On a class of topological groups

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ABSTRACT: A topological group is an SNS-group if its identity element possesses a fundamental system of neighborhoods formed by normal subgroups. In this paper we prove the existence of initial SNS-topologies, from which we derive that the class of SNS-groups is closed under the formation of products and projective limits, and we prove the existence of final SNS-topologies, from which we derive that the class of SNS-groups is closed under the formation of free products and inductive limits.

Key Words: topological groups, linearly topologized groups.

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1. Introduction

Although the class of linearly topologized groups (a linearly topologized group is an abelian topological group whose identity element possesses a fundamental system of neighborhoods formed by subgroups) is quite rich in examples, there exist important non-abelian SNS-groups which occur, for example, in Galois Theory. This fact has motivated us to write the present paper, where basic results concerning SNS-groups have been established.

The main purpose of the paper is the discussion of the fundamental constructions in the class of SNS-groups. In this regard, it is shown that this class is closed under the formation of products, projective limits, free products and inductive limits. Moreover, we emphasize the universal properties satisfied by the objects which have been constructed. It should also be mentioned that, although part of the results presented here are well-known in the mathematical folklore, the ones concerning topological free products and topological inductive limits are possibly new.

Throughout this paper $G$ is an arbitrary group whose identity element is denoted by $e$, unless otherwise specified. In the whole paper, the operation on a group has been written multiplicatively.

Definition 1. A topological group $(G, \tau)$ is said to be an SNS-group, and $\tau$ is said to be an SNS-topology on $G$, if $e$ admits a fundamental system of $\tau$-neighborhoods consisting of normal subgroups of $G$. 

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In the case where \( G \) is an abelian group, to say that \( \tau \) is an SNS-topology on \( G \) is equivalent to saying that \( \tau \) is a linear topology on \( G \); we refer to [5] and [6] for the study of such topologies.

It is easily seen that the bilateral completion ([9], §5) of a separated SNS-group is an SNS-group.

**Example 2.** The discrete topology and the chaotic topology on a group \( G \) are SNS-topologies on \( G \).

**Example 3.** If \((E, \tau)\) is a linearly topologized module ([9], Definition 31.4), then the underlying additive topological group is an SNS-group.

**Example 4.** If \((E, \tau)\) is a non-archimedean topological vector space over a non-trivially valued division ring ([7], Definition 3.2), then the underlying additive topological group is an SNS-group in view of Theorem 3.7 of [7].

**Example 5.** Let \( A \) be a non-empty set and \((G, \tau)\) an SNS-group. If \( F(A, G) \) is the group of all mappings from \( A \) into \( G \) and \( \tau_A \) is the topology of uniform convergence on \( F(A, G) \), then \((F(A, G), \tau_A)\) is an SNS-group.

In fact, it is easily seen that \( \tau_A \) is a group topology on \( F(A, G) \). Moreover, if \( V \) is a fundamental system of \( \tau_- \)neighborhoods of \( e \) consisting of normal subgroups of \( G \), then the sets

\[
\{ f \in F(A, G); f(A) \subset V \} \quad (V \in V)
\]

form a fundamental system of \( \tau_A \)-neighborhoods of the identity element of \( F(A, G) \) consisting of normal subgroups of \( F(A, G) \).

**Proposition 6.** If \( B \) is a filter base on a group \( G \) consisting of normal subgroups of \( G \), then there exists a unique SNS-topology on \( G \) for which \( B \) is a fundamental system of neighborhoods of \( e \).

**Proof.** Follows immediately from Corollary 1.5 of [9].

**Example 7.** Let \( G \) be an abelian group and let \( p \) be a positive prime number. It is easily seen that \( B = \{ p^n G; n \in \mathbb{N} \} \) is a filter base on \( G \) consisting of subgroups of \( G \) (observe that \( p^n G \cap p^m E = p^m G \) if \( n \geq m \)). By Proposition 6, there exists a unique SNS-topology on \( G \) for which \( B \) is a fundamental system of neighborhoods of \( e \), called the \( p \)-adic topology on \( G \).

**Example 8.** Let \( F, G \) and \( H \) be three abelian groups and let \( e_F \) (resp. \( e_G, e_H \)) be the identity element of \( F \) (resp. \( G, H \)). Let \( B: F \times G \to H \) be a \( \mathbb{Z} \)-bilinear mapping.

For each finite subset \( \{ y_1, \ldots, y_m \} \) of \( G \) put

\[
U_{\{y_1, \ldots, y_m\}} = \{ x \in F; B(x, y_i) = e_H \text{ for } i = 1, \ldots, m \}.
\]

It is easily seen that the set \( B \) formed by all the sets \( U_{\{y_1, \ldots, y_m\}} \) is a filter base on \( F \) consisting of subgroups of \( F \). By Proposition 6, there exists a unique SNS-topology on \( F \) for which \( B \) is a fundamental system of neighborhoods of \( e_F \).

Similarly, for each finite subset \( \{ x_1, \ldots, x_m \} \) of \( F \) put

\[
V_{\{x_1, \ldots, x_m\}} = \{ y \in G; B(x_i, y) = e_H \text{ for } i = 1, \ldots, m \}.
\]
Let us mention two basic examples of \(\mathbb{Z}\)-bilinear mappings. For \(F, G\) and \(H\) as above, let \(\text{Hom}(F, G)\) (resp. \(\text{Hom}(G, H), \text{Hom}(F, H)\)) be the abelian group of all group homomorphisms from \(F\) into \(G\) (resp. \(G\) into \(H\), \(F\) into \(H\)). Then the mappings
\[
(x, u) \in F \times \text{Hom}(F, G) \mapsto u(x) \in G
\]
and
\[
(u, v) \in \text{Hom}(F, G) \times \text{Hom}(G, H) \mapsto v \circ u \in \text{Hom}(F, H)
\]
are \(\mathbb{Z}\)-bilinear.

**Example 9.** Let \(N\) be a Galois extension of a field \(K\) and \(\Gamma\) the Galois group of \(N\) over \(K\). For each intermediate field \(K \subset L \subset N\) which is a finite Galois extension of \(K\), let \(g(L)\) be the Galois group of \(N\) over \(L\); then \(g(L)\) is a normal subgroup of \(\Gamma\) and the set \(\mathcal{B}\) formed by all the sets \(g(L)\) constitutes a filter base on \(\Gamma\) (see Appendix II of [3] for the details). By Proposition 6, there exists a unique SNS-topology on \(\Gamma\) for which \(\mathcal{B}\) is a fundamental system of neighborhoods of the identity element \(1_N\) of \(\Gamma\).

### 2. Initial and final SNS-topologies

Now let us begin the discussion of initial SNS-topologies. **Theorem 10.** Let \(((G_i, \tau_i))_{i \in I}\) be a non-empty family of SNS-groups, \(G\) a group and, for each \(i \in I\), let \(u_i: G \to G_i\) be a group homomorphism. If \(\tau\) is the initial topology on \(G\) for the family \(((G_i, \tau_i), u_i)_{i \in I}\) ([2], p.28, Proposition 4), then \((G, \tau)\) is an SNS-group.

**Proof.** By Theorem 1.9 of [9], \((G, \tau)\) is a topological group. For each \(i \in I\) let \(\mathcal{V}_i\) be a fundamental system of \(\tau\)-neighborhoods of the identity element \(e_i\) of \(G_i\). For each finite subset \(\{i_1, \ldots, i_m\}\) of \(I\) and for each \(V_{i_1} \in \mathcal{V}_{i_1}, \ldots, V_{i_m} \in \mathcal{V}_{i_m}\), consider the set
\[
u_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap \nu_{i_m}^{-1}(V_{i_m}),
\]
which is a normal subgroup of \(G\). Since all these sets constitute a fundamental system of \(\tau\)-neighborhoods of \(e\), then \((G, \tau)\) is an SNS-group, as was to be shown.

**Remark 11.** Under the conditions of Theorem 10, if \(\tau_i\) is the chaotic topology on \(G_i\) for all \(i \in I\), then \(\tau\) is the chaotic topology on \(G\).

**Example 12.** Let \(X\) be a non-empty set and \(\mathcal{A}\) a set of non-empty subsets of \(X\). Let \((G, \tau)\) be an SNS-group and \(\mathcal{F}(X, G)\) the group of all mappings from \(X\) into \(G\). For each \(A \in \mathcal{A}\) consider the group homomorphism
\[
u_A: f \in \mathcal{F}(X, G) \mapsto f|_A \in \mathcal{F}(A, G).
\]
By Theorem 10, \(\mathcal{F}(X, G)\) endowed with the initial topology \(\tau_{\mathcal{A}}\) for the family \(((\mathcal{F}(A, G), \tau_A), u_A)_{A \in \mathcal{A}}\) is an SNS-group \(((\mathcal{F}(A, G), \tau_A)\) being as in Example 5). \(\tau_{\mathcal{A}}\) is the topology of \(\mathcal{A}\)-convergence on \(\mathcal{F}(X, G)\).

**Corollary 13.** Let \((H, \theta)\) be an SNS-group, \(G\) a group and \(u: G \to H\) a group homomorphism. If \(\tau\) is the inverse image of \(\theta\) under \(u\), then \((G, \tau)\) is an SNS-group.
In particular, every subgroup of an SNS-group is an SNS-group under the induced topology.

Proof. Follows immediately from Theorem 10.

Example 14. Let \((X, \tau)\) be a non-empty topological space, \(A\) a set of non-empty subsets of \(X\), \((G, \theta)\) an SNS-group and \(\mathcal{C}(X, G)\) the subgroup of \(\mathcal{F}(X, G)\) consisting of all continuous mappings from \((X, \tau)\) into \((G, \theta)\). Then, in view of Corollary 13, \(\mathcal{C}(X, G)\) is an SNS-group under the topology induced by \(\tau_{A}\) (\(\tau_{A}\) being as in Example 12).

Corollary 15. If \(\{(G_i, \tau_i)\}_{i \in I}\) is a non-empty family of SNS-groups, \(G\) is the product group \(\prod_{i \in I} G_i\) and \(\tau\) is the product topology \(\prod_{i \in I} \tau_i\) on \(G\), then \((G, \tau)\) is an SNS-group.

Proof. Follows immediately from Theorem 10.

Remark 16. Let \(\{(G_i, \tau_i)\}_{i \in I}\) and \((G, \tau)\) be as in Corollary 15 and, for each \(i \in I\), let \(pr_i : G \to G_i\) be the projection on the \(i\)-th factor. Then \((G, \tau)\) satisfies the following universal property: for each SNS-group \((H, \theta)\), the mapping

\[u \in \text{Hom}_{\mathcal{C}}(H, G) \mapsto (pr_i \circ u)_{i \in I} \in \prod_{i \in I} \text{Hom}_{\mathcal{C}}(H, G_i)\]

is a group isomorphism, where \(\text{Hom}_{\mathcal{C}}(H, G)\) is the group of all continuous group homomorphisms from \((H, \theta)\) into \((G, \tau)\). \(\text{Hom}_{\mathcal{C}}(H, G_i)\) is the group of all continuous group homomorphisms from \((H, \theta)\) into \((G_i, \tau_i)\) for all \(i \in I\) and \(\prod_{i \in I} \text{Hom}_{\mathcal{C}}(H, G_i)\) is the corresponding product group.

Corollary 17. If \(\{\tau_i\}_{i \in I}\) is a non-empty family of SNS-topologies on a group \(G\) and \(\tau = \sup \tau_i\), then \((G, \tau)\) is an SNS-group.

Proof. Follows immediately from Theorem 10.

Definition 18. Let \(\{(G_i, \tau_i), u_{ij}\}_{i \in I}\) be a projective system of SNS-groups (this means that \(I\) is a non-empty set endowed with a partial order \(\leq\), \((G_i, \tau_i)\) is an SNS-group for all \(i \in I\), \(u_{ij} : (G_j, \tau_j) \to (G_i, \tau_i)\) is a continuous group homomorphism for \(i, j \in I\) with \(i \leq j\), \(u_{ii} = 1_{G_i}\) for all \(i \in I\) and \(u_{ik} = u_{ij} \circ u_{jk}\) for \(i, j, k \in I\) with \(i \leq j \leq k\)).

\[
\begin{array}{ccc}
(G_k, \tau_k) & \xrightarrow{u_{ik}} & (G_i, \tau_i) \\
{u_{jk}} & \searrow & {u_{ij}} \\
(G_j, \tau_j)
\end{array}
\]

For each \(i \in I\) let \(pr_i : G = \prod_{i \in I} G_i \to G_i\) be the projection on the \(i\)-factor. Put

\[\lim G_i = \{x \in G; (u_{ij} \circ pr_j)(x) = pr_i(x) \text{ for all } i, j \in I \text{ with } i \leq j\},\]

which is a subgroup of the product group \(G = \prod_{i \in I} G_i\) (note that \(\lim G_i = G\) if \(\leq\) is the equality relation). For each \(i \in I\) let \(u_i = pr_i | (\lim G_i)\), which is called the canonical group homomorphism from \(\lim G_i\) into \(G_i\). If \(\tau\) is the initial topology on \(\lim G_i\) for the family \(\{(G_i, \tau_i), u_i\}_{i \in I}\), which makes \(\lim G_i\) an SNS-group in view
of Theorem 10, then \((\lim G_i, \tau)\) is said to be the topological projective limit of the system \(((G_i, \tau_i), u_{ij})_{i \in I}\).

**Remark 19.** The projective limit topology for the system \(((G_i, \tau_i), u_{ij})_{i \in I}\) is an SNS-topology since, by definition ([2], p.51), it is precisely the topology \(\tau\) considered in Definition 18.

**Proposition 20.** Let \((\lim G_i, \tau)\) be the topological projective limit of the projective system \(((G_i, \tau_i), u_{ij})_{i \in I}\) of SNS-groups. Then the topology \(\theta\) on \(\lim G_i\) induced by the product topology \(\prod_{i \in I} \tau_i\) coincides with \(\tau\).

**Proof.** Let \(H = \lim G_i\), and let \(pr_i\) and \(u_i\) be as in Definition 18. For each \(i \in I\) let \(V_i\) be the set of all \(\tau_i\)-neighborhoods of the identity element \(e_i\) of \(G_i\). If \(i_1, \ldots, i_m \in I\), \(V_{i_1} \in V_{i_1}, \ldots, V_{i_m} \in V_{i_m}\), we have

\[
\begin{align*}
[pr_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap pr_{i_m}^{-1}(V_{i_m})] \cap H &= (pr_{i_1}^{-1}(V_{i_1}) \cap H) \cap \cdots \cap (pr_{i_m}^{-1}(V_{i_m}) \cap H) \\
&= u_{i_1}^{-1}(V_{i_1}) \cap \cdots \cap u_{i_m}^{-1}(V_{i_m}).
\end{align*}
\]

Consequently, \(\theta = \tau\), as asserted.

**Example 21.** Let \(p\) be a positive prime number. Let \(Z\) be the additive group of integers and, for each positive integer \(m\), let \(G_m\) be the quotient group \(\mathbb{Z}/p^m\mathbb{Z}\) endowed with the discrete topology \(\tau_m\) \(((G_m, \tau_m)\) is an SNS-group by Example 2). For \(m \leq n\) let \(u_{mn}: G_n \to G_m\) be the canonical group homomorphism, which is obviously continuous from \((G_n, \tau_n)\) into \((G_m, \tau_m)\). Then \(((G_m, \tau_m), u_{mn})_{m \in \mathbb{N}}\) is a projective system of compact SNS-groups, and hence we can consider the topological projective limit \((\lim G_m, \tau)\) of this system. Since \(\lim G_m\) is closed in \(\left(\prod_{m \in \mathbb{N}} G_m, \prod_{m \in \mathbb{N}} \tau_m\right)\) and \(\left(\prod_{m \in \mathbb{N}} G_m, \prod_{m \in \mathbb{N}} \tau_m\right)\) is compact, it follows from Proposition 20 that \((\lim G_m, \tau)\) is compact. If \(\mathbb{Z}_p\) is the linearly topologized ring of \(p\)-adic integers ([8], p.11), then the underlying additive topological group is \((\lim G_m, \tau)\).

The topological projective limit \((\lim G_i, \tau)\) of the projective system \(((G_i, \tau_i), u_{ij})_{i \in I}\) of SNS-groups satisfies the following universal property:

**Proposition 22.** Let \((H, \theta)\) be an SNS-group and, for each \(i \in I\), let \(\alpha_i: (H, \theta) \to (G_i, \tau_i)\) be a continuous group homomorphism such that \(u_{ij} \circ \alpha_j = \alpha_i\) for \(i \leq j\). Then there exists a unique continuous group homomorphism \(u: (H, \theta) \to (\lim G_i, \tau)\) such that \(\alpha_i = u \circ u_i\) for all \(i \in I\) (\(u_i\) being as in Definition 18).

\[
\begin{array}{ccc}
(G_i, \tau_i) & \xrightarrow{u_{ij}} & (G_j, \tau_j) \\
\alpha_j & \xleftarrow{\alpha_j} & (H, \theta) \\
\alpha_i & \xrightarrow{u_i} & (\lim G_i, \tau) \\
\end{array}
\]

**Proof.** Let \(y \in H\) be arbitrary. It is clear that \(u(y) = (\alpha_i(y))_{i \in I}\) is the unique element of \(\prod_{i \in I} G_i\) satisfying \(\alpha_i(y) = pr_i(u(y))\) for all \(i \in I\). Moreover, \(u(y) \in \lim G_i\) because \((u_{ij} \circ \alpha_j)(y) = \alpha_i(y)\) for \(i \leq j\). It then follows that the mapping \(u: H \to \lim G_i\) so defined is the unique group homomorphism such that \(\alpha_i = u_i \circ u\) for all
Let \( i \in I \). Finally, \( u: (H, \theta) \rightarrow (\text{lim} G_i, \tau) \) is continuous because \( u_i \circ u: (H, \theta) \rightarrow (G_i, \tau_i) \) is continuous for all \( i \in I \).

**Corollary 23.** Let \( \{(\text{lim} G_i, \tau), u_{ij}\}_{i,j \in I} \) and \( \{(H_i, \theta_i), v_{ij}\}_{i,j \in I} \) be two projective systems of SNS-groups, and let \( \text{lim} G_i, \tau \) and \( \text{lim} H_i, \theta \) be the corresponding topological projective limits. For each \( i \in I \) let \( \beta_i: (G_i, \tau_i) \rightarrow (H_i, \theta_i) \) be a continuous group homomorphism such that \( v_{ij} \circ \beta_i = \beta_j \circ u_{ij} \) for all \( i \leq j \). Then there exists a unique continuous group homomorphism \( u: \text{lim} G_i \rightarrow \text{lim} H_i \) such that \( v_i \circ u = \beta_i \circ u_i \) for all \( i \in I \), where \( u_i: \text{lim} G_i \rightarrow G_i \) and \( v_i: \text{lim} H_i \rightarrow H_i \) are the canonical group homomorphisms \( i \in I \):

\[
\begin{array}{ccc}
(G_j, \tau_j) & \xrightarrow{\beta_j} & (H_j, \theta_j) \\
\downarrow u_{ij} & & \downarrow v_{ij} \\
(G_i, \tau_i) & \xrightarrow{\beta_i} & (H_i, \theta_i) \\
\end{array}
\]

\[
\xrightarrow{u} \quad \xrightarrow{\text{lim} G_i, \tau} \quad \text{lim} H_i, \theta
\]

**Proof.** Put \( \alpha_i = \beta_i \circ u_i \) for \( i \in I \); then \( \alpha_i \) is a continuous group homomorphism from \( \text{lim} G_i, \tau \) into \( (H_i, \theta_i) \). Since

\[
v_{ij} \circ \alpha_j = (v_{ij} \circ \beta_j) \circ u_j = (\beta_i \circ u_{ij}) \circ u_j = \beta_i \circ (u_{ij} \circ u_j) = \beta_i \circ u_i = \alpha_i
\]

for \( i \leq j \), Proposition 22 guarantees the existence of a unique continuous group homomorphism \( u: \text{lim} G_i \rightarrow \text{lim} H_i \) such that \( \alpha_i = v_i \circ u \) for all \( i \in I \). This completes the proof.

Now let us turn to the discussion of final SNS-topologies.

**Theorem 24.** Let \( \{(G_i, \tau_i)\}_{i \in I} \) be a non-empty family of SNS-groups and let \( G \) be a group. For each \( i \in I \) let \( u_i: G_i \rightarrow G \) be a group homomorphism. Then there exists a unique SNS-topology \( \tau \) on \( G \) which is final for the family \( \{(G_i, \tau_i), u_i\}_{i \in I} \), in the following sense: for every SNS-group \( (H, \theta) \) and for every group homomorphism \( u: G \rightarrow H \), we have that \( u: (G, \tau) \rightarrow (H, \theta) \) is continuous if and only if \( u \circ u_i: (G_i, \tau_i) \rightarrow (H, \theta) \) is continuous for all \( i \in I \).

**Proof.** For each \( i \in I \) let \( \mathcal{V}_i \) be the set of all \( \tau_i \)-neighborhoods of the identity element \( e_i \) of \( G_i \). Put

\[
\mathcal{B} = \{ U \subset G; U \text{ is a normal subgroup of } G \text{ and } u_i^{-1}(U) \in \mathcal{V}_i \text{ for all } i \in I \}.
\]

Clearly, \( \mathcal{B} \) is a filter base on \( G \). Let \( \tau \) be the SNS-topology on \( G \) for which \( \mathcal{B} \) is a fundamental system of \( \tau \)-neighborhoods of \( e \) (Proposition 6). By construction, \( u_i: (G_i, \tau_i) \rightarrow (G, \tau) \) is continuous for all \( i \in I \).

We claim that \( \tau \) is final for the family \( \{(G_i, \tau_i), u_i\}_{i \in I} \). Indeed, let \( (H, \theta) \) be an SNS-group and let \( u: G \rightarrow H \) be a group homomorphism. If \( u: (G, \tau) \rightarrow (H, \theta) \) is continuous, then \( u \circ u_i: (G_i, \tau_i) \rightarrow (H, \theta) \) is continuous for all \( i \in I \). Conversely, assume that \( u \circ u_i: (G_i, \tau_i) \rightarrow (H, \theta) \) is continuous for all \( i \in I \), and let \( V \) be a \( \theta \)-neighborhood of the identity element \( f \) of \( H \) which is a normal subgroup of \( H \). Then \( u_i^{-1}(V) \) is a normal subgroup of \( G \) and \( u_i^{-1}(u_i^{-1}(V)) = (u \circ u_i)^{-1}(V) \in \mathcal{V}_i \) for all \( i \in I \); thus \( u^{-1}(V) \in \mathcal{B} \). Therefore \( u: (G, \tau) \rightarrow (H, \theta) \) is continuous, proving our claim.
In order to prove the uniqueness, let \( \tilde{\tau} \) be an SNS-topology on \( G \) such that \( u_i: (G_i, \tau_i) \to (G, \tilde{\tau}) \) is continuous for all \( i \in I \), and consider the identity mapping \( 1_G: (G, \tau) \to (G, \tilde{\tau}) \). Since \( 1_G \circ u_i: (G_i, \tau_i) \to (G, \tilde{\tau}) \) is continuous for all \( i \in I \), it follows that \( 1_G: (G, \tau) \to (G, \tilde{\tau}) \) is continuous. Thus \( \tilde{\tau} \) is the finest SNS-topology on \( G \) which makes all the \( u_i \) continuous, and hence the uniqueness is established. This completes the proof.

**Remark 25.** Under the conditions of Theorem 24, if \( \tau_i \) is the discrete topology on \( G_i \) for all \( i \in I \), then \( \tau \) is the discrete topology on \( G \).

**Corollary 26.** Every non-empty family of SNS-topologies on a group \( G \) admits an infimum in the partially ordered set of all SNS-topologies on \( G \).

**Proof.** Follows immediately from Theorem 24.

**Corollary 27.** Let \((G, \tau)\) be an SNS-group, \(H\) a normal subgroup of \(G\) and \( \pi: G \to G/H \) the canonical surjection. Then the quotient topology \( \tau' \) on \( G/H \) coincides with the final SNS-topology \( \tau'' \) for the pair \(((G, \tau), \pi)\).

**Proof.** It is easily seen that \((G/H, \tau')\) is an SNS-group and that \( \tau' \) is finer than \( \tau'' \). Therefore \( \tau' = \tau'' \), as asserted.

**Definition 28.** Let \( ((G_i, \tau_i))_{i \in I} \) be a non-empty family of SNS-groups and let \( G \) be the free product of the family \( (G_i)_{i \in I} \) ([1]; [4], p.24). For each \( i \in I \) let \( u_i: G_i \to G \) be the canonical group homomorphism. If \( \tau \) is the final SNS-topology on \( G \) for the family \( ((G_i, \tau_i), u_i)_{i \in I} \), \((G, \tau)\) is said to be the topological free product of the family \( ((G_i, \tau_i))_{i \in I} \).

**Remark 29.** Let \( ((G_i, \tau_i))_{i \in I} \), \( G \), \( u_i \) and \( \tau \) be as in Definition 28. Then \((G, \tau)\) satisfies the following universal property: for each SNS-group \((H, \theta)\), the mapping

\[
u \in \text{Hom}_c(G, H) \mapsto (u \circ u_i)_{i \in I} \in \prod_{i \in I} \text{Hom}_c(G_i, H)
\]

is a group isomorphism, where \( \text{Hom}_c(G, H) \) is the group of all continuous group homomorphisms from \((G, \tau)\) into \((H, \theta)\), \( \text{Hom}_c(G_i, H) \) is the group of all continuous group homomorphisms from \((G_i, \tau_i)\) into \((H, \theta)\) for all \( i \in I \) and \( \prod_{i \in I} \text{Hom}_c(G_i, H) \) is the corresponding product group.

Before we proceed let us establish an auxiliary result which is probably known:

**Proposition 30.** Let \( T \) be a non-empty subset of a group \( G \). Then the normal subgroup of \( G \) generated by \( T \) (that is, the smallest normal subgroup of \( G \) containing \( T \)) is the set \( N \) of all finite products of elements of the form \( gtg^{-1} \), where \( g \in G \) and \( t \) is either an element of \( T \) or the inverse of an element of \( T \).

**Proof.** Obviously, \( e \in N \) and \( ab^{-1} \in N \) for all \( a, b \in N \). Let \( g \in G \) and \( a = g t_1 g_1^{-1} g t_2 g_2^{-1} \ldots g t_m g_m^{-1} \in N \). Then \( g a g^{-1} = (g t_1)(g_1^{-1} (g t_2)(g_2^{-1} \ldots (g t_m)(g_m^{-1}) \in N \). Thus \( N \) is a normal subgroup \( G \), which clearly contains \( T \). Finally, it is clear that any normal subgroup of \( G \) containing \( T \) also contains \( N \).

**Proposition 31.** Let \( (G_i, u_{ij})_{i \in I} \) be an inductive system of groups and let \( G \) be the free product of the family \( (G_i)_{i \in I} \). Let \( T \) be the subset of \( G \) formed by all elements of the form \( u_i(x_i)(u_j \circ u_{ij})(x_j^{-1}) \) \( (i \leq j, x_i \in E_i), \) \( u_i \) being as in Definition 28, and let \( N \) be the normal subgroup of \( G \) generated by \( T \). Let \( \text{lim}_G \), \( \pi: G \to \text{lim}_G \), the canonical surjection and, for each
\(i \in I\), let \(v_i = \pi \circ u_i\) be the canonical group homomorphism. Then \(v_j \circ u_{ji} = v_i\) for \(i \leq j\). Moreover, if \(H\) is a group and, for each \(i \in I\), \(\alpha_i : G_i \to H\) is a group homomorphism such that \(\alpha_i \circ u_{ji} = \alpha_i\) for \(i \leq j\), then there exists a unique group homomorphism \(u : \lim \longrightarrow G_i \to H\) such that \(\alpha_i = u \circ v_i\) for all \(i \in I\).

\[
\begin{array}{cccc}
G_i & \xrightarrow{v_i} & \lim G_i & \xrightarrow{\pi} \\
G & \alpha_i & G_j & \xrightarrow{v_i} H \\
G_i & \xrightarrow{\alpha_j} & G_i & \xrightarrow{\alpha_i}
\end{array}
\]

**Proof.** Firstly, let us verify that \(v_j \circ u_{ji} = v_i\) for \(i \leq j\). In fact, for all \(x_i \in E_i\),

\[
v_i(x_i)( (v_j \circ u_{ji})(x_i) )^{-1} = (\pi \circ u_i)(x_i)( (\pi \circ u_j \circ u_{ji})(x_i) )^{-1} \\
= (\pi \circ u_i)(x_i)( (\pi \circ u_j \circ u_{ji})(x_i^{-1}) ) = \pi(u_i(x_i)(u_j \circ u_{ji})(x_i^{-1})) \\
= \pi(\pi(x_i)(\alpha_j \circ u_{ji})(x_i^{-1})) = \pi(x_i)(\alpha_i \circ (x_i^{-1})) = f.
\]

Therefore it follows from Proposition 30 that \(N \subseteq \text{Ker}(\bar{u})\).

By the isomorphism theorem, there exists a unique group homomorphism \(u : \lim \longrightarrow G_i \to H\) such that \(\bar{u} = u \circ \pi\). Moreover,

\[
u \circ v_i = (u \circ \pi) \circ u_i = \bar{u} \circ u_i = \alpha_i
\]

for all \(i \in I\).

Finally, let \(v : \lim \longrightarrow G_i \to H\) be a group homomorphism such that \(v \circ v_i = \alpha_i\) for all \(i \in I\), and put \(\bar{v} = v \circ \pi\). Then

\[
\bar{v} \circ u_i = v \circ (\pi \circ u_i) = v \circ v_i = \alpha_i
\]

for all \(i \in I\). Consequently, \(\bar{v} = \bar{u}\), and hence \(v = u\). This completes the proof.

**Definition 32.** Let \((G_i, \tau_i, u_{ji})_{i \in I}\) be an inductive system of SNS-groups (this means that \(I\) is a non-empty set endowed with a partial order \(\leq\), \((G_i, \tau_i)\) is an SNS-group for all \(i \in I\), \(u_{ji} : (G_i, \tau_i) \to (G_j, \tau_j)\) is a continuous group homomorphism for \(i, j \in I\) with \(i \leq j\), \(u_{ii} = 1_{G_i}\) for all \(i \in I\) and \(u_{ki} = u_{kj} \circ u_{ji}\) for \(i, j, k \in I\) with \(i \leq j \leq k\). Let \(\lim G_i\) and \(v_i (i \in I)\) be as in Proposition 31. If \(\tau\) is the final SNS-topology on \(\lim G_i\) for the family \((G_i, \tau_i, v_i)_{i \in I}\), then \((\lim G_i, \tau)\) is said to be the topological inductive limit of the system \((G_i, \tau_i, u_{ji})_{i \in I}\).

**Proposition 33.** Let \((\lim G_i, \tau)\) be the topological inductive limit of the inductive system \((G_i, \tau_i, u_{ji})_{i \in I}\) of SNS-groups and let \((G, \tilde{\tau})\) be the topological free product of the family \(((G_i, \tau_i))_{i \in I}\). Then the quotient topology \(\tilde{\tau}\) on \(G/N(= \lim G_i)\) coincides with \(\tau\), \(N\) being as in Proposition 31.
there exists a unique continuous group homomorphism that satisfies the following universal property: for all complete the proof.

Conversely, let $\tau$ be a normal subgroup of $\lim G_i$ such that $u_i^{-1}(\tau)$ is a $\tau_i$-neighborhood of the identity element of $G_i$ for all $i \in I$ ($U$ is a basic $\tau$-neighborhood of the identity element of $\lim G_i$). Since $v_i = \pi \circ u_i$, $u_i^{-1}(\pi^{-1}(U))$ is a $\tau_i$-neighborhood of the identity element of $G_i$ for all $i \in I$. Therefore the normal subgroup $\pi^{-1}(U)$ of $G$ is a $\tau$-neighborhood of the identity element of $G$, and hence $U$ is a $\tau$-neighborhood of the identity element of $\lim G_i$. Thus $\tau$ is coarser than $\overline{\tau}$, and the equality $\overline{\tau} = \tau$ is established.

The topological inductive limit $(\lim G_i, \tau)$ of the inductive system $((G_i, \tau_i), u_{ji})_{i \in I}$ of SNS-groups satisfies the following universal property:

**Proposition 34.** Let $(H, \theta)$ be an SNS-group and, for each $i \in I$, let $\alpha_i: (G_i, \tau_i) \to (H, \theta)$ be a continuous group homomorphism such that $\alpha_j \circ u_{ji} = \alpha_i$ for $i \leq j$. Then there exists a unique continuous group homomorphism $u: (\lim G_i, \tau) \to (H, \theta)$ such that $\alpha_i = u \circ v_i$ for all $i \in I$ ($v_i$ being as in Proposition 31).

\[
\begin{array}{ccc}
(G_i, \tau_i) & \xrightarrow{u_{ji}} & (G_j, \tau_j) \\
\alpha_i & \leftarrow & \alpha_j \\
\downarrow & & \downarrow \\
(H, \theta) & \xrightarrow{u} & (G_i, \tau_i)
\end{array}
\]

**Proof.** By Proposition 31, there exists a unique group homomorphism $u: \lim G_i \to H$ such that $\alpha_i = u \circ v_i$ for all $i \in I$. Moreover, since $u \circ v_i: (G_i, \tau_i) \to (H, \theta)$ is continuous for all $i \in I$, then $u: (\lim G_i, \tau) \to (H, \theta)$ is continuous. This completes the proof.

**Corollary 35.** Let $((G_i, \tau_i), u_{ji})_{i \in I}$ and $((H_i, \theta_i), v_{ji})_{i \in I}$ be two inductive systems of SNS-groups, and let $(\lim G_i, \tau)$ and $(\lim H_i, \theta)$ be the corresponding topological inductive limits. For each $i \in I$ let $\beta_i: (\lim G_i, \tau) \to (H_i, \theta_i)$ be a continuous group homomorphism such that $v_{ji} \circ \beta_i = \beta_j \circ u_{ji}$ for $i \leq j$. Then there exists a unique continuous group homomorphism $u: (\lim G_i, \tau) \to (\lim H_i, \theta)$ such that $u \circ v_i = w_i \circ \beta_i$ for all $i \in I$, where $v_i: G_i \to \lim G_i$ and $w_i: H_i \to \lim H_i$ are the canonical group homomorphisms ($i \in I$).

\[
\begin{array}{ccc}
(G_i, \tau_i) & \xrightarrow{u_{ji}} & (H_i, \theta_i) \\
\downarrow & & \downarrow \\
(G_j, \tau_j) & \xrightarrow{\beta_j} & (H_j, \theta_j) \\
\downarrow & & \downarrow \\
(\lim G_i, \tau) & \xrightarrow{u} & (\lim H_i, \theta)
\end{array}
\]

**Proof.** For each $i \in I$ put $\alpha_i = w_i \circ \beta_i$; then $\alpha_i$ is a continuous group homomorphism from $(G_i, \tau_i)$ into $(\lim H_i, \theta)$. Since

\[
\alpha_j \circ u_{ji} = w_j \circ (\beta_j \circ u_{ji}) = w_j \circ (v_{ji} \circ \beta_i) = (w_j \circ v_{ji}) \circ \beta_i = w_i \circ \beta_i = \alpha_i
\]

for $i \leq j$, Proposition 34 guarantees the existence of a unique continuous group homomorphism $u: (\lim G_i, \tau) \to (\lim H_i, \theta)$ such that $\alpha_i = u \circ v_i$ for all $i \in I$. This completes the proof.
References

[6] D.P. Pombo Jr., Linear topologies on groups: basic constructions, Atas do 53º Seminário Brasileiro de Análise (2001), 67–84. [Page 74, line 8: read “$E = \lim_{i \in I} E_i$, and let” in place of “$E$,”.]

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