One Sided Generalized \((\sigma, \tau)\)-derivations on Rings

Evrim Guven

ABSTRACT: Let \( R \) be a prime ring with characteristic not 2 and \( \sigma, \tau, \lambda, \mu, \alpha, \beta \) be automorphisms of \( R \). Let \( h \) be a nonzero left (resp. right)-generalized \((\sigma, \tau)\)–derivation of \( R \) and \( I, J \) nonzero ideals of \( R \) and \( a \in R \). The main object in this article is to study the situations. (1) \( h(I) \subseteq C_{\lambda, \mu}(J) \) and \( ah(I) \subseteq C_{\lambda, \mu}(J) \); (2) \( h(I) \subseteq C_{\lambda, \mu}(J) \), \( h(I, a)_{\lambda, \mu} = 0 \) or \( h(I, a)_{\lambda, \mu} = 0 \); \( \forall \lambda, \mu \); \( x \in I, y \in J \). Every \((\sigma, \tau)\)–derivation of \( R \) is called a \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d \). The mapping defined by \( h(r) = [r, a]_{\sigma, \tau}, \forall r \in R \) is a right-generalized derivation associated with derivation \( d(r) = [r, \sigma(a)], \forall r \in R \) and left-generalized derivation associated with derivation \( d_1(r) = [r, \sigma(a)], \forall r \in R \). The mapping \( h(r) = (a, r)_{\sigma, \tau}, \forall r \in R \) is a left-generalized \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d_2(r) = [a, r]_{\sigma, \tau}, \forall r \in R \) and right-generalized \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d_2 \).

\textbf{Keywords:} \((\sigma, \tau)\)–Lie ideal, Prime ring, Commutativity.

\textbf{Contents}

\begin{tabular}{ll}
\hline
1 & Introduction \hspace{1cm} 41 \\
2 & Results \hspace{1cm} 42 \\
\hline
\end{tabular}

1. Introduction

Let \( R \) be an associative ring with center \( Z \). Recall that \( R \) is prime if \( aRb = 0 \) implies that \( a = 0 \) or \( b = 0 \). For any \( x, y \in R \) the symbol \([x, y]\) represents commutator \( xy - yx \) and the Jordan product \((x, y) = xy + yx \). Let \( \sigma \) and \( \tau \) be any two endomorphisms of \( R \). For any \( x, y \in R \) we set \([x, y]_{\sigma, \tau} = x\sigma(y) - \tau(y)x \) and \((x, y)_{\sigma, \tau} = x\sigma(y) + \tau(y)x \). Let \( h \) and \( d \) be additive mappings of \( R \). If \( d(xy) = d(x)y + xd(y), \forall x, y \in R \) then \( d \) is called a derivation of \( R \). If there exists a derivation \( d \) such that \( h(xy) = h(x)y + xd(y), \forall x, y \in R \) then \( h \) is called a generalized derivation of \( R \) (see [3]). If \( h(xy) = d(x)\sigma(y) + \tau(x)d(y), \forall x, y \in R \) then \( h \) is called a \((\sigma, \tau)\)–derivation of \( R \). Obviously every derivation \( d : R \to R \) is a \((1, 1)\)–derivation of \( R \), where \( 1 : R \to R \) is an identity mapping. If \( h(xy) = d(x)\sigma(y) + \tau(x)d(y), \forall x, y \in R \) then \( h \) is said to be a left-generalized \((\sigma, \tau)\)–derivation with \( d \) and if \( h(xy) = h(x)\sigma(y) + \tau(x)d(y), \forall x, y \in R \) then \( h \) is said to be a right-generalized \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d \), (see [4]). Every \((\sigma, \tau)\)–derivation associated with \( d \) is a right (and left)-generalized \((\sigma, \tau)\)–derivation associated with \( d \).

The mapping defined by \( h(r) = [r, a]_{\sigma, \tau}, \forall r \in R \) is a right-generalized derivation associated with derivation \( d(r) = [r, \sigma(a)], \forall r \in R \) and left-generalized derivation associated with derivation \( d_1(r) = [r, \sigma(a)], \forall r \in R \). The mapping \( h(r) = (a, r)_{\sigma, \tau}, \forall r \in R \) is a left-generalized \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d_2(r) = [a, r]_{\sigma, \tau}, \forall r \in R \) and right-generalized \((\sigma, \tau)\)–derivation associated with \((\sigma, \tau)\)–derivation \( d_2 \).
The following result is proved by Posner in (see [12]). Let \( R \) be a prime ring and \( d \neq 0 \) derivation of \( R \) such that \( [d(x), x] = 0, \forall x \in R \). Then \( R \) is commutative. Ashraf and Rehman (see [1]) generalized Posner’s result as follows. Let \( R \) be a 2-torsion free prime ring. Suppose there exists a \((\sigma, \tau)\)-derivation \( d : R \to R \) such that \( [d(x), x]_{\sigma, \tau} = 0, \forall x \in R \). Then either \( d = 0 \) or \( R \) is commutative. Taking an ideal of \( R \) instead of \( R \), Marubayashi H. and Ashraf M., Rehman N., Ali Shakir, generalized Rehman’s result in (see [10]). On the other hand, Rehman (see [13]) gave another generalization of Posner’s Theorem as follows. Let \( R \) be a prime ring. If \( R \) admits a nonzero generalized derivation \( h \) with \( d \) such that 
\[ h(x), x = 0, \forall x \in R, \] 
and if \( d \neq 0 \), then \( R \) is commutative.

In this paper, using left-generalized \((\sigma, \tau)\)-derivation of \( R \), we have given another generalization of Ashraf and Rehman’s result (see [1]) as in Theorem 3. Also, we discuss the commutativity of prime rings admitting a left-generalized \((\sigma, \tau)\)-derivation \( h : R \to R \) satisfying several conditions on ideals.

Throughout the paper, \( R \) will be a prime ring with characteristic not 2 and \( \sigma, \tau, \lambda, \mu, \alpha, \beta \) be automorphisms of \( R \). Let \( J \) be an ideal of \( R \) and \( h \) be a nonzero generalized derivation associated with a nonzero \((\sigma, \tau)\)-derivation of \( R \).

Let \( C_{\sigma, \tau}(J) = \{ r \in R \mid \tau(r) = \tau(r)r, \forall x \in J \} \) and will make extensive use of the following basic commutator identities.

\[
\begin{align*}
[x, y]_{\sigma, \tau} &= x[y, \sigma]_{\sigma, \tau} + [x, \tau(z)]y = x[y, \sigma(z)] + [x, z]_{\sigma, \tau}y \\
[x, y]_{\sigma, \tau} &= \tau(y)[x, z]_{\sigma, \tau} + [x, y]_{\sigma, \tau}(z) \\
(x, yz)_{\sigma, \tau} &= \tau(y)(x, z)_{\sigma, \tau} + [x, y]_{\sigma, \tau}(z) = -\tau(y)[x, z]_{\sigma, \tau} + (x, y)_{\sigma, \tau}(z) \\
(xy, z)_{\sigma, \tau} &= (x, y)_{\sigma, \tau}(z) - [x, \tau(z)]y = x[y, \sigma(z)] + (x, z)_{\sigma, \tau}y. 
\end{align*}
\]

2. Results

We begin with the following known results which will be used to prove our theorems.

**Lemma 2.1.** [2, Lemma1] Let \( R \) be a prime ring and \( d : R \to R \) be a \((\sigma, \tau)\)-derivation. If \( U \) is a nonzero right ideal of \( R \) and \( d(U) = 0 \) then \( d = 0 \).

**Lemma 2.2.** [11, Lemma2] If a prime ring contains a nonzero commutative right ideal then it is commutative.

**Lemma 2.3.** [6, Lemma5] Let \( I \) be a nonzero ideal of \( R \) and \( a, b \in R \). If \([a, I]_{\alpha, \beta} \subset C_{\lambda, \mu}(R)\) or \((a, I)_{\alpha, \beta} \subset C_{\lambda, \mu}(R)\) then \( a \in C_{\alpha, \beta}(R) \) or \( R \) is commutative.

**Lemma 2.4.** [5, Corollary 1] If \( I \) is a nonzero ideal of \( R \) and \( a \in R \) such that \([I, a]_{\alpha, \beta} \subset C_{\lambda, \mu}(R)\), then \( a \in Z \).

**Lemma 2.5.** [7, Lemma 2.16] Let \( R \) be a prime ring and \( h : R \to R \) be a nonzero left-generalized \((\sigma, \tau)\)-derivation associated with a nonzero \((\sigma, \tau)\)-derivation \( d \). If \( I \) is a nonzero ideal of \( R \) and \( a \in R \) such that \((h(I), a)_{\lambda, \mu} = 0\) then \( a \in Z \) or \( d(\tau^{-1}\mu(a)) = 0 \).

**Lemma 2.6.** [7, Theorem 2.7] Let \( h : R \to R \) be a nonzero right-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d \) and \( I, J \) be nonzero ideals of \( R \). If \( a \in R \) such that \( ah(I) \subset C_{\lambda, \mu}(J) \) then \( a \in Z \) or \( d = 0 \).
Lemma 2.7. Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $h : R \rightarrow R$ is a nonzero left-generalized $(\sigma, \tau)$-derivation associated with $(\sigma, \tau)$-derivation $d$ such that $[h(I)a, b]_{\lambda, \mu} = 0$ then $a[a, \lambda(b)] = 0$ or $d(\tau^{-1}\mu(b)) = 0$.

Proof. Using hypothesis we have,

$$0 = [b(\tau^{-1}\mu(b))a, b]_{\lambda, \mu} = [d(\tau^{-1}\mu(b))\sigma(x)a + \mu(b)h(x)a, b]_{\lambda, \mu}$$

$$= d(\tau^{-1}\mu(b))[\sigma(x)a, \lambda(b)] + [d(\tau^{-1}\mu(b)), b]_{\lambda, \mu}\sigma(x)a$$

$$+ \mu(b)[h(x)a, b]_{\lambda, \mu} + [\mu(b), \mu(b)]h(x)a$$

$$= d(\tau^{-1}\mu(b))[\sigma(x)a, \lambda(b)] + [d(\tau^{-1}\mu(b)), b]_{\lambda, \mu}\sigma(x)a, \forall x \in I$$

That is,

$$k[\sigma(x)a, \lambda(b)] + [k, b]_{\lambda, \mu}\sigma(x)a = 0, \forall x \in I \text{ where } k = d(\tau^{-1}\mu(b)). \quad (2.1)$$

Replacing $x$ by $x\sigma^{-1}(a)y$ in (1) and using (1) we get,

$$0 = k[\sigma(x)a\sigma(y)a, \lambda(b)] + [k, b]_{\lambda, \mu}\sigma(x)a\sigma(y)a$$

$$= k\sigma(x)a[\sigma(y)a, \lambda(b)] + k[\sigma(x)a, \lambda(b)]\sigma(y)a + [k, b]_{\lambda, \mu}\sigma(x)a\sigma(y)a$$

$$= k\sigma(x)a[\sigma(y)a, \lambda(b)], \forall x, y \in I.$$ 

That is $k\sigma(I)a[\sigma(I)a, \lambda(b)] = 0$. Since $\sigma(I)$ is a nonzero ideal of $R$ then we have

$$d(\tau^{-1}\mu(b)) = 0 \text{ or } a[\sigma(I)a, \lambda(b)] = 0. \quad (2.2)$$

If $a[\sigma(I)a, \lambda(b)] = 0$ in (2) then we get,

$$0 = a[\sigma(\sigma^{-1}(a)x)a, \lambda(b)] = a[\sigma(x)a, \lambda(b)]$$

$$= aa[\sigma(x)a, \lambda(b)] + a[a, \lambda(b)]\sigma(x)a = a[a, \lambda(b)]\sigma(x)a, \forall x \in I.$$

From the last relation we obtain that $a[a, \lambda(b)] = 0$ for two case. \qed

Remark 2.8. Let $J$ be a nonzero ideal of $R$. If $b \in C_{\lambda, \mu}(J)$ then $b \in C_{\lambda, \mu}(R)$.

Proof. If $b \in C_{\lambda, \mu}(J)$ then we have $0 = [b, x\tau]_{\lambda, \mu} = \mu(x)[b, r]_{\lambda, \mu} + [b, x]_{\lambda, \mu}\lambda(r) = \mu(x)[b, r]_{\lambda, \mu}, \forall x \in J, r \in R$. That is $\mu(J)b, \lambda(r)_{\lambda, \mu} = 0$. This gives that $b \in C_{\lambda, \mu}(R)$. \qed

Theorem 2.9. Let $h : R \rightarrow R$ be a nonzero left-generalized $(\sigma, \tau)$-derivation associated with nonzero $(\sigma, \tau)$-derivation $d$ and $a, b \in R$. Let $I, J$ be nonzero ideals of $R$.

(i) If $h(I)a \subset C_{\lambda, \mu}(J)$ then $a \in Z$.

(ii) If $ah(I) \subset C_{\lambda, \mu}(J)$ then $a \in Z$ or $ad\tau^{-1}(a) = 0$. 

Proof. (i) If $h(I)a \subseteq C_{\lambda,\mu}(J)$ then we have $[h(I)a, x]_{\lambda,\mu} = 0, \forall x \in J$. Using this relation and Lemma 7 we get, for any $x \in J$,

$$a[a, \lambda(x)] = 0 \text{ or } d\tau^{-1}\mu(x) = 0$$

Let $K = \{x \in J \mid a[a, \lambda(x)] = 0\}$ and $L = \{x \in J \mid d\tau^{-1}\mu(x) = 0\}$. Then $K$ and $L$ are subgroups of $J$ and $J = K \cup L$. A group can not write the union of its proper subgroups. Hence we have $K = J$ or $L = J$. That is,

$$a[a, \lambda(J)] = 0 \text{ or } d(\tau^{-1}\mu(J)) = 0$$

Since $d \neq 0$ then $d(\tau^{-1}\mu(J)) \neq 0$ by Lemma 1. If $a[a, \lambda(J)] = 0$ then we get

$$0 = a[a, \lambda(xr)] = a\lambda(x)[a, \lambda(r)] + a[a, \lambda(x)]\lambda(r)$$

$$= a\lambda(x)[a, \lambda(r)], \forall x \in J, r \in R$$

and so $a\lambda(J)[a, R] = 0$. From this relation we obtain that $a \in Z$. (ii) If $ah(I) \subseteq C_{\lambda,\mu}(J)$ then we have $ah(I) \subseteq C_{\lambda,\mu}(R)$ by Remark 1. Using this relation we get

$$0 = [ah(\tau^{-1}(a)y), \mu^{-1}(a)]_{\lambda,\mu} = [ad(\tau^{-1}(a))\sigma(y) + aah(y), \mu^{-1}(a)]_{\lambda,\mu}$$

$$= ad(\tau^{-1}(a))[\sigma(y), \lambda^{-1}(a)] + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y)$$

$$+ a[ah(y), \mu^{-1}(a)]_{\lambda,\mu} + [a, a]ah(y)$$

$$= ad(\tau^{-1}(a))[\sigma(y), \lambda^{-1}(a)] + [ad(\tau^{-1}(a)), \mu^{-1}(a)]_{\lambda,\mu}\sigma(y), \forall y \in I,$$

and so

$$k[\sigma(y), p] + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y) = 0, \forall y \in I, \text{ where } k = ad(\tau^{-1}(a)) \text{ and } p = \lambda\mu^{-1}(a).$$

Replacing $y$ by $yx, x \in I$ in (3) we obtain that

$$0 = k\sigma(y)[\sigma(x), p] + k[\sigma(y), p]\sigma(x) + [k, \mu^{-1}(a)]_{\lambda,\mu}\sigma(y)\sigma(x)$$

$$= k\sigma(y)[\sigma(x), p], \forall x, y \in I.$$
Proof. (i) Let \( h(r) = [r, b]_{\sigma, \tau}, \forall r \in R \) and \( d(r) = [r, \tau(b)], \forall r \in R \). Since,

\[
    h(rs) = [rs, b]_{\sigma, \tau} = r[s, b]_{\sigma, \tau} + [r, \tau(b)]s = d(r)s + rh(s), \forall r, s \in R,
\]

then \( h \) is a left-generalized derivation associated with derivation \( d \). If \( h = 0 \) then \( d = 0 \) (and so \( b \in Z \)) is obtained by the relation (5).

If \( [I, b]_{\sigma, \tau}a \subset C_{\lambda, \mu}(J) \) then we can write \( h(I)a \subset C_{\lambda, \mu}(J) \). If \( h \neq 0 \) and \( d \neq 0 \) then we have \( a \in Z \) by Theorem 1(i).

(ii) The mapping defined by \( d_1(r) = [b, r]_{\sigma, \tau}, \forall r \in R \) is a \((\sigma, \tau)\)-derivation and so, left (and right)-generalized \((\sigma, \tau)\)-derivation with \( d_1 \). If \( d_1 = 0 \) then we have \( b \in C_{\sigma, \tau}(R) \).

Let \( d_1 \neq 0 \). If \([b, I]_{\sigma, \tau}a \subset C_{\lambda, \mu}(J) \) then we can write \( d_1(I)a \subset C_{\lambda, \mu}(J) \). This gives that \( a \in Z \) by Theorem 1(i). Finally we obtain that \( a \in Z \) or \( b \in C_{\sigma, \tau}(R) \).

(iii) The mapping defined by \( g(r) = (b, r)_{\sigma, \tau}, \forall r \in R \) is a left-generalized \((\sigma, \tau)\)-derivation associated with \((\sigma, \tau)\)-derivation \( d_1(r) = [b, r]_{\sigma, \tau}, \forall r \in R \). If \( g = 0 \) then \( d_1 = 0 \) and so \( b \in C_{\sigma, \tau}(R) \) is obtained. Let \( g \neq 0 \) and \( d_1 \neq 0 \). If \( a(b, I)_{\sigma, \tau} \subset C_{\lambda, \mu}(J) \) then we have \( ag(I) \subset C_{\lambda, \mu}(J) \). This implies that \( a \in Z \) or \( ad_1\tau^{-1}(b) = 0 \) by Theorem 1(ii). That is \( a \in Z \) or \( a[b, \tau^{-1}(b)]_{\sigma, \tau} = 0 \).

\[
\text{Lemma 2.11. Let } I \text{ be a nonzero ideal of } R \text{ and } h : R \rightarrow R \text{ be a nonzero left-generalized } (\sigma, \tau) \text{-derivation associated with a nonzero } (\sigma, \tau) \text{-derivation } d. \text{ If } a \in R \text{ such that } [h(I), a]_{\lambda, \mu} = 0 \text{ then } a \in Z \text{ or } d(\tau^{-1}u(a)) = 0.
\]

Proof. Using hypothesis we get,

\[
\begin{align*}
0 &= [h(\tau^{-1}u(a)x), a]_{\lambda, \mu} = [d(\tau^{-1}u(a))\sigma(x) + \mu(a)h(x), a]_{\lambda, \mu} \\
&= d(\tau^{-1}u(a))[\sigma(x), \lambda(a)] + [d(\tau^{-1}u(a)), a]_{\lambda, \mu}\sigma(x) + \mu(a)[h(x), a]_{\lambda, \mu} + [\mu(a), a]_{\lambda, \mu}h(x) \\
&= d(\tau^{-1}u(a))[\sigma(x), \lambda(a)] + [d(\tau^{-1}u(a)), a]_{\lambda, \mu}\sigma(x), \forall x \in I.
\end{align*}
\]

That is,

\[
k[\sigma(x), \lambda(a)] + [k, a]_{\lambda, \mu}\sigma(x) = 0, \forall x \in I, \text{ where } k = d(\tau^{-1}u(a)).
\]

Replacing \( x \) by \( xr, r \in R \) in (6) and using (6) we get

\[
\begin{align*}
0 &= k[\sigma(x), \lambda(a)] + k[\sigma(x), \lambda(a)]\sigma(x) + [k, a]_{\lambda, \mu}\sigma(x)\sigma(x) \\
&= k[\sigma(x), \lambda(a)], \forall x \in I, r \in R.
\end{align*}
\]

and so \( k[\sigma(I)[R, \lambda(a)] = 0 \). Since \( \sigma(I) \neq 0 \) is an ideal and \( R \) is prime then we have \( a \in Z \) or \( d(\tau^{-1}u(a)) = 0 \).

\[
\text{Theorem 2.12. Let } h \text{ be a nonzero left-generalized } (\sigma, \tau) \text{-derivation associated with } (\sigma, \tau) \text{-derivation } 0 \neq d \text{ and } I, J \text{ be nonzero ideals of } R.
\]

(i) If \( h(I) \subset C_{\lambda, \mu}(J) \) then \( R \) is commutative.

(ii) If \([h(I), J]_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \text{ or } (h(I), J)_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \) then \( R \) is commutative.

(iii) If \([J, h(I)]_{\alpha, \beta} \subset C_{\lambda, \mu}(R) \) then \( R \) is commutative.
Proof. (i) If \( h(I) \subseteq C_{\lambda,\mu}(J) \) then we have \([h(I), x]_{\lambda,\mu} = 0, \forall x \in J\). This means that, for any \( x \in J\), \[ x \in Z \text{ or } d(\tau^{-1}\mu(x)) = 0 \] (2.7) by Lemma 8. Using (7), let us consider the following sets, \( K = \{ x \in J \mid x \in Z \} \) and \( L = \{ x \in J \mid d\tau^{-1}\mu(x) = 0 \} \). Considering as in the proof of Theorem 1 we obtain that \( J \subseteq Z \) or \( d(\tau^{-1}\mu(J)) = 0 \). Since \( d \neq 0 \) then we have \( d(\tau^{-1}\mu(J)) \neq 0 \) by Lemma 1. Hence, we obtain that \( K = J \) and so \( J \subseteq Z \). This means that \( R \) is commutative by Lemma 2.

(ii) If \([h(I), J]_{\alpha,\beta} \subseteq C_{\lambda,\mu}(R)\) or \((h(I), J)_{\alpha,\beta} \subseteq C_{\lambda,\mu}(R)\) then we have \( h(I) \subseteq C_{\alpha,\beta}(R) \) or \( R \) is commutative by Lemma 3. On the other hand \( h(I) \subseteq C_{\alpha,\beta}(R) \) means that \( R \) is commutative by (i).

(iii) If \([J, h(I)]_{\alpha,\beta} \subseteq C_{\lambda,\mu}(R)\) then we have \( h(I) \subseteq Z \) by Lemma 4 and so \( R \) is commutative by (i). \( \square \)

**Corollary 2.13.** [8, Lemma 2] Let \( U \) be a nonzero ideal of \( R \). If \( d : R \rightarrow R \) is a nonzero \((\sigma, \tau)\)-derivation such that \( d(U) \subseteq C_{\lambda,\mu}(R) \). Then \( R \) is commutative.

**Theorem 2.14.** Let \( h : R \rightarrow R \) be a nonzero left-generalized \((\sigma, \tau)\)-derivation associated with a nonzero \((\sigma, \tau)\)-derivation \( d \). If \( I \neq 0 \) is an ideal of \( R \) such that \([h(x), x]_{\lambda,\tau} = 0, \forall x \in I \) then \( R \) is commutative.

**Proof.** Linearizing the hypothesis, we get

\[ [h(x), y]_{\lambda,\tau} + [h(y), x]_{\lambda,\tau} = 0, \forall x, y \in I. \] (2.8)

Replacing \( x \) by \( gx \) in (8) and using (8) we have

\[ 0 = [h(gx), y]_{\lambda,\tau} + [h(y), gx]_{\lambda,\tau} \]
\[ = [d(y)\sigma(x) + \tau(y)h(x), y]_{\lambda,\tau} + [h(y), gx]_{\lambda,\tau} \]
\[ = d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau} \]
\[ + [\tau(y), x]_{\lambda,\tau} + [h(y), x]_{\lambda,\tau} + [h(y), y]_{\lambda,\tau} \lambda(x) \]
\[ = d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau} \sigma(x), \forall x, y \in I. \]

That is

\[ d(y)[\sigma(x), \lambda(y)] + [d(y), y]_{\lambda,\tau} \sigma(x) = 0, \forall x, y \in I. \] (2.9)

Taking \( xr, r \in R \) instead of \( x \) in (9) and using (9) then we arrive

\[ 0 = d(y)[\sigma(x)\sigma(r), \lambda(y)] + d(y)[\sigma(x), \lambda(y)]\sigma(r) + [d(y), y]_{\lambda,\tau} \sigma(x)\sigma(r) \]
\[ = d(y)[\sigma(x)\sigma(r), \lambda(y)], \forall x, y \in I, r \in R \]

which leads to

\[ d(y)\sigma(I)[R, \lambda(y)] = 0, \forall y \in I. \] (2.10)

Since \( \sigma(I) \neq 0 \) an ideal then, for any \( y \in I \), we have \([R, \lambda(y)] = 0 \) or \( d(y) = 0 \) by (10) and so \( y \in Z \) or \( d(y) = 0 \).
Let $K = \{ y \in I \mid y \in Z \}$ and $L = \{ y \in I \mid d(y) = 0 \}$. Considering as in the proof of Theorem 1 we have, $I \subset Z$ or $d(I) = 0$. Since $I \neq 0$ an ideal and $d \neq 0$ then we obtain that $K = I$ by Lemma 1 and so $I \subset Z$. This means that $R$ is commutative by Lemma 2.

Corollary 2.15. [1, Theorem 1] Let $R$ be a prime ring and $I$ be a nonzero ideal of $R$. If $R$ admits a nonzero $(\alpha, \beta)$–derivation $d$ such that $\{d(x), x\}_{\alpha, \beta} = 0, \forall x \in I$, then $R$ is commutative.

Theorem 2.16. Let $R$ be a prime ring and $0 \neq a \in R$. If $h : R \rightarrow R$ is a nonzero left-generalized $(\sigma, \tau)$–derivation associated with a nonzero $(\sigma, \tau)$–derivation $d$ and $I \neq 0$ an ideal of $R$ such that $\{h(x)a, x\}_{\lambda, \tau} = 0, \forall x \in I$ then $R$ is commutative.

Proof. Replacing $x$ by $x + y$ in hypothesis we have

$$[h(x)a, y]_{\lambda, \tau} + [h(y)a, x]_{\lambda, \tau} = 0, \forall x, y \in I. \quad (11)$$

If we take $yx$ instead of $x$ in (11) and using (11) we get

$$0 = [h(yx)a, y]_{\lambda, \tau} + [h(y)a, yx]_{\lambda, \tau}$$

$$= [d(y)\sigma(x)a + \tau(y)h(x)a, y]_{\lambda, \tau} + [h(y)a, yx]_{\lambda, \tau}$$

$$= d(y)[\sigma(x)a, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)a + \tau(y)[h(x)a, y]_{\lambda, \tau}$$

$$+ [\tau(y), \tau(y)]h(x)a + \tau(y)[h(y)a, x]_{\lambda, \tau} + [h(y)a, y]_{\lambda, \tau}\lambda(x), \forall x, y \in I.$$  

That is

$$d(y)[\sigma(x)a, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)a = 0, \forall x, y \in I. \quad (12)$$

Replacing $x$ by $x\sigma^{-1}(a)$ in (12) and using (12) we have

$$0 = d(y)[\sigma(x)aa, \lambda(y)] + [d(y), y]_{\lambda, \tau}\sigma(x)aa$$

$$= d(y)\sigma(x)a[a, \lambda(y)] + d(y)[\sigma(x)a, \lambda(y)]a + [d(y), y]_{\lambda, \tau}\sigma(x)aa$$

$$= d(y)\sigma(x)a[a, \lambda(y)], \forall x, y \in I.$$  

That is

$$d(y)\sigma(I)a[a, \lambda(y)] = 0, \forall y \in I. \quad (13)$$

Since $\sigma(I)$ a nonzero ideal of $R$ then, for any $y \in I$, we obtain that

$$a[a, \lambda(y)] = 0 \text{ or } d(y) = 0$$

by (13). Hence, the additive group $I$ is a union of subgroups $K = \{ y \in I \mid a[a, \lambda(y)] = 0 \}$ and $L = \{ y \in I \mid d(y) = 0 \}$. Considering as in the proof of the Theorem 1, we obtain that $K = I$ and so $a[a, \lambda(I)] = 0$. Using this result we get,

$$0 = a[a, \lambda(yr)] = a\lambda(y)[a, \lambda(r)] + a[a, \lambda(y)]\lambda(r)$$

$$= a\lambda(y)[a, \lambda(r)], \forall r \in R, y \in I.$$
That is $a\lambda(I)[a, R] = 0$. This means that $a \in Z$. On the other hand, considering that $a \in Z$ and hypothesis, we get

\[0 = [h(x)]a, x\lambda, \tau = h(x)[a, \lambda(x)] = h(x)[x\lambda, \tau a \text{ for all } x \in I].\]

That is $[h(x), x]_{\lambda, \tau} a = 0, \forall x \in I$. Since $a \in Z$ and $a \neq 0$ we have $[h(x), x]_{\lambda, \tau} = 0$ for all $x \in I$. This gives that $R$ is commutative by Theorem 3. \hfill \Box

**Remark 2.17.** Let $I$ be a nonzero ideal of $R$ and $a, b \in R$. If $(I, a)_{\lambda, \mu} b = 0$ or $b(I, a)_{\lambda, \mu} = 0$ then $a \in Z$ or $b = 0$.

**Proof.** If $(I, a)_{\lambda, \mu} b = 0$ then we have $0 = (r, x)_{\lambda, \mu} b = (r, x, a)_{\lambda, \mu} b - (r, \mu(a)) x b = -[r, \mu(a)] x b, \forall r \in R, x \in I$. That is $(R, \mu(a)) x b = 0$. This gives that $a \in Z$ or $b = 0$.

Let $b(I, a)_{\lambda, \mu} = 0$. Then $0 = b(x, a)_{\lambda, \mu} = b(x, a)_{\lambda, \mu} r = b x [r, \lambda(a)], \forall r \in R, x \in I$.

This gives that $b I[R, \lambda(a)] = 0$ and so $a \in Z$ or $b = 0$. \hfill \Box

**Lemma 2.18.** Let $I$ be a nonzero ideal of $R$ and $a$ be a noncentral element of $R$. Let $h : R \rightarrow R$ be a nonzero right-generalized derivation associated with $d$. If $h(I, a)_{\lambda, \mu} = 0$ or $(h(I, a))_{\lambda, \mu} = 0$ then $d\lambda(a) = 0$.

**Proof.** If $h(I, a)_{\lambda, \mu} = 0$ then using that $h$ is a right general derivation we get

\[0 = h(x)\lambda(a), a)_{\lambda, \mu} = h\{x\lambda(a), \lambda(a)\} + (x, a)_{\lambda, \mu} \lambda(a)\} = h\{x, a)_{\lambda, \mu} \lambda(a)\}

which leads to

\[(I, a)_{\lambda, \mu} d\lambda(a) = 0.\] (2.14)

Using Remark 2 and (14) we have $a \in Z$ or $d\lambda(a) = 0$. Since $a$ be a noncentral then $d\lambda(a) = 0$ is obtained.

If $(h(I, a))_{\lambda, \mu} = 0$ then we have

\[0 = (h(x)\lambda(a), a)_{\lambda, \mu} = (h(x)\lambda(a) + x d\lambda(a), a)_{\lambda, \mu}

\[= h(x)\lambda(a), \lambda(a)\} + (h(x), a)_{\lambda, \mu} \lambda(a) + x d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)] d\lambda(a)

\[= x(d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)] d\lambda(a), \forall x \in I.\]

That is,

\[x(d\lambda(a), a)_{\lambda, \mu} - [x, \mu(a)] d\lambda(a) = 0, \forall x \in I.\] (2.15)

Replacing $x$ by $xy, y \in I$ in (15) and using (15) we get

\[0 = xy d\lambda(a), a)_{\lambda, \mu} - y d\lambda(a) - [x, \mu(a)] y d\lambda(a)

\[= -[x, \mu(a)] y d\lambda(a), \forall x, y \in I.\]

and so $(I, a) d\lambda(a) = 0$. Since $R$ is prime and $a$ be a noncentral element then we obtain that $d\lambda(a) = 0$. \hfill \Box
Lemma 2.19. Let $I$ be a nonzero ideal of $R$ and $a$ is a noncentral element of $R$. Let $h : R \rightarrow R$ be a nonzero left generalized derivation associated with derivation $d_1 : R \rightarrow R$. If $h((I, a)_{\lambda, \mu}) = 0$ or $(h(I), a)_{\lambda, \mu} = 0$ then $d_1(\mu(a)) = 0$.

Proof. If $h(I, a)_{\lambda, \mu} = 0$ then using that $h$ is a left-generalized derivation we get
\[ 0 = h(\mu(a)x, a)_{\lambda, \mu} = h(\mu(a)(x, a)_{\lambda, \mu} - [\mu(a), \mu(a)]x) \]
\[ = h(\mu(a)(x, a)_{\lambda, \mu}) = d_1(\mu(a))(x, a)_{\lambda, \mu} + \mu(\mu(a)h((x, a)_{\lambda, \mu})) \]
\[ = d_1(\mu(a))(x, a)_{\lambda, \mu}, \forall x \in I. \]

That is,
\[ d_1(\mu(a))(I, a)_{\lambda, \mu} = 0. \quad (2.16) \]

Since $a$ be noncentral then using Remark 2 and (16) we obtain that $d_1(\mu(a)) = 0$.

On the other hand, If $(h(I), a)_{\lambda, \mu} = 0$ then we have $d_1(\mu(a)) = 0$ by Lemma 5.

Theorem 2.20. Let $I$ be a nonzero ideal of $R$ and $a$ is a noncentral element of $R$. Let $h : R \rightarrow R$ be a nonzero right-generalized derivation associated with $d_1$ and left-generalized derivation associated with $d_1$. Then $h((I, a)_{\lambda, \mu}) = 0$ if and only if $(h(I), a)_{\lambda, \mu} = 0$.

Proof. If $h((I, a)_{\lambda, \mu}) = 0$ or $(h(I), a)_{\lambda, \mu} = 0$ then $d(\lambda(a)) = 0$ and $d_1(\mu(a)) = 0$ are obtained by Lemma 9 and Lemma 10.

Using these results we get
\[ h((I, a)_{\lambda, \mu}) = 0 \iff h(x\lambda(a) + \mu(a)x) = 0, \forall x \in I. \]
\[ \iff h(x)\lambda(a) + xd(\lambda(a)) + d_1(\mu(a))x + \mu(a)h(x) = 0, \forall x \in I. \]
\[ \iff h(x)\lambda(a) + \mu(a)h(x) = 0, \forall x \in I. \]
\[ \iff (h(I), a)_{\lambda, \mu} = 0. \]

Corollary 2.21. [9, Theorem 7] Let $R$ be a prime ring of characteristic different from two, $d : R \rightarrow R$ be a nonzero derivation and $a \in R$. Then $(d(R), a) = 0$ if and only if $d(R, a) = 0$.

References


Evrim Guven,
Kocaeli University,
Department of Mathematics,
Faculty of Science and Art,
TURKEY.
E-mail address: evrim@kocaeli.edu.tr