Stabilization of Navier-Stokes Equations

Viorel Barbu (Iași, Romania)

ABSTRACT: We survey here a few recent stabilization results for Navier-Stokes equations.

Key Words: Navier-Stokes equations, Hamilton-Jacobi equation, Brownian motion.

Contents

1 Introduction 107
2 The nonlinear optimal control problem 109
3 Stabilization by noise 112

1. Introduction

We are concerned here with the stabilization of steady state solutions to Navier-Stokes equation

\[ \begin{align*}
    y_t - \nu \Delta y + (y \cdot \nabla) y &= \nabla p + mu + f_e \quad \text{in } D \times R^+ \\
    \nabla \cdot y &= 0 \quad \text{in } D \times R^+ \\
    y &= 0 \quad \text{on } \partial D \times R^+ \\
    y(0) &= y_0 \quad \text{in } D
\end{align*} \]  

(1)

where \( D \) is a bounded and smooth domain of \( R^d \), \( d = 2, 3 \) and \( f_e \in (L^2(D))^d \). Here \( m \in C_1^0(O) \) and \( m > 1 \) on \( O_1 \subset \subset O \) where \( O_1, O \) are open subdomains of \( D \).

There is a large number of recent works devoted to feedback stabilization of Navier-Stokes equations of the form (1) with internal and boundary controllers.

Let \( y_e \) be a solution to stationary equation

\[ \begin{align*}
    -\nu \Delta y_e + (y_e \cdot \nabla) y_e &= \nabla p_e + f_e \quad \text{in } D \\
    \nabla \cdot y_e &= 0 \quad \text{in } D \\
    y_e &= 0 \quad \text{on } \partial D.
\end{align*} \]  

(2)

The main result established in [4] (see also [1], [5], [6]) is that there is a stabilizing feedback controller \( u = u(t, x) \) of the form

\[ u(t, x) = -\sum_{i=1}^{M} \psi_i(x) \int_D R(y - y_e)\psi_im \, dx \]  

(3)

2000 Mathematics Subject Classification: 16N60, 16U80, 16W25
where $R$ is a self-adjoint operator to be precised below and $\{\psi_i\}_{i=1}^M$ is a given system of functions.

We set $y - y_e \mapsto y$, $H = \{y \in (L^2(D))^d; \nabla \cdot y = 0, y \cdot n = 0$ on $\partial D\}$, $P : (L^2(D))^d \to H$ is the Leray projector and

$$Ay = P \Delta y, \quad By = P(y \cdot \nabla) y,$$

with $D(A) = \{y \in H \cap (H^1_0(D))^d; \Delta y \in (L^2(D))^d\}$. Then we may write equation for $y \mapsto y - y_e$ as

$$\frac{dy}{dt}(t) + \nu Ay(t) + A_0 y(t) + By(t) = Fu$$

$$y(0) = y_0 - y_e = x$$

where $A_0 y = P((y_e \cdot \nabla)y + (y \cdot \nabla)y_e)$ and $Fu = \sum_{i=1}^M P(m \psi_i) u_i, \{\psi_i\}_{i=1}^M \subset D(A)$. We shall denote by $|\cdot|$ the norm of $H$, $(\cdot, \cdot)$ the scalar product and by $A^\alpha, 0 < \alpha < 1$, the fractional power of order $\alpha$ of operator $A$. We set also $|u|_\alpha = |A^\frac{\alpha}{2} u|$ for all $\alpha \in (0, 1)$.

The operator $R$ arising in (3) is the symmetric solution to Riccati equation

$$((\nu A + A_0)y, Ry) + \frac{1}{2} |F^* Ry|^2 = \frac{1}{2} |A^\frac{\alpha}{2} y|^2, \quad \forall y \in D(A^\frac{\alpha}{2})$$

and has the following properties (see [4])

$$R \in L(D(A^\frac{\alpha}{2}), (D(A^\frac{\alpha}{2}))'), \quad (Ry, y) \geq \delta |A^\frac{\alpha}{2} y|^2, \quad \forall y \in D(A^\frac{\alpha}{2}).$$

Here $\{\psi_i\}_{i=1}^M$ is a system of functions which belongs to space $\text{lin}\{\varphi_j\}_{j=1}^N$ of eigenfunctions for $\mathcal{A} = \nu A + A_0$ ($N$ is the number of unstable eigenvalues) and the dimension $M$ of the system is given by spectral properties of $\mathcal{A}$ ($M = 2$ if all unstable eigenvalues are simple and is maximum $2N$ in general case). As shown in [4], the feedback operator (3) exponentially stabilizes zero solution of (4) in a neighborhood

$$\mathcal{U}_\rho = \{x \in D(A^\frac{\alpha}{2}); \ |x|_2 < \rho\}.$$  

The optimal radius $\rho$ of stability domain $\mathcal{U}_\rho$ is determined by formula

$$\max_{|y|_2 \leq \rho} \frac{2|b(y, y, Ry)|}{|y|^2_2} < 1$$

where $b(y, z, w) = \int_D (y \cdot \nabla) zw \, dx, \forall y, z, w \in D(A^\frac{\alpha}{2})$. It follows that for $d = 3$, $\rho$ might be taken as

$$0 < \rho < \frac{1}{2} \|R\|_{L(D(A^\frac{\alpha}{2}), H)}.$$
Stabilization of Navier-Stokes Equations

This result is of course not optimal because it uses a linear stabilizing feedback for the linearized equation in nonlinear equation (4) and so our aim here is to find the maximal domain of stability via a nonlinear feedback control.

One might speculate that an optimal control feedback for equation (4) with cost functional

\[ J(y, u) = \frac{1}{2} \int_0^\infty (|y(t)|^2 + |u(t)|^2) dt \]  

(11)
has a wider domain of stability. The construction of such a feedback law is our aim. Here \(| \cdot |_M\) is the norm in \(R^M\).

We shall denote by \(A : D(A) = D(A) \to H\) the operator

\[ Ay = \nu Ay + A_0 y, \quad D(A) = D(A) \]  

(12)
and by \(B : D(B) \subset H \to H\) the operator defined by

\[ (By, w) = b(y, y, w), \quad \forall w \in D(A^{1/2}). \]  

(13)

We shall denote by \(W\) the space \(D(A^{1/4})\) with the norm denoted \(\| \cdot \|_{1/2}\).

In Section 2 below we shall develop this approach and refer to [2] for complete proofs. Section 3 is devoted to a different technique, stabilization by noise developed in forthcoming paper [3].

2. The nonlinear optimal control problem

Consider the control system (4), i.e.,

\[ \frac{dy}{dt} + Ay + By = Fu, \quad \forall t \geq 0, \]
\[ y(0) = x, \]  

(14)
where \(u(t) = \{u_i(t)\}_{i=1}^M\) and \(Fu = \sum_{i=1}^M u_i P(m\psi_i)(x)\). By solution to (14) on \([0, T]\) we mean a function \(y \in C([0, T]; H) \cap L^2(0, T; D(A^{1/2}))\) with \(\frac{dy}{dt} \in L^2(0, T; (D(A^{1/2}))')\) which satisfies a.e. the equation. For \(d = 2\) and \(x \in H\) there is a unique such a solution while for \(d = 3\) it exists only locally or globally (in time) for \(x\) in a suitable chosen neighborhood of origin. We shall denote by \(D\) the stabilizability domain of (14) with respect to cost functional (11), i.e.,

\[ D = \{ x \in W; \exists (y, u) \in L^2(0, \infty; D(A^{1/2})) \times L^2(0, \infty; R^M) \text{ satisfying (14)} \}. \]  

(15)
As mentioned earlier \(D \neq \emptyset\) and it contains \(U_\rho\) given by (8).

Define the function \(\varphi : D \to R\),

\[ \varphi(x) = \inf_{(y, u)} \{ J(y, u) \}, \quad \forall x \in D. \]  

(16)

We have
Proposition 1  For each \( x \in \mathcal{D} \) there is at least one pair \( (y^*, u^*) \) such that
\[
\varphi(x) = J(y^*, u^*). \tag{17}
\]
Moreover, \( y^*(t) \in \mathcal{D}, \ \forall t \geq 0. \)

Proof. Existence is standard and so it will be omitted.

Theorem 2 below is a maximum principle type result for problem (16). For the sake of simplicity we shall assume from now on that \( d = 2 \). The extension to \( d = 3 \) is however straightforward.

**Theorem 2**  Let \( (y^*, u^*) \) be optimal in problem (16). Then
\[
 u^*(t) = F^*p(t) = \left\{ \int_{\mathcal{D}} m(x)\psi_i(x)p(t, x)dx \right\}_{i=1}^{M}, \ \forall t > 0, \tag{18}
\]
where \( p \in L^2(0, \infty; H) \cap C([0, \infty); H) \cap L^\infty(0, \infty; D(A^{\frac{1}{2}})) \cap L^2(0, \infty; D(A^{\frac{1}{2}})) \) is the solution to equation
\[
\frac{dp}{dt} - A^*p - (B'(y^*))^*p = A^{\frac{3}{2}}y^* \text{ a.e. } t \geq 0. \tag{19}
\]
Here \( A^* \) is the adjoint of \( A \) in \( H \) and \( (B'(y^*))^* \) is defined by
\[
((B'(y^*))^*p, w) = b(y^*, w, p) + b(w, y^*, p), \ \forall w \in D(A^{\frac{1}{2}}). \tag{20}
\]

**Proof:** If \((y^*, u^*)\) is optimal in (16), then it is also optimal for problem
\[
\begin{align*}
\min \left\{ \int_0^\infty \left( ||y(t)||^2 + |v(t) - F^*Ry(t)|^2 \right) dt; \right. \\
\left. \frac{dy}{dt} + \mathcal{A} + FF^*Ry + By = Fv, \ y(0) = x, \ v \in L^2(0, \infty; R^M) \right\}, \tag{21}
\end{align*}
\]
too, where \( R \in L(D(A^{\frac{1}{2}})), (D(A^{\frac{1}{2}})) \cap L(D(A^{\frac{1}{2}}), H) \) is the solution to the algebraic Riccati equation (5).

Next we consider the operator \( \mathcal{L} : L^2(0, \infty; H) \rightarrow L^2(0, \infty; H) \) defined by
\[
(\mathcal{L}z)(t) = \frac{dz}{dt} + Az(t) + B'((y^*)^*)z(t) + FF^*Rz(t), \ \forall z \in D(\mathcal{L}) \tag{22}
\]
\[
D(\mathcal{L}) = \left\{ z \in L^2(0, \infty; D(A^{\frac{1}{2}})) \cap C([0, \infty); D(A^{\frac{1}{2}})); \right. \\
\left. \frac{dz}{dt} \in L^2(0, \infty; (D(A^{\frac{1}{2}}))^\prime), \\
\frac{dz}{dt} + Az \in L^2(0, \infty; H), \ z(0) = 0 \right\}. \tag{23}
\]
We have also that, if $z \in D(\mathcal{L})$, then $z \in L^2_{\text{loc}}(0, \infty; D(A))$, $\frac{dz}{dt} \in L^2_{\text{loc}}(0, \infty; H)$.

(By $L^2_{\text{loc}}(0, \infty; X)$ we mean the space of measurable functions $u : (0, \infty) \to X$ such that $u \in L^2(\delta, T; X)$ for all $0 < \delta < T < \infty$.) We set

$$W^{1,2}(0, \infty; H) = \{ z \in L^2_{\text{loc}}(0, \infty; H); \frac{dz}{dt} \in L^2_{\text{loc}}(0, \infty; H) \},$$

and it turns out that (see [2]) the operator $\mathcal{L}$ is surjective. \(\square\)

**Proof of Theorem 2.** For each $f \in L^2(0, \infty; H)$, the solution $q \in L^2(0, H; H)$ to equation

$$\frac{dq}{dt} - A^*q - (B'(y^*))^*q - (FF^*R)^*q = f, \quad t \geq 0$$

is defined by

$$\langle q, \psi \rangle_{L^2(0, \infty; H)} = -\langle f, \mathcal{L}^{-1}\psi \rangle_{L^2(0, \infty; H)}, \quad \forall \psi \in L^2(0, \infty; H)$$

and so $\mathcal{L}^*(q) = -f$ where $\mathcal{L}^*$ is the adjoint of $\mathcal{L}$.

According to this definition, the solution $p$ to equation (19) is defined by

$$\langle p, \psi \rangle_{L^2(0, \infty; H)} = (A^2 y^* - RFF^*p, \mathcal{L}^{-1}\psi)_{L^2(0, \infty; H)}, \quad \forall \psi \in L^2(0, \infty; H).$$

Since, as seen earlier, $FF^*Rz = FF^*R\mathcal{L}^{-1}\psi \in L^2(0, \infty; H)$, (26) makes sense. Now, coming back earlier, $FF^*Rz = FF^*R\mathcal{L}^{-1}\psi \in L^2(0, \infty; H)$, (26) makes sense. Now, coming back to problem (21), we see that for $v^*(t) = u^*(t) + F^*Ry^*(t)$ (optimal) we have

$$\int_0^\infty ((y^*(t), z(t))_2 + (v^*(t) - F^*Ry^*(t), v(t) - F^*Rz(t)))dt = 0$$

for all $v \in L^2(0, \infty; R^M)$, where $z$ is the solution to equation

$$\mathcal{L}(z) = Fv.$$

(Here $(\cdot, \cdot)_2$ is the scalar product in $D(A^{\frac{3}{2}})$.) Then, if $p$ is the solution to equation

$$\mathcal{L}p^* = -(A^2 y^* - R Fu^*),$$

we obtain by (27) that

$$\langle \mathcal{L}^*p, z \rangle_{L^2(0, \infty; H)} - \langle u^*, v \rangle_{L^2(0, \infty; H)} = 0, \quad \forall v \in L^2(0, \infty; H)$$

and we get (18) as claimed.

**Theorem 3** For $x \in \mathcal{U}_\rho$ and $\rho$ sufficiently small the solution $(y^*, u^*)$ to problem (16) is unique and $\varphi : W \to R$ is Gâteaux differentiable on $\mathcal{U}_\rho$. Moreover, the semigroup $t \to y^*(t, x)$ leaves invariant the set $\mathcal{U}_\rho$ and

$$u^*(t) = -F^*\nabla \varphi(y^*(t)), \quad \forall t \geq 0.$$
**Proof.** The proof of uniqueness for solution \((y^*, p)\) to system \((14), (19), (20)\) for \(x \in \mathcal{U}_\rho\) with \(\rho\) small enough follows as in [7] by standard estimates of the type used above for the solutions to system \((4), (20)\). Moreover, by \((18)\) we see that for all \(h \in \mathcal{U}_\rho\),

\[
\lim_{\lambda \downarrow 0} \frac{\varphi(x) + \lambda h - \varphi(x)}{h} = \int_0^\infty \left( |(y^*(t), z(t))|^2 + (u^*(t), v(t))_{\mathbb{R}^m} \right) dt
\]

where \((z, v)\) is the solution to system

\[
\begin{align*}
\frac{dz}{dt} + Az + B^t(y^*)z &= Fv, \quad t \geq 0 \\
z(0) &= h.
\end{align*}
\]

Then, we obtain that

\[
-(p(0), h) = \lim_{h \downarrow 0} \frac{1}{h} (\varphi(x + h) - \varphi(x)).
\]

Hence \(-p(0) = \nabla \varphi(x)\), where \(p\) is the solution to equation \((20)\). By the dynamic programming principle the latter implies also that \((30)\) holds and the flow \(t \rightarrow y^*(t, x)\) leaves invariant \(\mathcal{U}_\rho\).

**Corollary 1** The function \(\varphi\) is the unique solution on \(\mathcal{U}_\rho\) to operatorial \((\text{Hamilton–Jacobi})\) equation

\[
(Ax + Bx, \nabla \varphi(x)) + \frac{1}{2} \left| F^* \nabla \varphi(x) \right|^2 = \frac{1}{2} |x|^2, \quad \forall x \in \mathcal{U}_\rho \cap D(A). \tag{32}
\]

Moreover, \(\varphi\) is convex for a sufficiently small \(\rho\), \(\varphi(x) \geq \gamma |x|^2, \quad \forall x \in \mathcal{U}_\rho\) and \((D^2 \varphi(0) h, h) = (Rh, h) \geq \gamma |h|^2, \quad \forall h \in W\), where \(\gamma\) is a positive constant.

**Proof.** Equation \((32)\) follows by \((30)\) and the obvious relation

\[
\varphi(y^*(t)) = \frac{1}{2} \int_t^\infty (|y^*(s)|^2 + |u^*(s)|^2) ds, \quad \forall t \geq 0. \tag{33}
\]

Conversely, if \(\varphi\) is a solution to \((32)\), then \((33)\) holds and this proves uniqueness of solution \(\varphi\) to \((32)\).

We note also that \(D^2 \varphi(0) = R \in L(W, W^*) \cap L(D(A)_{\frac{1}{2}}; H)\) is the solution to the algebraic Riccati equation \((5)\) and \(D^2 \varphi \in C_b(\mathcal{U}_\rho; L(D(A)_{\frac{1}{2}}), H)\). In particular this implies that \(\varphi\) is convex in the neighborhood \(\mathcal{U}_\rho\) of the origin for \(\rho\) sufficiently small.

**3. Stabilization by noise**

Throughout in the following \(\beta_i, i = 1, \ldots\), are independent Brownian motions in a probability space \(\{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}_{t \geq 0}\).
Consider the Navier-Stokes equation (1), i.e.,
\[ X_t - \nu \Delta X + (X \cdot \nabla) X = f_e + \nabla p \quad \text{in } D \times (0, \infty) \]
\[ \nabla \cdot X = 0, \quad X|_{\partial D} = 0 \]
\[ X(0) = x, \quad D \subset \mathbb{R}^d, \quad d \geq 2. \] (34)

Let \( X_e \) be a steady-state to (34), i.e. (see (2)),
\[ -\nu \Delta X_e + (X_e \cdot \nabla) X_e = f_e + \nabla p_e \quad \text{in } D \]
\[ \nabla \cdot X_e = 0, \quad X_e|_{\partial D} = 0. \] (35)

If \( X \Rightarrow X - X_e \), equation (34) reduces to
\[ X_t - \nu \Delta X + (X \cdot \nabla) X_e + (X_e \cdot \nabla) X + (X \cdot \nabla) X = \nabla p \]
\[ \nabla \cdot X = 0, \quad X|_{\partial D} = 0 \]
\[ X(0) = x \] (36)

where \( X \) is \( x - X_e \).

Or, in the space \( H \),
\[ \dot{X}(t) + A X(t) + B X(t) = 0, \quad t \geq 0, \]
\[ X(0) = x. \] (37)

To stabilize the linearized part of (37), we associate the control stochastic problem
\[ dX + AX dt = v dW_t \]
\[ X(0) = x \] (38)

where \( W_t \) is a Wiener process in a probability space \( \{\Omega, \mathbb{P}, \mathcal{F}, \mathcal{F}_t\}_{t \geq 0} \) and \( v \) a control input to be precised below.

We recall a few properties of the Stokes–Oseen operator \( A \) defined above.

We shall denote by \( \bar{H} \) the complexified space \( \mathbb{H} + i \mathbb{H} \) with scalar product denoted \( \langle \cdot , \cdot \rangle \) and norm \( | \cdot |_{\bar{H}} \). Denote again \( A \) the extension of \( \mathcal{A} \) to this space. The operator \( A \) has a compact resolvent \( (I + A)^{-1} \) and \( -A \) generates a \( C_0 \)-analytic semigroup \( e^{-At} \) in \( \bar{H} \). Consequently, it has a countable number of eigenvalues \( \{\lambda_j\}_{j=1}^\infty \) with corresponding eigenfunctions \( \varphi_j \) each with finite algebraic multiplicity \( m_j \).

We shall denote by \( N \) the number of eigenvalues \( \lambda_j \) with \( \text{Re} \lambda_j \leq -\gamma \), \( j = 1, \ldots, N \), where \( \gamma \) is a fixed positive number.

Denote by \( \bar{P}_N \) the projector on the finite dimensional space
\[ \bar{X}_u = \text{lin span} \{ \varphi_j \}_{j=1}^N. \] (39)

We have \( \bar{X}_u = \bar{P}_N \bar{H} \) and
\[ \bar{P}_N = \frac{1}{2\pi i} \int_{\Gamma} (\lambda I + A)^{-1} d\lambda, \]
where \( \Gamma \) is a closed smooth curve in \( \mathbb{C} \) which is the boundary of a domain containing in interior the eigenvalues \( \{\lambda_j\}_{j=1}^N \).

Let \( A_u = P_N A, A_s = (I - P_N)A \). Then \( A_u, A_s \) leave invariant spaces \( \tilde{X}_u, \tilde{X}_s = (I - P_N)H \) and the spectra \( \sigma(A_u), \sigma(A_s) \) are given by (see [9])
\[
\sigma(A_u) = \{\lambda_j\}_{j=1}^N, \quad \sigma(A_s) = \{\lambda_j\}_{j=N+1}^\infty.
\]
Since \( \sigma(A_s) \subseteq \{\lambda \in \mathbb{C}; \text{Re} \lambda > -\gamma\} \) and \( A_s \) generates an analytic \( C_0 \)-semigroup on \( H \), we have
\[
|e^{-A_s t}|_H \leq C e^{-\gamma t}|x|_H, \quad \forall x \in H, \quad t \geq 0.
\]  (40)

We set
\[
\psi_j^1 = \text{Re} \varphi_j, \quad \psi_j^2 = \text{Im} \varphi_j, \quad j = 1, ..., N,
\]  (41)
and
\[
X_u = \text{lin span}\{\{\psi_j^1\} \cup \{\psi_j^2\}\}_{j=1}^N,
\]
\[
X_s = \text{lin span}\{\{\psi_j^1\} \cup \{\psi_j^2\}\}_{j=N+1}^\infty.
\]
Clearly, \( A \) leaves invariant the real spaces \( X_u \) and \( X_s \). More precisely, we have
\[
\tilde{X}_u = X_u + iX_u, \quad \tilde{X}_s = X_s + iX_s
\]
and therefore, \( H = X_u \oplus X_s \), the direct sum of \( X_u \) and \( X_s \) (see, e.g., [9]). Since the system \( \{\lambda_j\}_{j=1}^N \) is of the form
\[
\{\xi_j \pm i\eta_j\}_{j=1}^{2M}, \quad \{\delta_j\}_{j=1}^{M_0}, \quad 2M + M_0 = N, \quad \xi_j, \eta_j, \delta_j \in \mathbb{R}
\]
it follows that
\[
X_u = \text{lin span}\{\psi_j^1\}_{j=1}^N,
\]  (42)
where
\[
\psi_j = \psi_j^1, \quad 1 \leq j \leq M, \quad \psi_j = \psi_j^2, \quad M < j \leq 2M, \quad \psi_j = \varphi_j \quad (\text{real eigenfunctions}), \quad 2M < j \leq N.
\]  (43)
The infinite dimensional space \( X_s \) is similarly generated and estimate (40) remains valid in the \( \cdot |_N \)-norm for \( A_s \) defined on \( X_s \subseteq H \). We shall denote by \( P_N \) the projector corresponding to the decomposition \( H = X_u \oplus X_s \), i.e.,
\[
X_u = P_N H, \quad X_s = (I - P_N)H.
\]

Consider the orthonormal system \( \{\phi_j\}_{j=1}^N \) obtained from \( \{\psi_j\}_{j=1}^N \) by the Schmidt orthogonalization procedure.

Consider the following stochastic perturbation of the linearized system (5) (see (6))
\[
dX + AX \, dt = n \sum_{i=1}^N (X(t), \phi_i) P(m\phi_i) d\beta_i(t)
\]
\[
X(0) = x,
\]  (44)
where $\eta \in R$ and $m = \chi_{O_0}$ is the characteristic of the open subset $O_0 \subset D$.

Equation (44) can be seen as a closed loop system associated to the controlled equation

$$dX + AX \, dt = \eta \sum_{i=1}^{N} v_i(t) P(m\phi_i) \, d\beta_i(t), \quad t \geq 0,$$

$$X(0) = x,$$

with the feedback controller $v_i = (X(t), \phi_i), \; i = 1, ..., N$.

**Theorem 4** Let $X = X(t, x)$ be the solution to (45). Then, for $|\eta|$ sufficiently large, we have

$$P \left\{ \lim_{t \to \infty} e^{\gamma t} X(t, x) = 0 \right\} = 1, \; \forall \; x \in H,$$

The closed loop system (44) can be equivalently written as

$$dX(t) - \nu \Delta X(t) \, dt + (X(t) \cdot \nabla) X_e \, dt + (X_e \cdot \nabla) X(t) \, dt$$

$$= \eta m \sum_{i=1}^{N} (X(t), \phi_i) \phi_i \, d\beta_i(t) + \nabla p(t) \, dt \quad \text{in} \; (0, \infty) \times D$$

$$\nabla \cdot X(t) = 0, \; X(t) |_{\partial D} = 0$$

$$X(0) = x \; \text{in} \; D.$$  

In particular, it follows by Theorem 4 that the feedback controller $u_i = \eta m(X - X_e, \phi_i) \phi_i, \; i = 1, ..., N$, stabilizes exponentially the stationary solution $X_e$, i.e., we have

**Corollary 2.1.** The solution $X$ to closed loop system

$$dX(t) - \nu \Delta X(t) \, dt = \eta m \sum_{i=1}^{N} (X(t) - X_e, \phi_i) \phi_i d\beta_i(t) + \nabla p(t) \, dt, \quad t \geq 0$$

$$\nabla \cdot X = 0, \; X|_{\partial D} = 0$$

$$X(0) = x$$

satisfies

$$P \left[ \lim_{t \to \infty} (X(t) - X_e) e^{\gamma t} = 0 \right] = 1, \; \forall \; x \in H.$$

The proof of Theorem 4 follows from same sharp arguments involving the martingale theory and is given in [3].

**References**

3. V. Barbu, The internal stabilization by noise of the linearized Navier-Stokes equation (submitted).


Viorel Barbu  
Iaşi, Romania  
“Al. I. Cuza” University, Iasi