On Ternary Left Almost Semigroups

Pairote Yiarayong

ABSTRACT: A ternary LA-semigroup is a nonempty set together with a ternary multiplication which is non associative. Analogous to the theory of LA-semigroups, a regularity condition on a ternary LA-semigroup is introduced and the properties of ternary LA-semigroups are studied. Some characterizations of quasi-prime and quasi-ideals were obtained.

Key Words: Ternary LA-semigroup, Quasi-prime ideal, Quasi-ideal, Ternary LA-subsemigroup, Left ideal.

Contents

1 Introduction 189
2 Preliminaries 189
3 Left ideals of ternary LA-semigroups 193
4 Quasi-ideals of ternary LA-semigroups 197

1. Introduction

The theory of ternary algebraic systems was introduced by Lehmer [3] in 1932. The notion of ternary semigroup was known to Banach [4] who is credited with example of a ternary semigroup which can not reduce to a semigroup. The quasi-ideal theory in ternary semigroups was studied by Sioson [5] in the year 1965. Los [4] studied some properties of ternary semigroup and proved that every ternary semigroup can be embedded in a semigroup. Dixit and Dewan [1,2] studied quasi-ideals and bi-ideals in ternary semigroups. Recently, Bashir and Shabir [4] defined the concepts of weakly pure ideal and purely prime ideal in a ternary semigroup without order.

In this study we followed lines as adopted in [7,8,9] and established the notion of ternary LA-semigroups. Specifically we characterize the quasi-prime and quasi-ideals in ternary LA-semigroups with left identity.

2. Preliminaries

In this section, we give some preliminary results of ternary LA-semigroups which will be required for our later discussions.

2010 Mathematics Subject Classification: 20M10, 20N99.
Submitted June 26, 2017. Published February 19, 2018
Definition 2.1. Let $S$ be a nonempty set. Then $S$ is called a ternary left almost semigroup (or simply a ternary LA-semigroup) if there exists a ternary operation $S \times S \times S \rightarrow S$, written as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$, such that

$$(x_1x_2x_3)(x_4x_5) = ((x_4x_5)x_3)(x_1x_2) = ((x_3x_2)x_1)(x_4x_5)$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Example 2.2. Let $S = \{0, i, -i\}$. Then by defining $S \times S \times S \rightarrow S$, as $(x_1, x_2, x_3) \mapsto x_1x_2x_3$, for all $x_1, x_2, x_3 \in S$. It can be easily verified that $S$ is a ternary LA-semigroup under complex number multiplication while $S$ is not an LA-semigroup.

Example 2.3. Let $S = \mathbb{Z}$. Define a mapping $S \times S \times S \rightarrow S$ by $(x_1, x_2, x_3) \mapsto -x_1 + x_2 - x_3$, for all $x_1, x_2, x_3 \in S$, where $-i$ is a usual subtraction of integers. Then $S$ is a ternary LA-semigroup while $S$ is not a ternary semigroup. Indeed

$$(x_1x_2x_3)(x_4x_5) = (-x_1 + x_2 - x_3)(x_4x_5)$$

and

$$(x_1x_2x_3)(x_4x_5) = ((x_4x_5)x_3)(x_1x_2) = ((x_3x_2)x_1)(x_4x_5)$$

which implies $((x_1x_2)x_3)(x_4x_5) = ((x_4x_5)x_3)(x_1x_2) = ((x_3x_2)x_1)(x_4x_5)$, for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Proposition 2.4. If $S$ is a ternary LA-semigroup, then $((x_1x_2)x_3)((x_4x_5)x_6) = ((x_1x_2)(x_4x_5))x_6$, for all $x_1, x_2, x_3, x_4, x_5, x_6 \in S$.

Proof: Let $x_1, x_2, x_3, x_4, x_5, x_6 \in S$. Then by Definition of ternary LA-semigroup, we get

$$(x_1x_2)(x_4x_5)x_6 = (((x_4x_5)x_6)x_3)(x_1x_2)$$

and

$$(x_3x_2)(x_4x_5)x_6 = ((x_3x_6)(x_4x_5))(x_1x_2).$$

Hence $((x_1x_2)x_3)((x_4x_5)x_6) = ((x_1x_2)(x_4x_5))(x_3x_6)$. □

Proposition 2.5. If $S$ is a ternary LA-semigroup, then

$$[((x_1x_2)x_3)(x_4x_5)][(x_6x_7)(x_8x_9)x_{10}] = [((x_1x_2)x_3)(x_6x_7)][(x_4x_5)(x_8x_9)x_{10}],$$

for all $x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10} \in S$. 
On Ternary LA-semigroups

Proof: Let \( x_1, x_2, x_3, x_4, x_5, x_6 \in S \). Then by Definition of ternary LA-semigroup, we have

\[
\begin{align*}
[((x_1x_2)x_3)(x_4x_5)] & \cdot [(x_6x_7)((x_8x_9)x_{10})]
\end{align*}
\]

Definition 2.6. An element \( e \) of a ternary LA-semigroup \( S \) is called a left identity if\( (ee)x = x \), for all \( x \in S \).

Note 1. By \( (AB)C (A(BC)) \) we mean the set

\[
\{ (ab) : a \in A, b \in B, c \in C \} \quad \{ (a(bc) : a \in A, b \in B, c \in C \}
\]

for non empty subsets \( A, B, C \) of \( S \). If \( A = \{ a \} \), then we write \( (\{ a \} B)C \) as \( (aB)C \) and similarly if \( B = \{ b \} \) or \( C = \{ c \} \), we write \( (AB)c \) and \( (AB)c \) respectively.

Lemma 2.7. If \( S \) is a ternary LA-semigroup with left identity \( e \), then \( (SS)S = S \) and \( (ee)S = S \).

Proof: Let \( x \in S \). Then \( x = (ee)x \in (SS)S \) so \( S \subseteq (SS)S \). Hence \( S = (SS)S \).

Lemma 2.8. If \( S \) is a ternary LA-semigroup with left identity, then \( x_1(x_2x_3) = x_2(x_1x_3) \), for all \( x_1, x_2, x_3 \in S \).

Proof: Let \( x_1, x_2, x_3 \in S \). Then by Definition of ternary LA-semigroup, we get

\[
\begin{align*}
x_1(x_2x_3) &= ((ee)x_1)(x_2x_3) \\
&= (x_2x_3)(x_1)(ee) \\
&= (x_2x_3)(x_1)(ee) \\
&= (ee)(x_2)(x_1x_3) \\
&= x_2(x_1x_3).
\end{align*}
\]

Hence \( x_1(x_2x_3) = x_2(x_1x_3) \).

Proposition 2.9. If \( S \) is a ternary LA-semigroup with left identity, then

\[
(x_1x_2)((x_3x_4)x_5) = (x_5(x_3x_4))(x_2x_1),
\]

for all \( x_1, x_2, x_3, x_4, x_5 \in S \).
Proof: Let \( x_1, x_2, x_3, x_4, x_5 \in S \). Then by Definition of ternary LA-semigroup, we get
\[
(x_1 x_2)(x_3 x_4) = (((x_2 x_3) x_4) x_5)
\]
Hence \( (x_1 x_2)(x_3 x_4) = (x_5(x_3 x_4))(x_2 x_1) \). \( \blacksquare \)

Lemma 2.10. If \( S \) is a ternary LA-semigroup with left identity, then
\[
(x_1 x_2)(x_3 x_4) = (x_1 x_3)(x_2 x_4),
\]
for all \( x_1, x_2, x_3, x_4 \in S \).

Proof: Let \( x_1, x_2, x_3, x_4 \in S \). Then by Definition of ternary LA-semigroup, we have
\[
(x_1 x_2)(x_3 x_4) = (((x_2 x_3) x_4) x_5)
\]
Hence \( (x_1 x_2)(x_3 x_4) = (x_1 x_3)(x_2 x_4) \). \( \blacksquare \)

Lemma 2.11. If \( S \) is a ternary LA-semigroup with left identity, then
\[
(x_1 x_2)(x_3 x_4) = (x_4 x_3)(x_2 x_1),
\]
for all \( x_1, x_2, x_3, x_4 \in S \).

Proof: Let \( x_1, x_2, x_3, x_4 \in S \). Then by Definition of ternary LA-semigroup, we have
\[
(x_1 x_2)(x_3 x_4) = (((x_2 x_3) x_4) x_5)
\]
Hence \( (x_1 x_2)(x_3 x_4) = (x_4 x_3)(x_2 x_1) \). \( \blacksquare \)

Lemma 2.12. If \( S \) is a ternary LA-semigroup with left identity, then
\[
(x_1 x_2)x_3 = (x_3 x_2)x_1,
\]
for all \( x_1, x_2, x_3 \in S \).
Proof: Let \(x_1, x_2, x_3 \in S\). Then by Definition of ternary LA-semigroup, we have
\[
(x_1 x_2) x_3 = (x_1 x_2)((ee)x_3) = (x_1(ee))(x_2 x_3) = (x_3 x_2)((ee)x_1) = (x_3 x_2)x_1.
\]
Hence \((x_1 x_2)x_3 = (x_3 x_2)x_1\).
\[\square\]

3. Left ideals of ternary LA-semigroups

The results of the following lemmas seem to play an important role to study ternary LA-semigroups; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 3.1. A non-empty subset \(A\) of a ternary LA-semigroup \(S\) is said to be

1. ternary LA-subsemigroup if \((AA)A \subseteq A\);

2. left ideal if \((SS)A \subseteq A\).

Remark 3.2. It is easy to see that every left ideal is ternary LA-subsemigroup.

Example 3.3. Let \(S = \{0, -1, -2, -3, -4\}\) the binary operation \(\cdot\) be defined on \(S\) as follows:

\[
\begin{array}{cccccc}
  \cdot & 0 & -1 & -2 & -3 & -4 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  -1 & 0 & 1 & 2 & 3 & 4 \\
  -2 & 0 & 2 & 4 & 1 & 3 \\
  -3 & 0 & 3 & 1 & 4 & 2 \\
  -4 & 0 & 4 & 3 & 2 & 1 \\
\end{array}
\]

\[
\begin{array}{cccccc}
  \cdot & 0 & 1 & 2 & 3 & 4 \\
  0 & 0 & 0 & 0 & 0 & 0 \\
  -1 & 0 & -1 & 2 & -3 & -4 \\
  -2 & 0 & -2 & -4 & 1 & -1 \\
  -3 & 0 & -3 & -1 & -4 & -2 \\
  -4 & 0 & -4 & -3 & -2 & -1 \\
\end{array}
\]

Define a mapping \(S \times S \times S \rightarrow S\) by \((x_1, x_2, x_3) \mapsto x_1^{-1} \cdot x_2 \cdot x_3^{-1}\), for all \(x_1, x_2, x_3 \in S\) and \(x_1 \cdot x_1^{-1} = x_1^{-1} \cdot x_1 = 1 = x_3 \cdot x_3^{-1} = x_3^{-1} \cdot x_3\). Then \(S\) is a ternary LA-semigroup. It is easy to see that \(\{0, -1\}\) is a ternary LA-subsemigroup of \(S\). But \(\{0, -1\}\) is not a left ideal of \(S\), since \((SS)\{0, -1\} = S \not\subseteq \{0, -1\}\).

Example 3.4. Let \(S = \{0, -a, -b, -c\}\) the binary operation \(\cdot\) be defined on \(S\) as follows:
Define a mapping $S \times S \times S \rightarrow S$ by

\[
(x_1, x_2, x_3) \mapsto \begin{cases} \quad x_1^{-1} \cdot x_2 \cdot x_3^{-1} & \text{if } \exists x_1^{-1}, x_3^{-1} \in S \text{ s.t. } x_1 \cdot x_1^{-1} = a = x_3 \cdot x_3^{-1} \\ 0 & \text{otherwise.} \end{cases}
\]

Then $S$ is a ternary LA-semigroup. It is easy to see that $\{0, -b\}$ is a left ideal of $S$.

**Lemma 3.5.** Let $S$ be a ternary LA-semigroup with left identity. If $A$ is a left ideal of $S$, then $B(AA)$ is a left ideal of $S$, where $\emptyset \neq A \subseteq S$.

**Proof:** Suppose that $S$ is a ternary LA-semigroup with left identity. Let $A$ be a left ideal of $S$. Then by Lemma 2.12, we have

\[
(SS)(B(AA)) = B((SS)(AA)) = B(A((SS)A)) \subseteq B(AA)
\]

By Definition of left ideal, we get $B(AA)$ is a left ideal of $S$. \hfill \Box

**Corollary 3.6.** Let $S$ be a ternary LA-semigroup with left identity. If $A$ is a left ideal of $S$, then $a(AA)$ is a left ideal of $S$, where $a \in S$.

**Proof:** It is straightforward by Lemma 3.5. \hfill \Box

**Definition 3.7.** If $A$ is a left ideal of a ternary LA-semigroup $S$ and $\emptyset \neq B, C \subseteq S$, then

\[
(A : B : C) = \{s \in S : (BC)s \subseteq A\}
\]

and it is called the extension of $A$ by $B, C$. Let $(A : a : b)$ stand for $(A : \{a\} : \{b\})$.

**Proposition 3.8.** Let $A$ be a left ideal of a ternary LA-semigroup $S$ and $\emptyset \neq B, C, D, E \subseteq S$. Then the following statements hold.

1. $A \subseteq (A : a : b)$, where $a, b \in S$.
2. If $D \subseteq B$, then $(A : B : C) \subseteq (A : D : C)$.
3. If $E \subseteq C$, then $(A : B : C) \subseteq (A : B : E)$.
4. If $A \subseteq D$, then $(A : B : C) \subseteq (D : B : C)$.
5. If $B \subseteq A$, then $(A : B : C) = S$. 

6. If \( C \subseteq A \), then \((A : B : C) = S\).

**Proof:** It is straightforward by Definition 3.7. 

**Proposition 3.9.** Let \( A \) be a left ideal of a ternary LA-semigroup with left identity \( S \) and \( \emptyset \neq B, C \subseteq S \). Then the following statements hold.

1. If \( B \subseteq A \), then \((A : B : C) = S\).
2. If \( C \subseteq A \), then \((A : B : C) = S\).

**Proof:** It is straightforward by Definition 3.7. 

**Lemma 3.10.** Let \( S \) be a ternary LA-semigroup with left identity. If \( A \) is a left ideal of \( S \), then \((A : a : b)\) is a left ideal of \( S \), where \( a, b \in S \).

**Proof:** Suppose that \( S \) is a ternary LA-semigroup with left identity. Let \( r, s \in S \) and \( x \in (A : a : b) \). Then \((ab)x \in A\). Then by Lemma 2.12, we have

\[
(ab)(rs)x = (rs)((ab)x) \\
\quad \quad \quad \subset (rs)A \\
\quad \quad \quad \subseteq A.
\]

Therefore \((rs)x \in (A : a : b)\) so that \((SS)(A : a : b) \subseteq (A : a : b)\). Hence \((A : a : b)\) is a left ideal in \( S \). 

**Corollary 3.11.** Let \( S \) be a ternary LA-semigroup with left identity. If \( A \) is a left ideal of \( S \), then \((A : B : C)\) is a left ideal of \( S \), where \( \emptyset \neq B, C \subseteq S \).

**Proof:** It is straightforward by Definition 3.10. 

**Lemma 3.12.** Let \( S \) be a ternary LA-semigroup with left identity. If \( A \) is a left ideal of \( S \), then \((SS)a\) is a left ideal of \( S \), where \( a \in S \).

**Proof:** Suppose that \( S \) is a ternary LA-semigroup with left identity. Then by Definition of ternary LA-semigroup, we have

\[
(SS)((SS)a) = (((ee)S)(SS))((SS)a) = ((SS)(ee))((SS)a) = (((SS)a)(ee))(SS) = (((ee)a)(SS))(SS) = ((SS)(SS))((ee)a) = (S((SS)S))((ee)a) = (SS)a
\]

where \( a \in S \). By Definition of left ideal, we get \((SS)a\) is a left ideal of \( S \). 

\[\square\]
Definition 3.13. Let $S$ be a ternary LA-semigroup. A left ideal $P$ is called quasi-prime if $(AB)C \subseteq P$ implies that $A \subseteq P$ or $B \subseteq P$ or $C \subseteq P$ for all left ideals $A, B$ and $C$ in $S$.

Example 3.14. By Example 3.15, $S = \{0, -a, -b, -c\}$ is a ternary LA-semigroup with left identity $a$. Then $\{0, -b\}$ is a quasi-prime ideal of $S$.

Theorem 3.15. Let $S$ be a ternary LA-semigroup with left identity. Then a left ideal $P$ of $S$ is quasi-prime if and only if $(ab)c \in P$ implies that $a \in P$ or $b \in P$ or $c \in P$, where $a, b, c \in S$.

Proof: Suppose that $S$ is a ternary LA-semigroup with left identity. Let $P$ be a left ideal of $S$. Then by Definition of left ideal, we get

\[
((SS)a)((SS)b)((SS)c) = (((SS)b)a)(SS)((SS)c) = (((ab)(SS))((SS)c) = (((SS)(SS))(ab))(SS)c = ((SS)c)((ab)(((SS)(SS)) = ((ab)c)(((SS)(SS)) = ((SS)(SS))(ab)c = ((SS)SS)((ab)c = (SS)((ab)c \subseteq (SS)P \subseteq P.
\]

By Lemma 3.12, we get $(SS)a, (SS)b, (SS)c$ are left ideals in $S$ so that $(ee)a \in (SS)a \subseteq P$ or $(ee)b \in (SS)b \subseteq P$ or $(ee)c \in (SS)c \subseteq P$. Conversely, assume that if $(ab)c \in P$, then $a \in P$ or $b \in P$ or $c \in P$, where $a, b, c \in S$. Suppose that $(AB)C \subseteq P$, where $A, B$ and $C$ are left ideals of $S$ such that $A \not\subseteq P$ and $B \not\subseteq P$. Then there exist $a \in A, b \in B$ such that $a, b \in P$. Now consider $(ab)c \in (AB)C \subseteq P$, for all $c \in C$. So by hypothesis, $c \in P$ for all $c \in C$ implies that $C \subseteq P$. Hence $P$ is a quasi-prime ideal in $S$.

Example 3.16. In the ternary LA-semigroup $\mathbb{Z}^-$ of all negative integers, the ideal $P = \{2x : x \in \mathbb{Z}^-\}$ is a quasi-prime ideal of $\mathbb{Z}^-$. But the ideal $Q = \{70x : x \in \mathbb{Z}^-\}$ is not a quasi-prime ideal of $\mathbb{Z}^-$, since $(-2)(-5)(-7) = -70 \in Q$ but $-2 \not\in Q, -5 \not\in Q$ and $-7 \not\in Q$.

Corollary 3.17. Let $S$ be a ternary LA-semigroup with left identity. Then a left ideal $P$ of $S$ is quasi-prime if and only if $a, bc \not\in P$, implies that $(ab)c \not\in P$, where $a, b, c \in S$.

Proof: It is straightforward by Theorem 3.15.

Theorem 3.18. Let $S$ be a ternary LA-semigroup with left identity. If $A$ is a left ideal of $S$ and $P$ is a quasi-prime ideal of $S$, then $A \cap P$ is a quasi-prime ideal of $A$.
Proof: Suppose that $S$ is a ternary $\mathcal{L}$A-semigroup with left identity. Clearly $A \cap P$ is a left ideal of $A$. Let $(ab)c \in A \cap P$. Then $(ab)c \in P$, since $A \cap P \subseteq P$. Since $S$ is a quasi-prime ideal of $S$, we have $a \in P$ or $b \in P$ or $c \in P$. Therefore $a \in A \cap P$ or $b \in A \cap P$ or $c \in A \cap P$. Consequently, by Theorem 3.15, we get $A \cap P$ is a quasi-prime ideal in $S$.

Theorem 3.19. Let $S$ be a ternary $\mathcal{L}$A-semigroup with left identity. If $P$ is a quasi-prime ideal of $S$, then $(P : a : b)$ is a quasi-prime ideal of $S$, where $a, b \in S$.

Proof: Assume that $P$ is a quasi-prime ideal of $S$. By Lemma 3.10, we have $(P : a : b)$ is a left ideal in $S$. Let $(xy)z \in (P : a : b)$. Then $(xy)((ab)z) = (ab)((xy)z) \in P$. By Theorem 3.15, we get $(ab)x \in (ab)P \subseteq P$ or $(ab)y \in (ab)P \subseteq P$ or $(ab)z \in P$. Therefore $x \in (P : a : b)$ or $y \in (P : a : b)$ or $z \in (P : a : b)$. Hence $(P : a : b)$ is a quasi-prime ideal of $S$.

Corollary 3.20. Let $S$ be a ternary $\mathcal{L}$A-semigroup with left identity. If $P$ is a quasi-prime ideal of $S$, then $(P : a : b)$ is a quasi-prime ideal of $S$, where $a, b \in S$.

Proof: It is straightforward by Definition 3.19.

Theorem 3.21. Let $S$ be a ternary $\mathcal{L}$A-semigroup with left identity. A left ideal $P$ of $S$ is a quasi-prime ideal if and only if $S - P$ is either ternary $\mathcal{L}$A-subsemigroup of $S$ or empty.

Proof: Suppose that $P$ is a quasi-prime ideal of $S$ and $S - P \neq \emptyset$. Let $a, b, c \in S - P$. Then $a, b, c \notin P$. By Corollary 3.17, we get $(ab)c \notin P$ so $(ab)c \in S - P$. Hence $S - P$ is a ternary $\mathcal{L}$A-subsemigroup of $S$. Conversely suppose that $S - P$ is either ternary $\mathcal{L}$A-subsemigroup of $S$ or empty. If $S - P$ is empty, then $S = P$ and hence $P$ is a quasi-prime ideal of $S$. Assume that $S - P$ is a ternary $\mathcal{L}$A-subsemigroup of $S$. Let $(ab)c \in P$. Then $(ab)c \notin S - P$. Since $S - P$ is a ternary $\mathcal{L}$A-subsemigroup, we get $c \notin S - P$ or $b \notin S - P$ or $a \notin S - P$. Therefore $a \in P$ or $b \in P$ or $c \in P$. By Theorem 3.15, we have $P$ of $S$ is a quasi-prime ideal of $S$.

4. Quasi-ideals of ternary $\mathcal{L}$A-semigroups

The results of the following theorems seem to play an important role to study quasi–ideals in ternary $\mathcal{L}$A-semigroups; these facts will be used frequently and normally we shall make no reference to this definition.

Definition 4.1. A non-empty subset $Q$ of a ternary $\mathcal{L}$A-semigroup $S$ is called a quasi-ideal of $S$ if

1. $(QS)S \cap (SQ)S \cap (SS)Q \subseteq Q$
2. $(QS)S \cap ((SS)Q)(SS) \cap (SS)Q \subseteq Q$. 
Remark 4.2. Let $S$ be a ternary LA-semigroup.

1. Each quasi-ideal $Q$ of $S$ is a ternary LA-subsemigroup. In fact, $(QQ)Q \subseteq (QS)S \cap (SQ)S \cap (SS)Q \subseteq Q$.

2. Every left ideal of $S$ is a quasi-ideal of $S$.

Proposition 4.3. Let $S$ be a ternary LA-semigroup. If $Q_i$ is a quasi-ideal of $S$, then $\bigcap_{i \in I} Q_i$ is a quasi-ideal of $S$.

Proof: Suppose that $A_i$ is a quasi-ideal of $S$. Then $(Q_iS)S \cap (SQ_i)S \cap (SS)Q_i \subseteq Q_i$ and $(Q_iS)S \cap ((SS)Q_i)(SS) \cap (SS)Q_i \subseteq Q_i$. Then by Definition of quasi-ideal, we get

$$((\bigcap_{i \in I} Q_i)S \cap (S(\bigcap_{i \in I} Q_i)))S \cap (SS) \left(\bigcap_{i \in I} Q_i\right) = \bigcap_{i \in I}((Q_iS)S \cap \bigcap_{i \in I}((SQ_i)S)$$

and

$$((\bigcap_{i \in I} Q_i)S \cap (SS) \bigcap_{i \in I} Q_i)\cap (SS)\left(\bigcap_{i \in I} Q_i\right) = \bigcap_{i \in I}((Q_iS)S \cap \bigcap_{i \in I}((SS)Q_i)(SS)\cap (SS)Q_i) \subseteq Q_i.$$

Therefore $((\bigcap_{i \in I} Q_i)S \cap (S(\bigcap_{i \in I} Q_i)))S \cap (SS)\left(\bigcap_{i \in I} Q_i\right) \subseteq \bigcap_{i \in I} Q_i$ and

$$((\bigcap_{i \in I} Q_i)S \cap (SS) \bigcap_{i \in I} Q_i)\cap (SS)\left(\bigcap_{i \in I} Q_i\right) \subseteq \bigcap_{i \in I} Q_i.$$

Hence $\bigcap_{i \in I} Q_i$ is a quasi-ideal of $S$. □

Theorem 4.4. Let $S$ be a ternary LA-semigroup with left identity. Then $S^2a\cap aS^2$ is a quasi-ideals of $S$, for every $a \in S$. 
Theorem 4.5. Let $S$ be a ternary LA-semigroup with left identity. Then by Definition of quasi-ideal, we get
\begin{align*}
((SS)a \cap a(SS))S &= (((SS)a)S \cap (a(SS))S)S \\
&= ((Sa)S^2)S \cap (SS^2)aS \\
&= (SS^2)(Sa) \cap (Sa)(SS^2) \\
&= (aS)(S^2S) \cap (S^2S)(aS) \\
&= (aS)S \cap S(aS) \\
&= (SS)a \cap a(SS).
\end{align*}

Therefore $(S^2a \cap aS^2)S \cap (S(S^2a \cap aS^2))S \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$ and $(S^2a \cap aS^2)S \cap (S(S^2a \cap aS^2))S^2 \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$. Hence $S^2a \cap aS^2$ is a quasi-ideals of $S$.

\textbf{Theorem 4.5.} Let $S$ be a ternary LA-semigroup with left identity. Then $A \cup S^2A$ is a quasi-ideals of $S$, for every $\emptyset \neq A \subseteq S$.

\textbf{Proof:} Let $S$ be a ternary LA-semigroup with left identity. Then
\begin{align*}
((SS)a \cap a(SS))S &= (((SS)a)S \cap (a(SS))S)S \\
&= ((Sa)S^2)S \cap (SS^2)aS \\
&= (SS^2)(Sa) \cap (Sa)(SS^2) \\
&= (aS)(S^2S) \cap (S^2S)(aS) \\
&= (aS)S \cap S(aS) \\
&= (SS)a \cap a(SS).
\end{align*}

Therefore $(S^2a \cap aS^2)S \cap (S(S^2a \cap aS^2))S \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$ and $(S^2a \cap aS^2)S \cap (S(S^2a \cap aS^2))S^2 \cap S(S(S^2a \cap aS^2)) \subseteq S^2a \cap aS^2$. Hence $S^2a \cap aS^2$ is a quasi-ideals of $S$. \hfill \Box

\textbf{References}


Parote Yiarayong,
Department of Mathematics
Faculty of Science and Technology
Pibulsongkram Rajabhat University
Phitsanuloke 65000, Thailand.
E-mail address: pairote0027@hotmail.com