On the index complex of a maximal subgroup and the group-theoretic properties of a finite group

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ABSTRACT: Let $G$ be a finite group, $S^p(G)$, $\Phi'(G)$ and $\Phi_1(G)$ be generalizations of the Frattini subgroup of $G$. Based on these characteristic subgroups and using Deskins index complex, this paper gets some necessary and sufficient conditions for $G$ to be a $p$-solvable, $\pi$-solvable, solvable, super-solvable and nilpotent group.

Key Words: index complex; solvable groups; super-solvable groups; nilpotent groups.

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1. Introduction

The relationship between the properties of maximal subgroups of a finite group and its structure has been studied extensively. The concept of index complex (see [1]) associated with a maximal subgroup plays an important role in the study of group theory.

Suppose that $G$ is a finite group, and $M$ is a maximal subgroup of $G$. A subgroup $C$ of $G$ is said to be a completion for $M$ in $G$ if $C$ is not contained in $M$ while every proper subgroup of $C$ which is normal in $G$ is contained in $M$. The set of all completions of $M$, denote it by $I(M)$, is called the index complex of $M$ in $G$. Clearly $I(M)$ contains a normal subgroup, and is a nonempty partially ordered set by set inclusion relation. If $C \in I(M)$ and $C$ is the maximal element of $I(M)$, $C$ is said to be a maximal completion for $M$. If moreover $C \triangleleft G$, $C$ then is said to be a normal completion for $M$. Clearly every normal completion of $M$
is a maximal completion of $M$. Furthermore, by $k(C)$ we denote the product of all normal subgroups of $G$ which are also proper subgroups of $C$, $k(C)$ is a proper normal subgroup of $C$.

In [2], Deskins studied the group-theoretic properties of the completions and its influences on the solvability of a finite group. He also raised a conjecture concerning super-solvability of a finite group in the same paper. Deskins’s conjecture and other investigations were continued by many successive works [3-5]. This paper will study the structure of a finite group $G$. Using the concept of index complex and applying Frattini-Like subgroups such as $S_p(G)$, $\Phi'(G)$ and $\Phi_1(G)$, the paper improves main results of [3-5] and obtains some necessary and sufficient conditions for the $G$ to be a $p$-solvable, $\pi$-solvable, solvable, super-solvable and nilpotent group.

Throughout this paper, $G$ denotes a finite group. The terminologies and notations agree with standard usage as in [6]. The notation $M \prec G$ means $M$ is a maximal subgroup of $G$, and $N \triangleleft G$ means that $N$ is a normal subgroup of $G$. If $p$ is a prime, then $p'$ denotes the complementary sets of primes and $\left| G : M \right|_p$ the $p$-part of $\left| G : M \right|$.

2. Preliminaries

For convenience, we give some notations and definitions firstly. Suppose that $p$ is a prime, put

\begin{align*}
F_c &= \{M : M \prec G \text{ and } \left| G : M \right| \text{ is composite}\}; \\
F_p &= \{M : M \prec G \text{ and } M \geq N_G(P) \text{ for a } P \in \text{Syl}_p(G)\}; \\
F_{pc} &= F_p \cap F_c; \\
F_G &= \bigcup_{p \in \pi(G)} F_p \\
F'_G &= F_G \cap F_{pc}.
\end{align*}

Using subgroups above, one can define Frattini-Like subgroups of $G$ as follows.

**Definition 2.1**

\begin{align*}
S^p(G) &= \bigcap_{M \in F_{pc}} \{M : M \prec G \} \text{ if } F_{pc} \text{ is nonempty, otherwise } S^p(G) = G; \\
\Phi_1(G) &= \bigcap_{M \in F_G} \{M : M \prec G \} \text{ if } F_G \text{ is nonempty, otherwise } \Phi_1(G) = G; \\
\Phi'(G) &= \bigcap_{M \in F'_G} \{M : M \prec G \} \text{ if } F'_G \text{ is nonempty, otherwise } \Phi'(G) = G.
\end{align*}

We begin with a preliminary result which will be used frequently in connection with induction arguments in the next section.

**Lemma 2.1** Let $M$ be a maximal subgroup of a group $G$ and $N$ a normal subgroup of $G$. If $C \in I(M)$ and $N \leq k(C)$, then $C/N \in I(M/N)$ and $k(C/N) = k(C)/N$.

**Proof.** Since $C \in I(M)$, $C \not\leq M$. Also $C/N \not\leq M/N$. And if $A/N < C/N$, $A/N \not\prec G/N$, then $A < C$ and $A \not\triangleleft G$. Since $A \leq M$, $A/N \leq M/N$, and $C/N \in
Also $C \not\subseteq M$ means $k(C) \neq C$. Then $k(C/N) \subseteq C/N$ and moreover $k(C)/N \leq M/N$. So $k(C)/N < k(C)/N$.

On the other hand, let $k(C/N) = H/N$, then $H < G$ and $H/N < C/N$. Thus, $H < C$ and $k(C/N) = H/N \leq k(C)/N$. Therefore, $k(C/N) = k(C)/N$. □

**Lemma 2.2** Let $C$ and $D$ be normal completions of a maximal subgroup $M$ of $G$. Then $C/k(C) \cong D/k(D)$.

The order of $C/k(C)$, where $C$ is a normal completion of $M$, is called the normal index of $M$ in $G$, denoted by $\eta(G : M)$.

**Lemma 2.3** $\Phi_1(G)$ is a nilpotent group; $\Phi'(G)$ is a Sylow tower group.

**Lemma 2.4** If $G$ is a group with a maximal core-free subgroup, the following are equivalent:

1. There exists a nontrivial solvable normal subgroup of $G$.
2. There exists a unique minimal normal subgroup $N$ of $G$ and the index of all maximal subgroups of $G$ in $F_G$ with core-free are powers of a unique prime.

Proof. Using Ref. [7], it suffices to prove that (2) implies (1). Indeed for every $L \in F_G$ with core-free, let $p$ be the unique prime divisor of $|G : L|$. Since $N \not\subseteq L$, $G = L N$. Moreover $|G : L| \big| |N|$, thus $p \big| |N|$. Let $P \in Syl_p(N)$. If $P \not\subseteq G$, by the Frattini argument we have $G = N \cdot N_G(P)$. Suppose that $N_G(P) \leq M < G$, there exists $G_P \in Syl_p(G)$ satisfying $N_G(P) \geq N_G(G_P)$. This means $M \geq N_G(G_P)$ and therefore $M \in F_G$. But $N \not\subseteq M$, by the uniqueness of $N$ we get that $M$ is core-free. By the hypothesis, $p \big| |G : M|$. Since $M \geq N_G(G_P)$, $p \big| |G : M|$. This leads to a contradiction. Thus $P \not\subseteq G$ and $P = N$ is a nontrivial solvable normal subgroup of $G$. □

3. Main Results

The following is the main result of the paper which gives a description of $p$-solvable group.

**Theorem 3.1** Let $p$ be the largest prime divisor of the order of $G$. The $G$ is $p$-solvable if and only if for each non-nilpotent maximal subgroup $M$ of $G$ in $F^p_C$, there exists a normal completion $C$ in $I(M)$ such that $C/k(C)$ is a $p'$-group.

Proof. It suffices to prove the sufficient condition. Suppose that the result is false and let $G$ be a counterexample of minimal order, now we can claim that:

i) $F^p_C$ is not empty. Indeed if $F^p_C$ is empty, then $S^p(G) = G$. Using [9, Lemma 2.2], $S^p(G)$ is $p$-closed. So $P \in Syl_p(G) < G$ and $G$ is $p$-solvable. This leads to a contradiction.

ii) Every maximal subgroup $M$ of $G$ in $F^p_C$ must be non-nilpotent. Indeed if there exists a maximal subgroup $M$ in $F^p_C$ which is also nilpotent, then $|G : M|_p =$
1 and \( G \) is \( p \)-solvable. It is a contradiction.

iii) \( G \) has a unique minimal normal subgroup \( N \) such that \( G/N \) is \( p \)-solvable. Indeed if \( G \) is simple, then for every \( M \) of \( G \) in \( F^p \), \( G \) is the only normal completion in \( I(M) \) with \( k(G) = 1 \). By hypothesis, \( G = G/k(G) \) is a \( p' \)-group. This contradicts with the fact that \( p \) is the largest prime dividing \( |G| \), hence \( G \) is not simple. Let \( N \) be a minimal normal subgroup of \( G \), we will according to cases of \( N \leq k(C) \) or \( N \notin k(C) \) prove that \( G/N \) satisfies the hypothesis of the theorem.

If \( N \leq k(C) \), then \( N \leq C \) and \( C/N \) is a normal completion for \( M/N \) in \( G/N \). By Lemma 2.1, \( C/N \xrightarrow{k(C/N)} C/N \xrightarrow{k(C)/N} C/k(C) \). Again \( C/k(C) \) is a \( p' \)-group, so \( C/N \xrightarrow{k(C/N)} C/k(C) \) is a \( p' \)-group.

If \( N \notin k(C) \), then \( N \notin C \). For otherwise, either \( N = C \) or \( N < C \), so either \( G = MC = MN = M \) or \( N < k(C) \). Each of which is a contradiction. Since \( N \) is a minimal normal subgroup of \( G \), we have either \( C \cap N = N \) or \( C \cap N = 1 \). If \( C \cap N = N \), then \( N \leq C \). It is also a contradiction. So \( C \cap N = 1 \). Then \( CN/N \) is a normal completion for \( M/N \) in \( G/N \). We are to show that \( C/N \xrightarrow{k(C/N)} C/N \) is a \( p' \)-group. Since \( k(C) \leq C \cap N = 1 \), it follows that \( k(C)N/N < CN/N \). Also \( k(C)N/N \triangleleft G/N \), so we have \( k(C)N/N \leq k(CN/N) \).

We define a map \( \phi \colon C/k(C) \to CN/N \xrightarrow{k(CN/N)} k(CN/N) \), by

\[
\phi(xk(C)) = xNk(CN/N)
\]

for all \( xk(C) \in C/k(C) \). Now \( xk(C) = yk(C) \) implies that \( x^{-1}y \in k(C) \), so \( (xN)^{-1}(yN) = (x^{-1}y)N \in k(C)N/N \leq k(CN/N) \) and

\[
(xN)k(CN/N) = (yN)k(CN/N).
\]

That is to say, \( \phi(xk(C)) = \phi(yk(C)) \). Hence the map is well defined. It can be verified that \( \phi \) is an epimorphism and \( CN/N \xrightarrow{k(CN/N)} k(CN/N) \) is an epimorphic image of a \( p' \)-group. Thus \( G/N \) satisfies the hypothesis of the theorem. By the minimality of \( N \), \( G/N \) is \( p \)-solvable.

Similarly, it can be shown that \( G/N_1 \) is \( p \)-solvable if \( N \) is another minimal normal subgroup \( N_1 \) of \( G \). Thus \( G = G/N \bigcap N_1 \), which is isomorphic a subgroup of the \( p \)-solvable group \( G/N \times G/N_1 \), is \( p \)-solvable. So in the following suppose that \( N \) is the unique minimal normal subgroup of \( G \).

If \( p \nmid |N| \) or \( N \) is a \( p \)-group, then \( N \) is \( p \)-solvable and so \( G \) is \( p \)-solvable. It is a contradiction. Hence, \( |N|_p \neq 1 \) and \( N \neq N_p \in Syl_p(N) \). Let \( M \) be a maximal subgroup of \( G \) such that \( N_G(N_p) \leq M \). By the Frattini argument, we obtain that \( G = N \cdot N_G(N_p) \). Using [7, lemma 5], there exists a \( G_p \in Syl_p(G) \) with \( N_G(N_p) \geq N_G(G_p) \), so \( M \in F^p \) and \( [G : M]_p = 1 \). If \( |G : M| = q \) be a prime less than \( p \), then \( |G| \) divides \( q! \). This leads to another contradiction. Thus \( |G : M| \) is
composite and \( M \in F^p_{pc} \). By ii) and hypothesis, there exists a normal completion \( C \) in \( I(M) \) such that \( C/k(C) \) is a \( p' \)-group. Obviously \( N \) is a normal completion of \( M \). Combining with Lemma 2.2, we have \( C/k(C) \cong N/k(N) = N \). Thus \( N \) is a \( p' \)-group, which leads to the final contradiction. This completes the proof. \( \square \)

As we have known in [3], a group \( G \) is \( \pi \)-solvable if and only if for every maximal subgroup \( M \) of \( G \) there exists a normal completion \( C \) in \( I(M) \) such that \( C/k(C) \) is \( \pi \)-solvable. We now extend this result by considering a smaller class of maximal subgroups.

**Theorem 3.2** Let \( G \) be a finite group. \( G \) is \( \pi \)-solvable if and only if for every maximal subgroup \( M \) of \( G \) in \( F'_{G} \) there exists a normal completion \( C \) in \( I(M) \) such that \( C/k(C) \) is \( \pi \)-solvable.

Proof. \( \Leftarrow \) Let \( G \) be a group satisfying the hypothesis of the theorem. If \( F'_{G} \) is empty then \( \Phi'(G) = G \), and \( G \) is solvable. Thus assume that \( F'_{G} \) is not empty. If \( G \) is simple, then for every \( M \) in \( F'_{G} \), \( G \) is the only normal completion in \( I(M) \) with \( k(G) = 1 \) and thus \( G = G/k(G) \) is \( \pi \)-solvable. So suppose that \( G \) is not simple. Let \( N \) be a minimal normal subgroup of \( G \). Without loss of generality, one can suppose that \( F'_{G/N} \) is not empty. We will use induction on the order of \( G \). For each \( M/N \in F'_{G/N} \), by [7, Lemma 3], it follows that \( M \in F'_{G} \). So by hypothesis there exists a normal completion \( C \) in \( I(M) \) such that \( C/k(C) \) is \( \pi \)-solvable.

Similar to the proof in Theorem 3.1, \( CN/N \mod \Phi'(G) \) is \( \pi \)-solvable. Thus \( G/N \) satisfies the hypothesis of the theorem. Using the induction we obtain that \( G/N \) is \( \pi \)-solvable. Furthermore, we can assume that \( N \) is the unique minimal normal subgroup of \( G \). By the same way, \( G/N \) is still a \( \pi \)-solvable group.

Now if \( N \trianglelefteq \Phi'(G) \), then from Lemma 2.3 \( \Phi'(G) \) is solvable. Thus, \( N \) is \( \pi \)-solvable, and furthermore \( G \) is \( \pi \)-solvable. If \( N \not\trianglelefteq \Phi'(G) \), there exists a maximal subgroup \( M_0 \in F'_{G} \) with \( N \not\trianglelefteq M_0 \). Then \( \text{Core}_{G} M_0 = 1 \) and \( G = NM_0 \). So \( N \) is a normal completion in \( I(M_0) \). By hypothesis there exists a normal completion \( C \) in \( I(M_0) \) such that \( C/k(C) \) is \( \pi \)-solvable. By Lemma 2.2, \( N/k(N) = N \cong C/k(C) \). Again \( C/k(C) \) is \( \pi \)-solvable, therefore \( N \) is \( \pi \)-solvable and moreover, \( G \) is \( \pi \)-solvable.

\( \Rightarrow \) The converse is obvious. \( \square \)

The following theorem can be proved similarly as Theorem 3.2, and we omit it here.

**Theorem 3.3** Let \( G \) be a finite group. \( G \) is solvable if and only if for every maximal subgroup \( M \) of \( G \) in \( F'_{G} \) there exists a normal completion \( C \) in \( I(M) \) such that \( C/k(C) \) is solvable.

As we have known [4], if \( G \) is \( S_4 \)-free, then \( G \) is super-solvable if and only if for each maximal subgroup \( M \) of \( G \), there exists a maximal completion \( C \) in \( I(M) \) such that \( G = CM \) and \( C/k(C) \) is cyclic. The following theorem extends this result.

**Theorem 3.4** Suppose that \( G \) is \( S_4 \)-free. \( G \) is super-solvable if and only if for each
maximal subgroup $M$ of $G$ in $F_G$, there exists a maximal completion $C$ in $I(M)$ such that $G = CM$ and $C/k(C)$ is cyclic.

Proof. Let $G$ be a super-solvable group. Then every chief factor of $G$ is a cyclic group of prime order. \forall M \in F'_G$, it is clear that the set $S = \{ T \trianglelefteq G \mid T \leq M \}$ is not empty. Choose an $H$ to be the minimal element in $S$. Clearly, $H \in I(M)$ and $H/k(H)$ is a chief factor of $G$, hence $H/k(H)$ is cyclic.

Let $G$ be a group satisfying the hypothesis of the Theorem. If $F'_G$ is empty then $G = \Phi(G)$ and $G$ is super-solvable [9]. We now assume that $F'_G$ is not empty and then $G$ is solvable. In the remainder of the proof we will drop the maximality imposed on the completion $C$ in $I(M)$ in the hypothesis. For each maximal subgroup $M$ in $F'_G$, there exists a completion $C$ in $I(M)$ such that $G = CM$ and $C/k(C)$ is cyclic. From [5, Lemma 2], we can get a normal completion $A$ in $I(M)$ such that $A/k(A)$ is either cyclic or elementary abelian of order $2^2$.

First suppose that there exists an $M$ in $F'_G$ which has a normal completion $A$ such that $A/k(A)$ is elementary abelian of order $2^2$. Let $\overline{G} = G/\text{core}_G(M)$ and $\overline{M}$, $\overline{N}$ be the images of $C$, $M$ and $A$ in $\overline{G}$ respectively. Then $\overline{G} = \overline{C} \cdot \overline{M} = \overline{N}$. It is easy to verify that $k(A) = A \cap \text{core}_G(M)$, so $A/k(A) \cong A/\text{core}_G(M)/\text{core}_G(M) = \overline{C}$. Since $\text{core}_G(\overline{M}) = 1$, $k(\overline{A}) = 1$, $\overline{A}$ is a minimal normal subgroup of $\overline{C}$. $\overline{A}$ is an elementary abelian of order $2^2$ and $\overline{M} \cap \overline{A} = 1$. Considering the permutation representation of $G$ on 4 cosets of $\overline{M}$, $G$ is isomorphic to a subgroup of $S_4$. Again $S_4$ and $A_4$ are the only non-super-solvable subgroups of $S_4$, $A_4$ doesn’t satisfy the hypothesis of the theorem, and $G$ is $S_4$-free, so $G$ is super-solvable.

Now assume that for each maximal subgroup $M$ in $F'_G$, $M$ has a normal completion $A$ so that $A/k(A)$ is cyclic. Let $N$ be a minimal normal subgroup of $G$. Obviously, that $G$ is $S_4$-free is quotient-closed. By [4, Lemma 3] and [7, Lemma 3], we can assume that the hypothesis holds for $G/N$. Using induction, we obtain that $G/N$ is super-solvable. Similar to Theorem 3.1, we can suppose that $N$ is the unique minimal normal subgroup of $G$. If $N \leq \Phi(G)$, then $G$ is super-solvable. If $N \not\leq \Phi(G)$, there exists a maximal subgroup $M$ in $F'_G$ so that $G = NM$ and $\text{core}_G(M) = 1$. Obviously $N$ is a normal completion in $I(M)$. By hypothesis, there exists a normal completion $A$ so that $A/k(A)$ is cyclic. By Lemma 2.2, $A/k(A) \cong N/k(N) = N$. Thus $N$ is cyclic and $G$ is super-solvable.

Remark Let $G$ be a solvable group. To obtain the conclusion in Theorem 3.4, the condition of maximality imposed on the completion $C$ is nonsignificant. So we have the following result: If $G$ is $S_4$-free and solvable, $G$ is super-solvable if and only if for each maximal subgroup $M$ of $G$ in $F'_G$, there exists a completion $C$ in $I(M)$ so that $G = CM$ and $C/k(C)$ is cyclic.

Theorem 3.5 Let $G$ be a group and $M$ be an arbitrary maximal subgroup of $G$ in $F_G$. Then $G$ is nilpotent if and only if for each normal completion $C$ of $M$,

$$|C/k(C)| = |G:M|.$$ 

Proof. $\Leftarrow$ Let $G$ be a group satisfying the hypothesis of the theorem. If $F_G$ is
empty then $G/N = \Phi_1(G/N)$. Using [9, Lemma 2.3], $G/N$ is nilpotent. If $G$ is simple, then for every $M$ in $F_G$, $G$ is the only normal completion in $I(M)$ with $k(G) = 1$. By hypothesis $|G/k(G)| = G = |G : M|$, $M = 1$, hence $G$ is a cyclic group of prime order. So assume that $G$ is not simple. Let $N$ be a minimal normal subgroup of $G$. Without loss of generality, suppose that $F_{G/N}$ is not empty. For any maximal subgroup $M/N$ in $F_{G/N}$, suppose that $C/N$ is an arbitrary normal completion in $I(M/N)$. From [7, Lemma 3] we have $M$ in $F_G$. Obviously $C$ is a normal completion in $I(M)$ and $|C/k(C)| = |G : M|$. Using Lemma 2.1,
\[
|C/N/k(C/N)| = |C/N/k(C)/N| = |C/k(C)| = |G : M| = |G/N/M/N|.
\]

Thus $G/N$ satisfies the hypothesis of the theorem. Applying induction one can see $G/N$ is nilpotent. Similar to the proof in Theorem 3.1, we may assume $N$ is the unique minimal subgroup of $G$.

If $N \leq \Phi_1(G)$, by [5, Lemma 2.3] $G$ is nilpotent. If $N \not\leq \Phi_1(G)$, there exists an $M$ in $F_G$ so that $G = NM$. Clearly, $N$ is a normal completion in $I(M)$. By hypothesis $|N/k(N)| = |N| = |G : M|$. For any $L$ in $F_G$ with $\text{core}_G(L) = 1$, obviously $N \not\leq L$ and $G = NL$. $N$ is also a normal completion in $I(M)$, so $|N/k(N)| = |N| = |G : L|$. By Lemma 2.4 $G$ has a nontrivial solvable subgroup $K$, so $N \leq K$ and $N$ is solvable. Since $G/N$ is nilpotent, $G$ is solvable. Thus $N$ is an elementary abelian $p$-group. If $G$ is not a $p$-group, we assume that $|G|$ has a prime factor $q$ different from $p$. If the subgroup $Q = \langle a | a \in G \rangle$ and $|a| = q \leq M$, this contradicts with the fact that $\text{core}_G M = 1$. So there exists an of order $q$ element $a$ in $G - M$. This implies that $G = \langle M, \langle a \rangle \rangle$. However, $|N| = |G : M|$ is a power of $p$. This leads to another contradiction. So $G$ must be a $p$-group and then is a nilpotent group.

$\Rightarrow$ The converse holds obviously. $\square$

References

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