, ds, G(u) = \int_0^u g(s) \, ds. We show that if u_0 ∈ C^1(Ω) satisfying u_0 ≥ \text{dist}(x, ∂Ω), for some 0 < η, is a local minimum of I in the C^1(Ω) ∩ C^0(Ω) topology, then it is also a local minimum in W^{1,N}(Ω) topology. This result is useful to prove the multiplicity of positive solutions to critical growth problems with co-normal boundary conditions.

Key Words: N-Laplace operator, singular equations, Neumann boundary condition, Variational methods, Local minimizers.

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1. Introduction

Let Ω ⊂ R^N, N ≥ 2 be a bounded smooth domain. Let f(x, s) = h(x, s) e^{bs N - 1}, b > 0 is a constant. Let h : Ω × R^+ → [0, +∞) be a C^1 function satisfying:

(h1) Nonnegative with h(x, 0) = 0. Moreover, f(x, t) = h(x, t) e^{bs N - 1} is nondecreasing in respect to t for t large.

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(h2) \( \forall \epsilon > 0, \liminf_{t \to \infty} h(x, t)e^{\epsilon|t|^{\frac{N}{N-1}}} = \infty, \liminf_{t \to \infty} h(x, t)e^{-\epsilon|t|^{\frac{N}{N-1}}} = 0 \) uniformly in \( x \in \overline{\Omega} \).

(h3) \( \forall \epsilon > 0, \liminf_{t \to \infty} h(x, t)t e^{\epsilon t^{\frac{N}{N-1}}} = \infty, \)

Let \( g : \mathbb{R}^+ \to \mathbb{R}^+ \) continuous on \((0, +\infty)\) satisfying

(g1) \( g \) is nonincreasing on \((0, +\infty)\),

(g2) \( c_1 \leq \liminf_{t \to 0^+} g(t) t^\delta \leq \limsup_{t \to 0^+} g(t) t^\delta = c_2 \) for some \( c_1, c_2 > 0 \) and \( 0 < \delta < 1 \).

From (g2), \( g \) is singular at the origin and \( \lim_{t \to 0^+} g(t) = +\infty \). We consider the singular functional \( I : W^{1,N}(\Omega) \to \mathbb{R} \) defined as

\[
I(u) \overset{\text{def}}{=} \frac{1}{N}\|u\|_{W^{1,N}(\Omega)}^N - \int_{\Omega} G(u^+) \, dx - \int_{\Omega} F(x, u^+) \, dx
- \frac{1}{q+1}\|u\|_{L^{q+1}(\partial\Omega)}^{q+1}
\]  

(1.1)

where \( F(x, u) = \int_0^s f(x, s) \, ds \), \( G(u) = \int_0^t g(s) \, ds \). Our aim in this paper is to show the following

**Theorem 1.1.** Suppose that the conditions (h1)-(h2) and (g1)-(g2) are satisfied. Let \( u_0 \in C^1(\Omega) \) satisfying

\[
u_0(x) \geq \eta d(x, \partial\Omega)
\]

be a local minimizer of \( I \) in \( C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}) \) topology; that is,

\[ \exists \epsilon > 0 \text{ such that } u \in C^1(\overline{\Omega}) \cap C_0(\overline{\Omega}), \|u - u_0\|_{C^1(\overline{\Omega})} < \epsilon \Rightarrow I(u_0) \leq I(u). \]

Then, \( u_0 \) is a local minimum of \( I \) in \( W^{1,N}(\Omega) \) also.

This result is useful to prove multiplicity of positive solutions to critical growth problems with co-normal boundary conditions. From Lemma A.2 in Appendix A, we remark that the conditions on \( u_0 \) in the above theorem implies that \( u_0 \) satisfies in the distributions sense the Euler-Lagrange equation associated to \( I \) that is

\[
\{P\} \begin{cases} -\Delta_N u + |u|^{N-2} u = g(u) + f(x, u) & u > 0 \quad \text{in } \Omega, \\ |\nabla u|^{N-2} \frac{\partial u}{\partial \nu} = |u|^{q-1} u & \text{on } \partial\Omega. \end{cases}
\]

It means that \( u_0 \in W^{1,N}(\Omega) \) is a weak solution to \( \{P\} \), i.e. satisfies \( \inf_K u_0 > 0 \) over every compact set \( K \subset \Omega \) and

\[
\int_{\Omega} |\nabla u_0|^{N-2} \nabla u_0 \cdot \nabla \phi \, dx + \int_{\Omega} |u_0|^{N-2} u_0 \phi \, dx = \int_{\Omega} g(u_0) \phi \, dx
+ \int_{\Omega} f(x, u_0) \phi \, dx + \int_{\partial\Omega} |u_0|^q \phi \, dx
\]  

(1.3)
for all $\phi \in C^\infty_c(\Omega)$. As usual, $C^\infty_c(\Omega)$ denotes the space of all $C^\infty$ functions $\phi : \Omega \to \mathbb{R}$ with compact support. We highlight that the condition (1.2) is natural. Indeed from Lemma A.4 in the Appendix A, any weak solution to (P) satisfies (1.2) for some $\eta > 0$. In particular, $u_0 \geq \bar{u}$ where $\bar{u}$ is the unique weak solution to the "pure singular" problem (PS):

\begin{equation}
\begin{aligned}
-\Delta_N u + |u|^{N-2}u &= g(u) \quad u > 0 \quad \text{in } \Omega, \\
|\nabla u|^{N-2} \frac{\partial u}{\partial \nu} &= |u|^{q-1}u \quad \text{on } \partial \Omega.
\end{aligned}
\end{equation}

given by Lemma A.3.

For proving Theorem 1.1, we will need uniform $L^\infty$-estimates for a family of solutions to $(P_{\epsilon})$. More precisely, we have the following result:

**Theorem 1.2.** Let $\{u_{\epsilon}\}_{\epsilon \in (0,1)}$ be a family of solutions to $(P_{\epsilon})$, where $u_0$ satisfies (1.2) and solves $(P)$. Let $\theta > 1$ be such that

$$
\sup_{\epsilon \in (0,1)} \left( ||f(x,u_{\epsilon})||_{L^\theta} + ||u_{\epsilon}||_{W^{1,N}(\Omega)} \right) < \infty.
$$

Then,

$$
\sup_{\epsilon \in (0,1)} ||u_{\epsilon}||_{L^\infty(\Omega)} < \infty.
$$

An important ingredient in our proof is the following Trudinger-Moser type inequality (see [5] and [6]):

$$
\sup_{||u||_{1,N} \leq 1} \int_{\Omega} \exp^{\alpha_N ||u||_{1,N}^{\frac{\alpha_N}{N-1}}} \, dx < \infty,
$$

(1.4)

where $\alpha_N = Nw_N^{\frac{1}{N-1}}$, $w_N = \text{volume of } S^{N-1}$. It follows immediately from (1.4) that the embedding $W^{1,N}(\Omega) \ni u \mapsto \exp^{\alpha_N} \in L^1(\Omega)$ is compact for all $\beta = (0,\frac{N}{N-1})$ and is continuous for $\beta = \frac{N}{N-1}$. The fact that this imbedding is not compact for $\beta = \frac{N}{N-1}$ can be shown using a sequence of Moser functions that are suitable truncations and dilations of the fundamental solution of $-\Delta_N$ on $W^{1,N}(\Omega)$. Thus the growth given by the map $t \mapsto \exp^{\alpha_N(t)}$ represents the critical growth for functions $u \in W^{1,N}(\Omega)$.

Theorem 1.1 was proved first in [1] for the case of critical growth functionals $I : H^1_0(\Omega) \to \mathbb{R}$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$, and later for critical growth functionals $I : W^{1,p}_0(\Omega) \to \mathbb{R}$, $1 < p < N$, $\Omega \subset \mathbb{R}^N$, $N \geq 3$ in [2] and for critical and singular functionals in [9]. The sub-critical case of $p(x)$-Laplacian is studied in [7]. In contrast, in our approach we use the equations involving only one p-Laplacian.

Using constraints based on $L^p$ norms rather than Sobolev norms as in [2], the equations for which uniform estimates are required can be simplified to a standard

\[ \text{W}^{1,N} \text{ Versus } C^1 \text{ Local Minimizers} \]
We now consider the following two cases:

Moreover, since $q, q^\prime$ are conjugate. Therefore, from (2.2) and since $W^{1,N}(\Omega) \hookrightarrow L^{1-\delta}(\Omega)$, and the trace imbedding $W^{1,N}(\Omega) \hookrightarrow L^N(\partial \Omega)$ we get the result. Moreover, we note that $S_\varepsilon$ is a convex set. Using Trudinger-Moser and trace embeddings we see that $S_\varepsilon$ is also a closed set in $W^{1,N}(\Omega)$ which implies that $S_\varepsilon$ is weakly closed in $W^{1,N}(\Omega)$, the facts that, $I$ is weakly lows semicontinuous in $W^{1,N}(\Omega)$, it follows that for $\varepsilon \in (0,1)$ $I_\varepsilon$ is achieved on some $u_\varepsilon \in S_\varepsilon$, that is

$$I(u_\varepsilon) = I_\varepsilon, \text{ and } I(u_\varepsilon) < I(u_0) \quad \forall \varepsilon \in (0,1).$$

Moreover, since $I(u_\varepsilon^+) \leq I(u_\varepsilon)$ and $u_\varepsilon^+ \in S_\varepsilon$, we may assume that $u_\varepsilon \geq 0$.

We now consider the following two cases:

1. Let $\rho(u_\varepsilon) < \varepsilon$.

Then $u_\varepsilon$ is also a local minimizer of $I$ in $W^{1,N}(\Omega)$. We now show that $I$ admits a Gâteaux-derivatives on $u_\varepsilon$ to derive that $u_\varepsilon$ satisfies the Euler-Lagrange equation associated with $I$. For this, according to Lemma A.2, in Appendix A, we need to prove that $\exists \tilde{\eta} > 0$ such that $u_\varepsilon \geq \tilde{\eta} dist(x, \partial \Omega)$ or equivalently

$$\exists \eta > 0 \text{ such that } u_\varepsilon \geq \eta \varphi_1;$$

$$2. \text{ Proof of Theorem 1.1}$$

We adapt the arguments in [8]. Assume that the conclusion of Theorem 1.1 is not true. Let $k : \mathbb{R} \to \mathbb{R}$ be defined as $k(s) = s^{p+1}e^{cs^{\alpha}}$ for $p > 1$ and for some constant $c > b$. We define the following constraint for each $\varepsilon > 0$:

$$S_\varepsilon \defeq \{ u \in W^{1,N}(\Omega) : \rho(u) \defeq \| k(u) \|_{L^1(\Omega)} + \| u \|_{L^{N,1}(\partial \Omega)}^{a+1} \leq \varepsilon \}, \quad a \defeq \max \{ p, q \}.$$  

(2.1)

We consider the following constraint minimization problem:

$$I_\varepsilon = \inf_{u \in S_\varepsilon} I(u).$$

Firstly, we have that $I_\varepsilon > -\infty$. Indeed, since $F(x, s) = O(s^{\frac{N}{N-1}})$ as $s \to 0$ (by (h1)) and from (1.4), we get for some constant $C, K$

$$\frac{1}{N} ||u||^N - \int_\Omega F(x, u) dx \geq \frac{1}{N} ||u||^N - K \int_\Omega u^{\frac{N}{N-1}} e^{b(1+\varepsilon)u^{\frac{\alpha}{1+\alpha}}} dx$$

$$\geq \frac{1}{N} ||u||^N - K \left( \int_\Omega e^{b(1+\varepsilon)u^{\frac{\alpha}{1+\alpha}}} dx \right)^{\frac{1}{\alpha} - \frac{1}{q}} \left( \int_\Omega u^{\frac{\alpha}{1+\alpha}q^\prime} dx \right)^{\frac{1}{q}}$$

$$\geq \frac{1}{N} ||u||^N - KC ||u||^{\frac{N}{N-1}q'}$$

$$\geq \frac{1}{N} ||u||^N - KC ||u||^N,$$

(2.2)

where $q, q'$ are conjugate. Therefore, from (2.2) and since $W^{1,N}(\Omega) \hookrightarrow L^{1-\delta}(\Omega)$, and the trace imbedding $W^{1,N}(\Omega) \hookrightarrow L^N(\partial \Omega)$ we get the result. Moreover, we note that $S_\varepsilon$ is a convex set. Using Trudinger-Moser and trace embeddings we see that $S_\varepsilon$ is also a closed set in $W^{1,N}(\Omega)$ which implies that $S_\varepsilon$ is weakly closed in $W^{1,N}(\Omega)$, the facts that, $I$ is weakly lows semicontinuous in $W^{1,N}(\Omega)$, it follows that for $\varepsilon \in (0,1)$ $I_\varepsilon$ is achieved on some $u_\varepsilon \in S_\varepsilon$, that is

$$I(u_\varepsilon) = I_\varepsilon, \text{ and } I(u_\varepsilon) < I(u_0) \quad \forall \varepsilon \in (0,1).$$

(2.3)

Moreover, since $I(u_{\varepsilon}^+) \leq I(u_\varepsilon)$ and $u_{\varepsilon}^+ \in S_\varepsilon$, we may assume that $u_\varepsilon \geq 0$.

We now consider the following two cases:

1. Let $\rho(u_\varepsilon) < \varepsilon$.

Then $u_\varepsilon$ is also a local minimizer of $I$ in $W^{1,N}(\Omega)$. We now show that $I$ admits a Gâteaux-derivatives on $u_\varepsilon$ to derive that $u_\varepsilon$ satisfies the Euler-Lagrange equation associated with $I$. For this, according to Lemma A.2, in Appendix A, we need to prove that $\exists \tilde{\eta} > 0$ such that $u_\varepsilon \geq \tilde{\eta} dist(x, \partial \Omega)$ or equivalently

$$\exists \eta > 0 \text{ such that } u_\varepsilon \geq \eta \varphi_1;$$

(2.4)
[φ₁] is the eigenfunction corresponding to the principal eigenvalue of the problem

\[
\begin{align*}
-\Delta_N u + u^{N-1} &= 0, \quad u > 0 \text{ in } \Omega, \\
|\nabla u|^{N-2} \frac{\partial u}{\partial \nu} &= \lambda u^{N-1} \text{ on } \partial \Omega.
\end{align*}
\]

To prove (2.4), we argue by contradiction: \(\forall \eta > 0\) let \(\Omega_\eta = \text{Supp}\{(\eta \varphi_1 - u_\epsilon)^+\}\) and suppose that \(\Omega_\eta\) has a non zero measure. Letting 
\[u_\eta = (\eta \varphi_1 - u_\epsilon)^+\] and for \(0 < t \leq 1\) set \(\xi(t) = I(u_\epsilon + tu_\eta)\). Then, there exists \(c(t)\) satisfying \(c(t) > \eta t\) such that \(\inf \frac{u_\epsilon + tu_\eta}{\varphi_1} \geq c(t)\) for \(t > 0\). Then, from Lemma A.3 \(\xi\) is differentiable for \(0 < t \leq 1\) and \(\xi'(t) = (I'(u_\epsilon + tu_\eta), u_\eta)\). Thus, 
\[
\xi'(t) = \int_\Omega |\nabla (u_\epsilon + tu_\eta)|^{N-2} \nabla (u_\epsilon + tu_\eta) \nabla u_\eta + \int_\Omega |u_\epsilon + tu_\eta|^{N-2} u_\eta \\
- \int_\Omega g(u_\epsilon + tu_\eta)u_\eta - \int_\Omega f(x, u_\epsilon + tu_\eta)u_\eta \\
- \int_\Omega |u_\epsilon + tu_\eta|^{q-1}(u_\epsilon + tu_\eta)u_\eta.
\]

From (h1) and (g2), we see that 
\[
\xi'(1) = \int_\Omega |\nabla \eta \varphi_1|^{N-2} \nabla (\eta \varphi_1) \nabla u_\eta + \int_\Omega |\eta \varphi_1|^{N-2} u_\eta \\
- \int_\Omega g(\eta \varphi_1)u_\eta - \int_\Omega f(x, \eta \varphi_1)u_\eta - \int_\Omega |\eta \varphi_1|^{q-1}(\eta \varphi_1)u_\eta < 0.
\]

for \(\eta > 0\) small enough. Now, since \(g(s) + f(x, s)\) is non increasing for \(0 < s\) small enough uniformly to \(x \in \Omega\) (by (h1), (g1)-(g2)) and from the monotonicity of the operator 
\(-\Delta_N u + |u|^{N-1} u\), we have that for \(0 < \eta\) small enough \(0 \leq \xi'(1) - \xi'(t)\). Therefore, from Taylor’s expansion and since \(\rho(u_\epsilon) \leq \epsilon\), there exists \(0 < \theta < 1\) such that

\[
0 \leq I(u_\epsilon + u_\eta) - I(u_\epsilon) = \langle I'(u_\epsilon + \theta u_\eta), u_\eta\rangle = \xi'(\theta).
\]

Letting \(t = \theta\) we have \(\xi'(\theta) \leq \xi'(1) < 0\). We obtain a contradiction with (2.5) and then \(u_\epsilon \geq \eta \varphi_1\) for some \(\eta > 0\) (which depends a priori on \(\epsilon\)). Since \(u_\epsilon\) is a local minimizer of \(I\), and \(I\) is Gâteaux differentiable in \(u_\epsilon\), we get \(I'(u_\epsilon)\) is defined and \(I'(u_\epsilon) = 0\). Recalling that \(u\) is the solution to (PS) given by Lemma A.4 and from the weak comparison principle, we have that \(\eta \varphi_1 \leq u \leq u_\epsilon\) for some \(\eta > 0\) (independent of \(\epsilon\)). Since \(u_\epsilon \in S_\epsilon\) and from the fact that \(u_\epsilon\) satisfies (P), we get that \(\{u_\epsilon\}_{\epsilon \geq 0}\) is uniformly bounded in \(W^{1,N}(\Omega)\). Now, using Theorem 1.2 and Theorem B.1 in [13], we get 

\[
|u_\epsilon|_{C^{1,N}(\Omega)} \leq C \text{ for some } \alpha \in (0, 1)
\]
and as $\epsilon \to 0^+$

$$u_\epsilon \to u_0$$

in $C^1(\overline{\Omega})$

which contradicts the fact that $u_0$ is a local minimizer in $C^1(\overline{\Omega}) \cap C_0(\overline{\Omega})$.

Now, we deal with the second case:

2. $\rho(u_\epsilon) = \epsilon$:

We again show that $u_\epsilon \geq \eta \varphi_1$ in $\Omega$ for some $\eta > 0$. Taking $u_\eta = (\eta \varphi - u_\epsilon)^+$, $\xi(t) = I(u_\epsilon + t\eta)$, we obtain as above that $\xi'(t) \leq \xi'(1) < 0$ for $0 < t < 1$ and $0 < \eta$ small enough.

Then $\xi(t) = I(u_\epsilon + t\eta)$ is decreasing. This implies that $I(u_\epsilon) > I(u_\epsilon + t\eta)$ for $t > 0$ and using (1.2)

$$\rho(u_\epsilon) = \rho(u_\epsilon + t\eta) < \rho(u_\epsilon) = \epsilon.$$

This yields a contradiction with the fact that $u_\epsilon$ is a global minimizer of $I$ on $S_\epsilon$.

In this case, using Lemma A.3 and from the Lagrange multiplier rule we have

$$I'(u_\epsilon) = \mu_\epsilon \rho'(u_\epsilon),$$

for some $\mu_\epsilon \in \mathbb{R}$. (2.7)

We first show that $\mu_\epsilon \leq 0$. We argue by contradiction. Suppose that $\mu_\epsilon > 0$, then there exists $\varphi \in W^{1,N}(\Omega)$ such that

$$\langle I'(u_\epsilon), \varphi \rangle < 0$$

and then for $t$ small we have

$$\begin{cases} I(u_\epsilon + t\varphi) < I(u_\epsilon), \\ \rho(u_\epsilon + t\varphi) < \rho(u_\epsilon) \leq \epsilon. \end{cases}$$

(2.8)

This contradicts the fact that $u_\epsilon$ is a global minimizer of $I$ in $S_\epsilon$.

We deal now with two following cases:

case (i): $\inf_{\epsilon \in (0,1)} \mu_\epsilon \overset{\text{def}}{=} l > -\infty$. In this case, we write (2.7) in its P.D.E form as (with $K(s) = \rho'(s)$).

$$(P_\epsilon) \quad \begin{cases} -\Delta_N u_\epsilon + u_\epsilon^{N-1} = g(u_\epsilon) + f(x, u_\epsilon) + \mu_\epsilon K(u_\epsilon), & u_\epsilon > 0 \quad \text{in } \Omega, \\ |\nabla u_\epsilon|^{N-2} \frac{\partial u_\epsilon}{\partial \nu} = u_\epsilon^{q-1} u_\epsilon + \mu_\epsilon |u_\epsilon|^{a-1} u_\epsilon \quad \text{on } \partial \Omega. \end{cases}$$

It is easy from the weak comparison principle to show that $\eta \varphi_1 \leq u_\epsilon$ with some $\eta > 0$, independent of $\epsilon > 0$. (Note that for $\eta$ small enough and for all $l \leq \mu_\epsilon \leq 0$, we have that $\eta \varphi_1$ is a strict subsolution to $P_\epsilon$.)

In this case, we show that (up to a subsequence) $u_\epsilon \to u_0$ in $W^{1,N}(\Omega)$. To see this, we define a new functional $J_\epsilon : W^{1,N}(\Omega) \to \mathbb{R}$ by

$$J_\epsilon(u) \overset{\text{def}}{=} I(u) - \mu_\epsilon \rho(u), \quad u \in W^{1,N}(\Omega), \quad \epsilon \in (0,1).$$

(2.9)
Then, we see that using (2.7), \( J'_\epsilon(u_\epsilon) = 0 \), \( \epsilon \in (0, 1) \). Since \( \{ J(u_\epsilon) \}_{\epsilon \in (0, 1)} \) is a bounded sequence (thanks to (2.2) and (2.3)) in \( \mathbb{R} \), we may choose a subsequence such that \( J_\epsilon(u_\epsilon) \to \tau \) as \( \epsilon \to 0 \). Now, using (h2) and Moser-Trudinger embedding, we get that

\[
\int_\Omega F(x, u_\epsilon) \, dx \to \int_\Omega F(x, u_0) \, dx.
\] (2.10)

Indeed, for \( q^* \) and \( q \) conjugate for some \( C_1, C_2 > 0 \) independent of \( u \in W^{1,N}(\Omega) \),

\[
\int_\Omega |F(x, u_\epsilon) - F(x, u_0)| \, dx \leq \int_\Omega e^{bu_\epsilon^N} |h(x, u_\epsilon) - h(x, u_0)| \, dx
\]

\[
+ \int_\Omega e^{bu_\epsilon^N} - e^{bu_0^N} |h(x, u_\epsilon)| \, dx
\]

\[
\leq o(1) + \left( \int_\Omega e^{bu_\epsilon^N} - e^{bu_0^N} \right)^{\frac{q^*}{q^* - 1}} \left( \int_\Omega C e^{bu_\epsilon^N} \, dx \right) \left( \int_\Omega e^{bq\epsilon u_\epsilon^N} \, dx \right)^{\frac{1}{q^* - 1}}.
\] (2.11)

The last quantity in (2.11) is bounded from the Moser-Trudinger inequality (1.4) and \( \epsilon \) small enough whereas

\[
\int_\Omega \left| e^{bu_\epsilon^N} - e^{bu_0^N} \right| \leq \int_{u_\epsilon \leq A} \left| e^{bu_\epsilon^N} - e^{bu_0^N} \right| q^*
\]

\[
+ K \left( \int_{u_\epsilon > A} e^{bq\epsilon u_\epsilon^N} \, dx + \int_{u_\epsilon > A} e^{bq u_0^N} \, dx \right)
\]

\[
= I + II.
\] (2.12)

\( I \) in (2.12) goes to 0 when \( \epsilon \to 0 \) by dominated convergence. \( II \) can be estimated as

\[
II \leq Ke^{-bA} \int_{u_\epsilon > A} e^{b(q^* + 1)u_\epsilon^N} \, dx + K \int_{u_\epsilon > A} e^{bq u_0^N} \, dx
\] (2.13)

From (2.11), (2.12), (2.13) and taking \( q^* \) such that \( (q^* + 1)r_0^N \leq 1 \) and letting \( A \to \infty \), (2.10) follows.

On the other hand, using the uniform estimate \( \eta \phi_1 \leq u_\epsilon \leq k \phi_1 \), we have

\[
\int_\Omega g(u_\epsilon) \, dx \to \int_\Omega g(u_0) \, dx \quad \text{when} \ \epsilon \to 0.
\] (2.14)

Then, since \( u_\epsilon \to u_0 \) in \( W^{1,N}(\Omega) \), by Fatou’s Lemma \( I(u_0) \leq \tau \). Since \( \tau = \lim_{\epsilon \to 0} J_\epsilon(u_\epsilon) \leq I(u_0) \) (from (2.3)), we obtain that \( \tau = I(u_0) \). From (2.10)-(2.14), and the fact that \( \int_{\partial \Omega} |u_\epsilon|^{q+1} \to \int_{\partial \Omega} |u_0|^{q+1} \) we obtain that \( ||u_\epsilon||_{W^{1,N}(\Omega)} \to 0 \) as claimed before.
Hence, using the Trudinger-Moser type inequality in (1.4) we can apply Theorem 1.2 to conclude that \( \sup_{x \in (0, 1)} \|v_{\epsilon}\|_{L^\infty(\Omega)} \leq C \). Using Lemma A.6 in [13], we deduce that \( v_{\epsilon} \leq k \phi_1 \) for some \( k > 0 \) independent of \( \epsilon \). From the uniform estimate \( \eta \phi_1 \leq v_{\epsilon} \leq k \phi_1 \), we can apply Theorem B.1 in [13] and get \( |v_{\epsilon}|_{C^{1, \alpha}(\overline{\Omega})} \leq C \) for some constant \( C > 0 \) independent of \( \epsilon \). Then we conclude as above.

Let us consider the case (ii): \( \inf_{\epsilon \in (0, 1)} \mu_\epsilon = -\infty \). From above, we can assume that \( \mu_\epsilon \leq -1 \) for \( 0 < \epsilon \) small enough. As above, we have that \( v_{\epsilon} \geq \eta \phi_1 \) for \( \eta > 0 \) small enough and independent of \( \epsilon \). Furthermore, since \( k \) is odd, we can find a number \( M > 0 \) independent of \( \epsilon > 0 \) and \( x \in \overline{\Omega} \), such that \( (g(s) + f(x, s) + \mu_\epsilon s) \) and \( (|s|^{q+1} + \mu_\epsilon |s|^\beta \) for all \( |s| \geq M \). Then, from the weak comparison principle we have that \( v_{\epsilon} \leq M \) for \( \epsilon > 0 \) small enough. From Lemma A.2, since \( u_0 \in W^{1, N}(\Omega) \) satisfies (1.2) and is a \( C^1 \) local minimizer, \( u_0 \) is a weak solution to \( (P) \), i.e. satisfies \( \epsilon \inf_{K} u_0 > 0 \) over every compact set \( K \subset \Omega \) and

\[
\int_\Omega |\nabla u_0|^{N-2} \nabla u_0 \nabla \phi \, dx + \int_\Omega |u_0|^{N-2} \phi \, dx - \int_\Omega g(u_0) \phi \, dx - \int_\Omega f(x, u_0) \phi \, dx - \int_\Omega |u_0|^q u_0 \phi \, dx
\]

for all \( \phi \in C_c^\infty(\Omega) \). From Lemma A.3, for every function \( w \in W^{1, N}(\Omega) \), \( u_0 \) satisfies

\[
\int_\Omega |\nabla u_0|^{N-2} \nabla u_0 \nabla w \, dx + \int_\Omega |u_0|^{N-2} w \, dx - \int_\Omega g(u_0) w \, dx - \int_\Omega f(x, u_0) w \, dx - \int_\Omega |u_0|^q u_0 w \, dx.
\]

Similarly,

\[
\int_\Omega |\nabla u_\epsilon|^{N-2} \nabla u_\epsilon \nabla w \, dx + \int_\Omega |u_\epsilon|^{N-2} w \, dx - \int_\Omega g(u_\epsilon) w \, dx - \int_\Omega f(x, u_\epsilon) w \, dx - \int_\Omega |u_\epsilon|^q u_\epsilon w \, dx.
\]

Now, subtracting the above relations with \( w = (u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}, \) with \( \beta \geq 1 \), as a test function in \( (P_\epsilon) \), integrate by parts and use the fact that \( u \mapsto -\Delta_N u + |u|^{N-1} u \) is a monotone operator to obtain,

\[
- \mu_\epsilon \left[ \int_\Omega k(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1}(u_\epsilon - u_0) \, dx + \int_{\partial \Omega} |u_\epsilon - u_0|^\alpha \beta \, d\sigma \right]
\leq \int_\Omega (g(u_\epsilon) - g(u_0))(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1} \, dx
+ \int_\Omega (f(x, u_\epsilon) - f(x, u_0))(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1} \, dx
+ \int_{\partial \Omega} (|u_\epsilon|^q u_\epsilon - |u_0|^q u_0)(u_\epsilon - u_0)|u_\epsilon - u_0|^{\beta-1} \, d\sigma.
\]
Using the bounds of $u_\epsilon, u_0$ we get

$$-\mu_\epsilon \left[ \int_\Omega k(u_\epsilon - u_0) |u_\epsilon - u_0|^{\beta - 1}(u_\epsilon - u_0) \, dx + \int_{\partial \Omega} |u_\epsilon - u_0|^{\alpha + \beta} \, dx \right]$$

$$\leq C \left[ \int_\Omega |u_\epsilon - u_0|^\beta \, dx + \int_{\partial \Omega} |u_\epsilon - u_0|^\beta \, dx \right]$$

where $C$ does not depend on $\beta$ and $\epsilon$. Now, using the inequality $k(s)s \geq c|s|^{p+1}$ for all $s \in \mathbb{R}, \alpha \geq p$ and the Hölder inequality we obtain

$$-\mu_\epsilon \left[ \int_\Omega |u_\epsilon - u_0|^{p+\beta} \, dx + \int_{\partial \Omega} |u_\epsilon - u_0|^{p+\beta} \, dx \right]$$

$$\leq C(|\Omega|) \left[ \int_\Omega |u_\epsilon - u_0|^{p+\beta} \, dx + \int_{\partial \Omega} |u_\epsilon - u_0|^{p+\beta} \, dx \right]^\frac{1}{p}.$$

Therefore, for any $\beta > 1$

$$-\mu_\epsilon \left[ \|u_\epsilon - u_0\|_{L^{p+\beta}(\Omega)}^p + \|u_\epsilon - u_0\|_{L^{p+\beta}(\partial \Omega)}^p \right] \leq C(|\Omega|). \tag{2.15}$$

Passing to the limit in (2.15) $\beta \to +\infty$ we get

$$\mu_\epsilon \left[ \|u_\epsilon - u_0\|_{L^{\infty}(\Omega)}^\infty + \|v_\epsilon - u_0\|_{L^{\infty}(\partial \Omega)}^\infty \right] \leq C. \tag{2.16}$$

Thus, using (2.16), the uniform $L^\infty$ bounds for $\{u_\epsilon\}_{\epsilon \in (0,1)}$ in $\Omega$ as well as $\partial \Omega$ and the fact that $k(s)s^{-p}$ is function bounded below in $\mathbb{R}$, we get that the right-hand side terms in $(P_\epsilon)$ are uniformly bounded in $L^\infty(\Omega)$ and in $L^\infty(\partial \Omega)$ from which as in the first case, we obtain that $u_\epsilon, (0 < \epsilon \leq 1)$ is bounded in $C^{1,\alpha}(\Omega)$ independently of $\epsilon$ and we conclude as above. \hfill $\square$

### 3. Uniform estimates

Consider the problems

$$(P_\epsilon) \quad \begin{cases} -\Delta_N u_\epsilon + u_\epsilon^{N-1} = g(u_\epsilon) + f(x, u_\epsilon) + \mu_\epsilon k(u_\epsilon), & u_\epsilon > 0 \quad \text{in } \Omega, \\ |\nabla u_\epsilon|^{N-2} \frac{\partial u_\epsilon}{\partial \nu} = u_\epsilon g^{-1} \mu_\epsilon + \mu_\epsilon |u_\epsilon|^{\alpha-1} u_\epsilon & \text{on } \partial \Omega, \end{cases}$$

where $\nu$ is the unit normal on $\partial \Omega$. In this section, we obtain the uniform $L^\infty$ estimates for a family of solutions to $(P_\epsilon)$. More precisely, we prove Theorem 1.2.

**Proof:** For $k > 0$ we consider the test function

$$T_k(s) \overset{\text{def}}{=} (s + k)\chi_{(-\infty, -k]} + (s - k)\chi_{[k, \infty)} \tag{3.1}$$

and define the two following sets

$$\Omega_k = \{ x \in \Omega_k \mid |u_\epsilon| \geq k \}, \quad \partial \Omega_k = \{ x \in \partial \Omega_k \mid |u_\epsilon| \geq k \}.$$
Using $T_k(u_e)$ as test function in (P$_k$), and the fact that $\mu_e \leq 0$ we get
\[
\int_{\Omega} |\nabla u_e|^{N-2}\nabla u_e \cdot \nabla (T_k(u_e)) + \int_{\Omega} |u_e|^{N-2} u_e T_k(u_e)
\]
\[
\leq \int_{\Omega} g(u_e) T_k(u_e) + \int_{\Omega} f(x, u_e) T_k(u_e) + \int_{\Omega} |u_e|^{q-1} u_e T_k(u_e).
\]  
(3.2)

The term on the right-hand side of (3.2) may be estimated using the Hölder inequality
\[
\int_{\Omega} g(u_e) T_k(u_e) + \int_{\Omega} f(x, u_e) T_k(u_e) + \int_{\Omega} |u_e|^{q-1} u_e T_k(u_e).
\]
\[
\leq (\int_{\Omega} (g(u_e))^\theta \big( \int_{\Omega} |T_k(u_e)|^{\theta} \big)^\frac{1}{\theta} |\Omega_k|^{1-\frac{1}{\theta} - \frac{\theta}{r}}
\]
\[
+ (\int_{\Omega} (f(u_e))^\theta \big( \int_{\Omega} |T_k(u_e)|^{\theta} \big)^\frac{1}{\theta} |\Omega_k|^{1-\frac{1}{\theta} - \frac{\theta}{r}}
\]
\[
+ (\int_{\partial \Omega} (|u_e|^q)^\theta \big( \int_{\partial \Omega} |T_k(u_e)|^{\theta} \big)^\frac{1}{\theta} |\partial \Omega_k|^{1-\frac{1}{\theta} - \frac{\theta}{r}} + |\partial \Omega_k|^{\frac{\theta}{r} - \frac{\theta}{1 - \frac{\theta}{r}}},
\]  
(3.3)

where, $\eta = \frac{N+1}{r}$ and $r = \theta \eta$, here, $|C|$ denotes the Lebesgue measure of the measurable set $C$. Now, using Sobolev and trace imbeddings we can estimate from below the term on the left-hand side of (3.2) as follows:
\[
\int_{\Omega} |\nabla u_e|^{N-2}\nabla u_e \cdot \nabla (T_k(u_e)) + \int_{\Omega} |u_e|^{N-2} u_e T_k(u_e)
\]
\[
\geq C \left( \int_{\Omega} |\nabla (T_k(u_e))|^{N} + \int_{\Omega} |T_k(u_e)|^{N} \right)
\]
\[
\geq C \left( \int_{\Omega} |T_k(u_e)|^{r} + \int_{\partial \Omega} |T_k(u_e)|^{r} \right)^{\frac{N}{r}}.
\]  
(3.4)

Substituting (3.3) and (3.4) in (3.2), we get
\[
\int_{\Omega} |T_k(u_e)|^{r} + \int_{\partial \Omega} |T_k(u_e)|^{r} \leq (|\partial \Omega_k| + |\partial \Omega_k|) \rightarrow \infty
\]  
(3.5)

Notice that for $0 < k < h, \Omega(h) \subset \Omega(k)$ since $T_k(s) = ([s]-k)(1-\chi_{[-k,k]}(s)) \forall s \in \mathbb{R}$ and then
\[
|\Omega_h|(h-k)^r = \int_{\Omega_h} (h-k)^r \leq \int_{\Omega_h} |u_e| - k)^r \leq \int_{\Omega_h} (|u_e| - k)^r = \int_{\Omega} |T_k(u_e)|^{r}.
\]  
(3.6)

In the same way we have
\[
|\partial \Omega_h|(h-k)^r \leq \int_{\partial \Omega} |T_k(u_e)|^{r}.
\]  
(3.7)
Substituting (3.6) and (3.7) in (3.5), we obtain
\[
\Phi(h) \leq C(h - k)^{-r}(\Phi(k))^{\frac{N}{N-1}} \quad 0 < k < h
\] (3.8)
where \(\Phi(k) \equiv |\Omega_k| + |\partial \Omega| \ k > 0\).

Now we have the following

**Claim:** Assume \(\Phi : [0, +\infty) \to [0, +\infty)\) is a non-increasing function such that if \(h > k > k_0\)
\[
\Phi(h) \leq C(h - k)^{-r}(\Phi(k))^{\frac{N}{N-1}} \quad 0 < k < h.
\] Then \(\Phi(d + k_0) = 0\) where \(d \equiv 2^N C^\frac{1}{r} \Phi(k_0)^{\frac{1}{N-1}}\).

By the Claim we get that \(\Phi(d) = |\Omega_d| + |\partial \Omega_d| = 0\) namely
\[
\sup_{c \in (0,1)} ||u_c||_{L^\infty} \leq d.
\]
To finish we need to prove the Claim.

**Proof of the claim:** Given \(d\) as above, define the sequence \(\{k_n\}\) by \(k_0 = 0\) and \(k_n = k_{n-1} + \frac{d}{2^n}\) for \(n = 1, 2, \ldots\). By recurrence we have that
\[
\Phi(k_n) \leq \frac{\Phi(k_0)}{2^{n(N-1)}} \to 0 \text{ as } n \to \infty.
\]
Then
\[
0 \leq \Phi(d + k_0) \leq \lim_{n \to \infty} \Phi(k_n) = 0.
\]
This gives the proof of the Claim and the proof of Theorem 1.2. \(\square\)

4. appendix

We start with an important technical tool which enables us to estimate the singularity in the Gâteaux derivative of the energy functional \(I : W^{1,N}(\Omega) \to \mathbb{R}\) defined in (1.1).

**Lemma 4.1.** Let \(0 < \delta < 1\). Then there exists a constant \(C_\delta > 0\) such that the inequality
\[
\int_0^1 |a + sb|^{-\delta} \, ds \leq C_\delta \left( \max_{0 \leq s \leq 1} |a + sb| \right)^{-\delta} \quad (4.1)
\]
holds true for all \(a, b \in \mathbb{R}^N\) with \(|a| + |b| > 0\).

An elementary proof of this lemma can be found in Takáč [15, Lemma A.1, p. 233].

We continue by showing the Gâteaux-differentiability of the energy functional \(I\) at a point \(u \in W_0^{1,N}(\Omega)\) satisfying \(u \geq \varepsilon \varphi_1\) in \(\Omega\) with a constant \(\varepsilon > 0\).
Lemma 4.2. Let the assumptions (h1)-(h2) and (g1)-(g2) be satisfied. Assume that \( u, v \in W^{1,N}(\Omega) \) and \( u \) satisfies \( u \geq \varepsilon \varphi_1 \) in \( \Omega \) with a constant \( \varepsilon > 0 \). Then we have

\[
\lim_{t \to 0} \frac{1}{t} (I(u + tv) - I(u)) = \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla v \, dx - \int_{\Omega} |u|^{N-1} v \, dx \\
- \int_{\Omega} g(u) v \, dx - \int_{\Omega} f(x, u) v \, dx - \int_{\partial \Omega} |u|^q-1 uv \, dx \tag{4.2}
\]

Proof: We show the result only for the singular term \( \int_{\Omega} g(u) v \, dx \); the other two terms are treated in a standard way. So let \( H(u) = \int_{\Omega} G(u(x)) \, dx \) for \( u \in W^{1,N}(\Omega) \).

For \( \xi \in \mathbb{R} \setminus \{0\} \) we define

\[
z(\xi) = \frac{d}{d\xi} G(\xi^+) = \begin{cases} 
g(\xi) & \text{if } \xi > 0; \\
0 & \text{if } \xi < 0.
\end{cases}
\]

Consequently,

\[
\frac{1}{t} (H(u + tv) - H(u)) = \int_{\Omega} \left( \int_0^1 z(u + stv) \, ds \right) v \, dx. \tag{4.3}
\]

Notice that for almost every \( x \in \Omega \) we have \( u(x) > 0 \) and

\[
\int_0^1 z(u(x) + stv(x)) \, ds \longrightarrow z(u(x)) = g(u(x)) \quad \text{as } t \to 0.
\]

Moreover, the integral on the left-hand side (with nonnegative integrand) is dominated by

\[
\int_0^1 z(u(x) + stv(x)) \, ds \leq C \int_0^1 |u(x) + stv(x)|^{-\delta} \, ds \\
\leq C_\delta \left( \max_{0 \leq s \leq 1} |u(x) + stv(x)| \right)^{-\delta} \\
\leq C_{\delta, x} \varphi_1(x)^{-\delta}
\]

with constants \( C, C_{\delta, x} > 0 \) independent of \( x \in \Omega \). Here, we have used the estimate \((4.1)\) from Lemma 4.1 above. Finally, we have \( v \varphi_1^{-\delta} \in L^1(\Omega) \), by \( v \in W^{1,N}(\Omega) \) and Hardy’s inequality. That’s

\[
\left( \int_0^1 z(u(x) + stv(x)) \, ds \right) v \leq C_{\delta, \varepsilon} \varphi_1(x)^{-\delta} v < \infty
\]

Hence, we are allowed to invoke the Lebesgue dominated convergence theorem in \((4.3)\) from which the lemma follows by letting \( t \to 0 \). \(\square\)
Corollary 4.3. Let the assumptions (h1)-(h2) and (g1)-(g2) be satisfied. Then the energy functional \( I : W^{1,N}(\Omega) \to \mathbb{R} \) is Gâteaux-differentiable at every point \( u \in W^{1,N}(\Omega) \) that satisfies \( u \geq \varepsilon \varphi_1 \) in \( \Omega \) with a constant \( \varepsilon > 0 \). Its Gâteaux derivative \( I'(u) \) at \( u \) is given by
\[
\langle I'(u), v \rangle = \int_{\Omega} |\nabla u|^{N-2} \nabla u \nabla v \, dx - \int_{\Omega} |u|^{N-1} v \, dx - \int_{\Omega} g(u) v \, dx
- \int_{\Omega} f(x, u) v \, dx - \int_{\partial \Omega} |u|^{q-1} uv \, dx
\]
for \( v \in W^{1,N}(\Omega) \).

We continue by proving the \( C^1 \)-differentiability of the cut off energy functional \( T \) defined below:

Lemma 4.4. Let the assumptions (h1)-(h2) and (g1)-(g2) be satisfied, and \( w \in W^{1,N}(\Omega) \) such that \( w \geq \varepsilon \varphi_1 \) with some \( \varepsilon > 0 \).

Define \( \tilde{g}_\lambda : \Omega \to \mathbb{R} \) by
\[
\tilde{g}(s) = \begin{cases} 
g(s) & s \geq w(x), 
g(w(x)) & s < w(x). \end{cases}
\]
\( \tilde{f}_\lambda : \Omega \times \mathbb{R} \to \mathbb{R} \) by
\[
\tilde{f}_\lambda(x, s) = \begin{cases} 
f(x, s) & s \geq w(x), 
f(x, w(x)) & s < w(x), \end{cases}
\]
and \( \tilde{h}_\lambda : \partial \Omega \times \mathbb{R} \to \mathbb{R} \) by
\[
\tilde{h}_\lambda(x, s) = \begin{cases} 
|s|^q & s \geq w(x), 
|w(x)|^q & s < w(x). \end{cases}
\]
Let \( \tilde{G}_\lambda(s) = \int_0^s \tilde{g}(t) \, dt, \tilde{F}_\lambda(x, s) = \int_0^s \tilde{f}(x, t) \, dt \) and \( \tilde{H}_\lambda(x, s) = \int_0^s \tilde{h}(x, t) \, dt \). Consider the functional \( \tilde{I} : W^{1,N}(\Omega) \to \mathbb{R} \) defined by
\[
\tilde{I}(u) = \frac{1}{N} \int_{\Omega} (|\nabla u|^N + |u|^N) - \int_{\Omega} \tilde{G}(u) \, dx - \int_{\Omega} \tilde{F}(x, u) \, dx - \int_{\partial \Omega} \tilde{H}_\lambda(x, u) \, dx. \quad (4.5)
\]

We have that \( \tilde{I} \) belongs to \( C^1(W^{1,N}(\Omega), \mathbb{R}) \).

Proof:
As in Lemma 4.2, we concentrate on the singular term, the others being standard. Let
\[
\tilde{g}(s) = \begin{cases} 
g(s) & s \geq w(x), 
g(w(x)) & s < w(x). \end{cases}
\]
\[ \tilde{G}(s) = \int_0^s g(t) \, dt, \text{ and } S(u) = \int_{\Omega} G(u) \, dx. \]

Proceeding as in Lemma 4.2, we obtain that for all \( u \in W^{1,N}(\Omega) \), \( S(u) \) has a Gâteaux derivative \( S'(u) \) given by
\[
\langle S'(u), v \rangle = \int_{\Omega} g((\max\{u(x), w(x)\})v(x) \, dx.
\]

Let \( u_k \in W^{1,N}(\Omega) \), \( u_k \to u_0 \). Then
\[
|\langle S'(u_k) - S'(u_0), v \rangle| \leq 2C \int_{\Omega} w^{-\delta}|v| \, dx \leq 2C\epsilon^{-\delta} \int_{\Omega} \varphi_1^{-\delta}|v| \, dx
\]
for all \( v \in W^{1,N}(\Omega) \). Again, as in Lemma 4.2, we use Hardy’s inequality to deduce that \( \varphi_1^{-\delta} \in L^1(\Omega) \), so that by Lesbegue’s dominated convergence theorem we conclude that the Gâteaux derivative of \( S \) is continuous which implies that \( S \in C^1(W^{1,N}(\Omega), \mathbb{R}) \).

We give now the existence of a subsolution to (P):

**Lemma 4.5.** Assume assumptions (g1)?(g2). Then problem \((PS)\) possesses a weak solution in \( W^{1,N}(\Omega) \) in the sense of distributions. This solution, denoted by \( u \), is the unique global minimizer to the energy functional \( \tilde{E} \) given by
\[
\tilde{E} \overset{\text{def}}{=} \frac{1}{N} \left( \int_{\Omega} |\nabla u|^N \, dx - \int_{\Omega} |u|^N \, dx \right) - \int_{\Omega} G(u^+) \, dx - \frac{1}{q+1} \int_{\Omega} |u|^{q+1} \, dx
\]
for all \( u \in W^{1,N}(\Omega) \). In addition, \( u \) is the unique solution to \((PS)\) in \( \Omega \overset{\text{def}}{=} \{ u \in W^{1,N}(\Omega) \text{ such that } u \geq \eta \varphi_1 \text{ for some } \eta > 0 \} \).

**Proof:** First, by Hölder’s inequality and Sobolev embedding and trace embedding \( W^{1,N}(\Omega) \hookrightarrow L^q(\partial\Omega) \) we get for some \( C_2 > 0 \),
\[
\int_{\partial\Omega} |u|^{q+1} \leq C_2 \|u\|_{L^{q+1}(\partial\Omega)}^{q+1} \leq C_2 \|u\|^{q+1}.
\]

Thus, from (4.6) and owing to the Poincaré inequality, assumption \( (g2) \) and \( 0 < 1 - \delta < 1 < N < \infty \), the functional \( \tilde{E} \) is coercive.

Now, we prove that \( \tilde{E} \) is weakly lower semicontinuous. To do this it is sufficient to show that for \( u_j \to u \) weakly in \( W^{1,N}(\Omega) \) we have
\[
\int_{\Omega} g(u_j) \, dx \to \int_{\Omega} g(u) \, dx \text{ when } j \to +\infty.
\]
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and

$$\int_{\partial\Omega} |u_j|^{q-1} u_j \, dx \to \int_{\partial\Omega} |u|^{q-1} u \, dx \text{ when } j \to +\infty. \quad (4.8)$$

(4.7) follows from the definition of weak convergence and using assumption $(g2)$. Finally, (4.8) follows from the trace embedding. It follows that $\tilde{E}$ possesses a global minimizer $\tilde{u} \in W^{1,N}(\Omega)$. We have $\tilde{u} \neq 0$ owing to $\tilde{E}(0) = 0 > \tilde{E}(\epsilon \varphi_1)$ for $\epsilon > 0$ small enough.

Second, the polar decomposition $u = u^+ - u^-$ of any function $u \in W^{1,N}(\Omega)$ gives $\nabla u = \nabla u^+ - \nabla u^-$. Thus, if $\tilde{u}$ is a global minimizer for $\tilde{E}$, then so is its absolute value $|\tilde{u}|$, by $\tilde{E}(||\tilde{u}||) \leq \tilde{E}(\tilde{u})$ holds if and only if $\tilde{u}^- = 0$ a.e. in $\Omega$, that is, if and only if $\tilde{u} \geq 0$ a.e. in $\Omega$. Thus, any global minimizer $\tilde{u}$ for $\tilde{E}$, must satisfy $\tilde{u} \geq 0$ a.e. in $\Omega$. Equivalently, $\tilde{u} \in W^{1,N}(\Omega)_+$ where

$$W^{1,N}(\Omega)_+ \overset{\text{def}}{=} \{ u \in W^{1,N}(\Omega) : u \geq 0 \text{ a.e. in } \Omega \}$$

stands for the positive cone in $W^{1,N}(\Omega)$.

From the fact $-\Delta_N u - |u|^{N-1} u$ is a monotone operator in the cone $\Omega_+ \overset{\text{def}}{=} \{ u \in W^{1,N}(\Omega) \text{ such that } u \geq \eta \varphi_1 \text{ for some } \eta > 0 \}$ and the weak comparison principle, we conclude that $\tilde{E}$ has a unique global minimizer denoted by $\tilde{u}$ in $W^{1,N}(\Omega)$ with the property $\operatorname{essinf}_K \tilde{u} > 0$ for any compact set $K \subset \Omega$. $\tilde{u}$ is then the unique weak solution to $(PS)$ in $\Omega$ and satisfies (1.2).

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References


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