Periodic Solutions of a Neutral Difference System

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Abstract: Sufficient conditions in terms of the matrix measure for the periodic solutions of a neutral type delay difference system

\[ \Delta [x(n) + cx(n-\tau)] = A(n, x(n)) x(n) + f(n, x(n-\sigma)) \]

are given.

Key words: Krasnolselskii fixed point theorem, periodic solution, neutral system

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1. Introduction

There are many studies related to periodic solutions of difference equations such as

\[ \Delta x(n) = A(n, x(n)) x(n) + f(n, x(n-\sigma)) \]

see e.g. [1,2,3,4,5]. One basic assumption behind such an equation is that the change \( x(n+1) - x(n) \) is, aside from a perturbation, 'proportional' to \( x(n) \). Yet there are cases when the effect of the change \( x(n-\tau+1) - x(n-\tau) \) is also important.

In this paper, we consider difference systems of the form

\[ \Delta [x(n) + cx(n-\tau)] = A(n, x(n)) x(n) + f(n, x(n-\sigma)) \]

where \( Z = \{0, \pm1, \pm2, \ldots \} \), \( \tau \) and \( \sigma \) are integers, \( c \in R \) and \( |c| < 1 \), \( A : Z \times R^s \to R^s \) and \( f : Z \times R^s \to R^s \) are continuous functions such that for some positive integer \( \omega \), \( A(n+\omega, x) = A(n, x) \) and \( f(n+\omega, x) = f(n, x) \) for \( (n, x) \in Z \times R^s \).

A solution of (1) is a real vector sequence of the form \( x = \{x(n)\}_{n\in Z} \) which renders (1) into an identity after substitution. As in the previous studies, we are concerned with the existence of solutions which are \( \omega \)-periodic, that is, solutions that satisfy \( x(n+\omega) = x(n) \) for \( n \in Z \).

We will invoke the Krasnolselskii fixed point theorem for finding \( \omega \)-periodic solutions of (1): Suppose \( B \) is a Banach space and \( G \) is a bounded, convex and closed subset of \( B \). Let \( S, P : X \to B \) satisfy the following conditions: (i) \( Sx + Py \in G \), for any \( x, y \in G \), (ii) \( S \) is a contraction mapping, and (iii) \( P \) is completely continuous. Then \( S + P \) has a fixed point in \( G \).

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2. Preliminaries

First of all, for any real (scalar) sequence \( \{u_n\}_{n \in \mathbb{Z}} \), we define a nonstandard summation operation:

\[
\bigoplus_{n=\alpha}^{\beta} u_n = \begin{cases} 
\sum_{n=\alpha}^{\beta} u_n, & \alpha \leq \beta \\
0, & \beta = \alpha - 1 \\
-\sum_{n=\beta+1}^{\alpha-1} u_n, & \beta < \alpha - 1
\end{cases}
\]

Next, we recall the matrix norms and matrix measures. Let \( C \) be the set of complex numbers. Let \( |\cdot| \) be the standard \( p \)-norm for the linear space \( C^* \). For each matrix \( A \in C^{s \times s} \), the quantity \( \|A\|_p \) defined by

\[
\|A\|_p = \sup_{|x|_p \neq 0} \frac{|Ax|_p}{|x|_p}
\]

is called the induced (matrix) norm of \( A \) corresponding to the vector norm \( |\cdot|_p \).

The matrix measure corresponding to \( \|\cdot\|_p \) is the function \( \mu_p : C^{s \times s} \rightarrow R \) defined by

\[
\mu_p (A) = \lim_{k \rightarrow +\infty} k \left( \left\| I + \frac{1}{k} A \right\|_p - 1 \right).
\]

It is known (see e.g. [6]) that \( \mu_p \) has the following properties:

(i) For each \( A \in C^{s \times s} \), the limit indicated in (3) exists and is well defined;

(ii) \( -\|A\|_p \leq -\mu_p (-A) \leq \mu_p (A) \leq \|A\|_p \) for \( A \in C^{s \times s} \);

(iii) \( \mu_p (\alpha A) = \alpha \mu_p (A) \) for \( \alpha \geq 0 \) and \( A \in C^{s \times s} \);

(iv) for \( A, B \in C^{s \times s} \),

\[
\text{max} \{ \mu_p (A) - \mu_p (-B) , -\mu_p (-A) + \mu_p (B) \} \leq \mu_p (A + B) \leq \mu_p (A) + \mu_p (B) ,
\]

(v) \( \mu_p \) is convex, that is, for \( \alpha \in [0, 1] \) and \( A, B \in C^{s \times s} \),

\[
\mu_p \{ \alpha A + (1 - \alpha) B \} \leq \alpha \mu_p (A) + (1 - \alpha) \mu_p (B) ,
\]

(iv) \( -\mu_p (-A) \leq \text{Re} \lambda \leq \mu_p (A) \) whenever \( \lambda \) is an eigenvalue of \( A \).

As examples (see e.g. [6]), let \( x = (x_1, \ldots, x_s)^T \), \( A = (a_{ij})_{s \times s} \in C^{s \times s} \), then

\[
|x|_\infty = \max_{0 \leq i \leq s} |x_i| , \|A\|_\infty = \max_{0 \leq i \leq s} \sum_j |a_{ij}| , \mu_\infty (A) = \max_{0 \leq i \leq s} \left\{ a_{ii} + \sum_{j \neq i} |a_{ij}| \right\} ,
\]

\[
|x|_1 = \sum_i |x_i| , \|A\|_1 = \max_{0 \leq j \leq s} \sum_i |a_{ij}| , \mu_1 (A) = \max_{0 \leq j \leq s} \left\{ a_{jj} + \sum_{i \neq j} |a_{ij}| \right\} .
\]

**LEMMA 1.** Let \( A = (a_{ij})_{s \times s} \in R^{s \times s} \) and \( |a_{ii}| \leq 1 \) for \( i = 1, 2, \ldots, s \). Then for all positive integer \( k \),

\[
\|I + A\|_p \leq k \left\| I + \frac{1}{k} A \right\|_p - (k - 1) , \ p = 1, \infty.
\]
**Proof:** By definition, for each positive integer $k$, there is an integer $i_0 \in \{1, 2, \ldots, s\}$ such that

\[
\left\| \frac{1}{k} I + \frac{1}{k} A \right\|_\infty = \frac{a_{i_0 i_0}}{k} + \frac{1}{k} \sum_{j \neq i_0} |a_{ij}| = 1 + \frac{a_{i_0 i_0}}{k} + \frac{1}{k} \sum_{j \neq i_0} |a_{ij}| - \frac{k - 1}{k} \leq \left\| I + \frac{1}{k} A \right\|_\infty - \frac{k - 1}{k}.
\]

It follows that

\[
\left\| I + A \right\|_p = k \left\| \frac{1}{k} I + \frac{1}{k} A \right\|_\infty \leq k \left\| I + \frac{1}{k} A \right\|_\infty - (k - 1).
\]

The other case where $p = 1$ may similarly be proved.

Next we recall some basic facts about linear periodic difference systems. Consider the system

\[
\Delta x(n) = A(n) x(n), \ n \in \mathbb{Z},
\]

where $A(n) = (a_{ij}(n))_{s \times s} \in R^{s \times s}$, $I + A(n)$ is nonsingular and $A(n + \omega) = A(n)$ for $n \in \mathbb{Z}$. Let $\Phi(n, n_0)$ be the fundamental matrix of (7) which satisfies $\Phi(n_0, n_0) = I$. Recall that

\[
\Phi(n, n_0) = \prod_{i = n_0}^{n-1} (I + A(i)), \ n > n_0
\]

and

\[
\Phi(n, n_0) = \prod_{i = n}^{n_0-1} (I + A(i))^{-1}, \ n < n_0,
\]

and any solution of (7) is of the form $x(n) = \Phi(n, n_0)x(n_0)$, and for $n, \delta, t \in \mathbb{Z}$,

\[
\Phi(n, \delta) \Phi(\delta, t) = \Phi(n, t),
\]

and

\[
\Phi(n + 1, \delta) - \Phi(n, \delta) = A(n) \Phi(n, \delta),
\]

As a consequence, if $\{x(n)\}_{n \in \mathbb{Z}}$ is any one nontrivial $\omega$-periodic solution of (7), then $x(0) \neq 0$ and

\[
(I - \Phi(\omega, 0)) x(0) = (\Phi(0, 0) - \Phi(\omega, 0)) x(0) = 0,
\]

so that

\[
\det (I - \Phi(\omega, 0)) = 0.
\]
Conversely, if \( \det (I - \Phi(\omega, 0)) = 0 \), then there is some \( x_0 \neq 0 \) such that \( Ix_0 = \Phi(\omega, 0)x_0 \). Let \( x = \{x(n)\}_{n \in \mathbb{Z}} \) be the unique solution of (7) which satisfies \( x(0) = x_0 \). Since \( x(\omega) = \Phi(\omega, 0)x_0 = x(0) \), \( x \) is a nontrivial \( \omega \)-periodic solution of (7).

**LEMMA 2.** Let \( \{x(n)\}_{n \in \mathbb{Z}} \) be a solution of (7). If \( A(n) = (a_{ij}(n))_{s \times s} \in \mathbb{R}^{s \times s} \) and \( |a_{ii}(n)| < 1 \) for \( 1 \leq i \leq s \) and \( n, m \in \mathbb{Z}, n \geq m \), then
\[
|x(n)|_\infty \leq |x(m)|_\infty \exp \left\{ \bigoplus_{i=m}^{n-1} \mu_\infty (A(i)) \right\}.
\]  

**Proof:** In view of (7), we have
\[
x(i + 1) = (I + A(i))x(i), \quad i \geq m.
\]  
By (11) and Lemma 1, we see that
\[
|x(i + 1)|_\infty \leq \|I + A(i)\|_\infty |x(i)|_\infty \leq \left( k \left\| I + \frac{1}{k} A(i) \right\|_\infty - (k - 1) \right) |x(i)|_\infty
\]
\[
\leq \exp \left\{ \left( k \left\| I + \frac{1}{k} A(i) \right\|_\infty - k \right) \right\} |x(i)|_\infty
\]
Taking limits on both sides as \( k \to +\infty \), we see that
\[
|x(i + 1)|_\infty \leq \exp (\mu_\infty (A(i))) |x(i)|_\infty, \quad i \geq m,
\]  
which implies (10). The proof is complete.

As an immediate consequence, the fundamental matrix of (7) satisfies
\[
\|\Phi(n, m)\|_\infty \leq \exp \left\{ \bigoplus_{i=m}^{n-1} \mu_\infty (A(i)) \right\}, \quad n \geq m.
\]  

Let us seek a solution \( x = \{x(n)\}_{n \in \mathbb{Z}} \) of the following nonhomogeneous system associated with (7):
\[
\Delta x(n) = A(n)x(n) + F(n), \quad n \in \mathbb{Z},
\]  
where \( F : \mathbb{Z} \to \mathbb{R}^s \) satisfies \( F(n + \omega) = F(n) \) for \( n \in \mathbb{Z} \). By the method of undetermined coefficients, we assume
\[
x(n) = \Phi(n, n_0)y(n), \quad n \in \mathbb{Z},
\]  
where \( \Phi(n, n_0) \) is the fundamental matrix of (7) satisfying \( \Phi(n_0, n_0) = I \) but \( y = \{y(n)\}_{n \in \mathbb{Z}} \) is to be sought. Since
\[
\Phi(n + 1, n_0) y(n + 1) = (I + A(n)) \Phi(n, n_0) y(n) + F(n),
\]  
and
\[
\Phi(n + 1, n_0) = (I + A(n)) \Phi(n, n_0),
\]  
we have
\[
\Phi(n + 1, n_0) \Delta y(n) = F(n).
\]
Thus
\[ \Delta y(n) = \Phi(n + 1, n_0)^{-1} F(n) = \Phi(n_0, n + 1) F(n), \]
so that
\[ y(n) = y(n_0) + \bigoplus_{i=n_0}^{n-1} \Phi(n_0, i + 1) F(i), \quad n \in \mathbb{Z}, \]
We have thus found a solution \( \{x(n)\}_{n \in \mathbb{Z}} \) of \( (14) \) defined by
\[
\begin{align*}
x(n) &= \Phi(n, n_0) x(n_0) + \Phi(n, n_0) \bigoplus_{i=n_0}^{n-1} \Phi(n_0, i + 1) F(i) \\
&= \Phi(n, n_0) x(n_0) + \bigoplus_{i=n_0}^{n-1} \Phi(n, i + 1) F(i)
\end{align*}
\]
for \( n \in \mathbb{Z}. \)

**THEOREM 1.** Suppose \( (7) \) does not have any nontrivial \( \omega \)-periodic solutions.

\[
\exp \left\{ \bigoplus_{i=0}^{\omega-1} \mu_{\infty}(A(i)) \right\} < 1.
\]

If the nonhomogeneous system \( (13) \) has an \( \omega \)-periodic solution \( \{x(n)\}_{n \in \mathbb{Z}} \), then \( \{x(n)\}_{n \in \mathbb{Z}} \) is an \( \omega \)-periodic solution of the system
\[
x(n) = (I - \Phi(n + \omega, n))^{-1} \bigoplus_{i=n}^{n+\omega-1} \Phi(n + \omega, i + 1) F(i), \quad n \in \mathbb{Z}.
\]

Conversely, if \( \{x(n)\}_{n \in \mathbb{Z}} \) is an \( \omega \)-periodic solution of \( (23) \), then it is also an \( \omega \)-periodic solution of \( (13) \).

Indeed, recall that \( (7) \) does not have any nontrivial \( \omega \)-periodic solutions if, and only if, \( \det (I - \Phi(\omega, 0)) \neq 0 \). Let \( \{x(n)\}_{n \in \mathbb{Z}} \) be an \( \omega \)-periodic solution of \( (14) \). Then in view of \( (21) \),
\[
x(n_0) = (I - \Phi(\omega, 0))^{-1} \bigoplus_{i=n_0}^{n+\omega-1} \Phi(n_0 + \omega, i + 1) F(i).
\]

By \( (21) \) again and relations \( (13) \) and \( (22) \),
\[
x(n) = (I - \Phi(n + \omega, n))^{-1} \bigoplus_{i=n}^{n+\omega-1} \Phi(n + \omega, i + 1) F(i), \quad n \in \mathbb{Z}.
\]

The converse is easily seen by reversing the arguments above. The proof is complete.

For the sake of simplicity, let the norm \( |\cdot|_{\infty} \), induced norm \( \|\cdot\|_{\infty} \) and the corresponding matrix measure \( \mu_{\infty}(\cdot) \) be denoted by \( |\cdot|, \|A\| \) and \( \mu(A) \) respectively. Let \( l^\omega \) be the Banach space of all real vector \( \omega \)-periodic sequences of the form \( x = \{x(n)\}_{n \in \mathbb{Z}} \) (where \( x(n) \in \mathbb{R}^s \) endowed with the usual linear structure as well as the norm \( \|x\|_2 = \|x\|^0 + \|x\|^1 \) where \( \|x\|^0 = \max_{0 \leq i \leq \omega - 1} |x(i)| \) and \( \|x\|^1 = \max_{0 \leq i \leq \omega - 1} |\Delta x(i)| \).

**LEMMA 3.** A subset \( D \) of \( l^\omega \) is relatively compact if and only if \( D \) is bounded.

**Proof:** It is easy to see that if \( D \) is relatively compact in \( l^s \), then \( D \) is bounded. Conversely, if the subset \( D \) of \( l^s \) is bounded, then there is a subset
\[
\Gamma := \{ x \in l^s \mid \|x\|^0 \leq H, \|x\|^1 \leq H \},
\]
where $H$ is a positive constant, such that $D \subset \Gamma$. It suffices to show that $\Gamma$ is relatively compact in $l^\omega$. To see this, note that for each $\varepsilon > 0$, we may choose numbers $y_0 < y_1 < ... < y_m$ such that $y_0 = -H$, $y_m = H$ and $y_{i+1} - y_i < \varepsilon/4$, for $i = 0, ..., m - 1$. Then the set $\Gamma_1$ of all real $\omega$-periodic vector sequence of the form
\[
\left\{ (v_1(n), v_2(n), ..., v_s(n))^T \right\}_{n \in \mathbb{Z}}
\]
that satisfies $v_j(i) \in \{y_0, y_1, ..., y_{m-1}\}$ for $j = 1, 2, ..., s$ and $i = 0, ..., \omega - 1$ is a finite $\varepsilon$-net of $\Gamma$. Indeed, it is easy to see that $\Gamma_1$ is a finite subset of $l^\omega$, furthermore, for any $x = \{x(n)\}_{n \in \mathbb{Z}} \in \Gamma$, we can let $\nu = \{v(n)\}_{n \in \mathbb{Z}} \in \Gamma_1$ such that $|x_j(n) - v_j(n)| < \varepsilon/4$ for $j = 1, 2, ..., s$ and $n = 0, ..., \omega - 1$. Then $|x(n) - \nu(n)| \leq \varepsilon/4$ and
\[
|x(n) - \Delta v(n)| \leq |x(n+1) - \nu(n+1)| + |x(n) - \nu(n)| \leq \varepsilon/2,
\]
for $n = 0, ..., \omega - 1$, so that
\[
\|x - \nu\|_2 = \|x - 0\| + \|x - \nu\| \leq \varepsilon/4 + \varepsilon/2 < \varepsilon.
\]
The proof is complete.

3. Main Results

We first recall the conditions imposed on (1): $|c| < 1$ and $A : Z \times R^s \to R^{s \times s}$ and $f : Z \times R^s \to R^s$ are continuous functions such that for some positive $\omega$, $A(n + \omega, x) = A(n, x)$ and $f(n + \omega, x) = f(n, x)$ for $(n, x) \in Z \times R^s$. Let $A(n, x) = (a_{ij}(n, x))_{s \times s}$.

**THEOREM 2.** Suppose there is a nontrivial $\omega$-periodic sequence $\{a(n)\}_{n \in \mathbb{Z}}$ such that
\[
\beta = \exp \left( \bigoplus_{i=0}^{\omega-1} a(i) \right) < 1
\]
and $|a_{ij}(n, x)| < 1$ for $1 \leq i, j \leq s$ and $(n, x) \in Z \times R^s$ and
\[
\mu(A(n, x)) \leq a(n), \ n \in \mathbb{Z}.
\] (26)

Suppose further that there is $M > 0$ such that
\[
\bigoplus_{n=0}^{\omega-1} \sup_{|x| \leq M} |f(n, x)| < \frac{(1 - \beta) M (1 - 2 |c|)}{M_0} - \frac{ML + b_0}{(1 - |c|)} |c| \omega
\] (27)
where
\[
L = \sup_{|x| < M, 0 \leq n \leq \omega} \|A(n, x)\|,
\] (28)
\[
b_0 = \sup_{0 \leq n \leq \omega, |x| \leq M} |f(n, x)|
\]
and
\[
M_0 = \sup_{0 \leq s \leq \omega - 1} \exp \left( \bigoplus_{i=0}^{t} a(i) \right).
\]
Then (1) has an ω-periodic solution.

**Proof:** For each \( u = \{u(n)\}_{n \in Z} \in l^w \), consider the periodic system of the form

\[
\Delta x(n) = A(n, u(n)) x(n), \quad n \in Z,
\]

and

\[
\Delta x(n) = A(n, u(n)) x(n) + f(n, u(n - \sigma)) - c\Delta u(n - \tau), \quad n \in Z.
\]

Since \( |a_{ij}(n, x)| < 1 \) for \( 1 \leq i, j \leq s \) and \( (n, x) \in Z \times R^s \), \( I + A(n, u(n)) \) is nonsingular for each \( n \in Z \). Let \( \Phi_u(n, n_0) \) be the fundamental matrix of (29) which satisfies \( \Phi_u(n_0, n_0) = I \). By (13) and our assumption, we have

\[
\|\Phi_u(n, 0)\| \leq \exp \left( \sum_{i=0}^{\omega-1} \mu(A(i, u(i))) \right) \leq \exp \left( \sum_{i=0}^{\omega-1} \alpha(i) \right) < 1,
\]

thus \((I - \Phi_u(n, 0))^{-1}\) exists, which shows that (29) has no nontrivial \( \omega \)-periodic solutions.

Define the mappings \( S : l^w \to l^w \) and \( P : l^w \to l^w \) by

\[
(Su)(n) = -cu(n - \tau),
\]

and

\[
(Pu)(n) = cu(n - \tau) + (I - \Phi_u(n + \omega, n))^{-1} \times \sum_{i=0}^{n+\omega-1} \{\Phi_u(n + \omega, i + 1) [f(i, u(i - \sigma)) - c\Delta u(i - \tau)]\}
\]

for \( n \in Z \). Then

\[
(Su + Pu)(n) = (I - \Phi_u(n + \omega, n))^{-1} \times \sum_{i=0}^{n+\omega-1} \{\Phi_u(n + \omega, i + 1) [f(i, u(i - \sigma)) - c\Delta u(i - \tau)]\}
\]

for \( n \in Z \). Thus if \( u \) is a fixed point of the operator \( S + P \), then by Theorem 1, it is also an \( \omega \)-periodic solution of (30).

We now show that the assumptions in the Krasnoselskii’s Theorem are satisfied, so that a fixed point of \( S + P \) can indeed be found. Let

\[
N = \frac{ML + b_0}{1 - |c|}.
\]

Define

\[
G = \left\{ x \in l^w : \|x\|^0 \leq M, \|x\|^1 \leq N \right\},
\]

it is easy to see that \( G \) is a bounded, closed and convex subset of \( l^w \).

It is easily seen that the condition \( |c| < 1 \) implies \( S \) is a contraction mapping. Next we assert that for any \( u, v \in G \), that satisfy \( \|Su + Pv\|^0 \leq M \). Indeed, since

\[
\|\Phi_u(n + \omega, s)\| \leq \exp \left( \sum_{i=s}^{n+\omega} \mu(A(i, u(i))) \right) \leq \exp \left( \sum_{i=s}^{n+\omega} \alpha(i) \right) < M_0, \quad n \leq s \leq n + \omega - 1,
\]
and by using (13) get
\[
\left\| (I - \Phi_u (n + \omega, n))^{-1} \right\| = \left\| \bigoplus_{i=0}^{\infty} (\Phi_u (n + \omega, n))^{(i)} \right\|
\leq \bigoplus_{i=0}^{\infty} \left\| (\Phi_u (n + \omega, n))^{(i)} \right\| \leq \bigoplus_{i=0}^{\infty} |\beta|^i = \frac{1}{1 - |\beta|}.
\]

From (27), (32), (33), (34), (35), (36) and (37), we have
\[
|\Delta ((Su) (n)) + (Pv) (n)| \leq |\Delta (Su) (n)| + |(Pv) (n)|
\leq 2 |c| M + \left\| (I - \Phi_v (n + \omega, n))^{-1} \right\| \bigoplus_{i=n+\omega-1}^{n} \left\| \Phi_u (n + \omega, i + 1) \right\| \left[ \sup_{|x| \leq M} |f (n, x)| + |c| N \right]
\leq 2 |c| M + \frac{M_0}{1 - |\beta|} \left\{ \frac{1 - |\beta|}{M_0} 2 |c| M + |c| N \omega + \bigoplus_{i=n+\omega-1}^{n} \sup_{|x| \leq M} |f (n, x)| \right\}
\leq \frac{M_0}{1 - |\beta|} \left[ \frac{1 - |\beta|}{M_0} 2 |c| M + \frac{ML + b_0}{1 - |c|} |c| \omega + M \left[ \frac{1 - |\beta|}{M_0} (1 - 2 |c|) - \frac{ML + b_0}{M (1 - |c|)} |c| \omega \right] \right]
= M.
\]

Since
\[
\Delta ((Su) (n)) = -c \Delta u (n - \tau).
\]
and
\[
\Delta (Pv) (n) = A (n, v (n)) \{(Pv) (n) + (Sv) (n)\} + f (n, v (n - \sigma)),
\]
we have
\[
|\Delta ((Su) (n) + Pv) (n))| \leq |\Delta (Su) (n)| + |(Pv) (n) + (Sv) (n))| + |f (n, v (n - \sigma))| + |c| \Delta u (n - \tau)
\leq LM + b_0 + |c| N = N,
\]
so that \(\|Su + Pv\|^1 \leq N\). We have now proved that for \(u, v \in G\), \(Su + Pv \in G\).

Next, we prove that \(P\) is a completely continuous operator from \(G\) into \(G\). For \(u, v \in G\), let \(V = Pu - Pv\). By (10), we know that
\[
\Delta (V (n)) = A (n, u (n)) \{(Pu) (n) + (Su) (n)\} + f (n, u (n - \sigma)) - A (n, v (n)) \{(Pv) (n) + (Sv) (n)\} - f (n, v (n - \sigma))
= A (n, u (n)) V (n) + A (n, u (n)) - A (n, v (n))] (Pv) (n)
+ A (n, v (n)) [(Su) (n) - (Sv) (n)]
+ [A (n, u (n)) - A (n, v (n))] (Sv) (n)
+ f (n, u (n - \sigma)) - f (n, v (n - \sigma)).
\]
Let
\[
    w(t, u(n), v(n)) = -A(n, u(n)) c[u(n - \tau) - v(n - \tau)] \\
    + [A(n, u(n)) - A(n, v(n))][Pu(n) + Sv(n)] \\
    + f(n, u(n - \sigma)) - f(n, v(n - \sigma)).
\]

(42)

Noting that \(A(n, x)\) and \(f(n, x)\) for \(0 \leq n \leq \omega - 1\) are continuous on \(G\) and \(Pv + Sv\) is bounded, we see that when \(\|u - v\|_2 \to 0\), \(\|w(t, u(n), v(n))\| \to 0\) holds for \(0 \leq n \leq \omega - 1\). By (41), we have
\[
    \Delta (V(n)) = A(n, u(n)) V(n) + w(t, u(n), v(n)).
\]

(43)

that is, \(V(n)\) is an \(\omega\)-periodic solution of (33). By Theorem 1 we have
\[
    |V(n)| \leq \left| (I - \Phi_u(n + \omega, n))^{-1} \right| \left\| \bigoplus_{i=n}^{n+\omega-1} \Phi_u(n + \omega, i + 1) |w(t, u(i), v(i))| \right\|
\]

\[
    \leq \frac{M_0}{1 - \beta} \left\| \bigoplus_{i=n}^{n+\omega-1} |w(t, u(i), v(i))| \right\|.
\]

(44)

Thus, we see that when \(\|u - v\| \to 0\), \(\|Pu - Pv\| = \|V\| \to 0\). On the other hand, in view of (11), we see that \(\|u - v\|^0 \to 0\) and \(\|Pu - Pv\|^1 = \|V\|^1 = \|\Delta V\|^0 \to 0\). Hence if \(\|u - v\|_2 \to 0\), then \(\|u - v\| \to 0\) and so \(\|Pu - Pv\|_2 = \|Pu - Pv\|_0 + \|Pu - Pv\|_1 \to 0\), that is, \(P\) is a continuous mapping on \(G\). On the other hand, note that \(PG \subset G\) and \(G\) is bounded, from Lemma 3, we know that \(PG\) is relatively compact. Thus \(P\) is a completely continuous mapping from \(G\) into \(G\). By means of the Krasnoselskii’s theorem, we know that \(P + S\) has a fixed point in \(G\). By Theorem 1, (11) has an \(\omega\)-periodic solution. The proof is complete.

**COROLLARY 1.** Suppose there is a nontrivial \(\omega\)-periodic sequence \(\{a(n)\}_{n \in \mathbb{Z}}\) such that
\[
    \beta = \exp \left( \bigoplus_{i=0}^{\omega-1} \alpha(i) \right) < 1,
\]

and \(|a_{ij}(n, x)| < 1\) for \(1 \leq i, j \leq s\) and \((n, x) \in \mathbb{Z} \times \mathbb{R}^s\) and
\[
    \mu(A(n, x)) \leq \alpha(n) \leq 0.
\]

Suppose further that there is \(M > 0\) such that
\[
    \bigoplus_{i=0}^{\omega-1} \sup_{|x| \leq M} |f(i, x)| < (1 - \beta) M (1 - 2|c|) - \frac{ML + b_0}{(1 - |c|)} |c| \omega,
\]

where
\[
    L = \sup_{|x| < M, 0 \leq n \leq \omega} \|A(n, x)\|
\]

and
\[
    b_0 = \sup_{0 \leq n \leq \omega - 1, |x| \leq M} |f(n, x)|.
\]

Then (11) has an \(\omega\)-periodic solution.
As an example, consider the two dimensional nonlinear neutral difference system of the form
\[
\Delta \left[ x(n) - \frac{1}{16} x(n - \tau) \right] = A(n, x(n)) x(n) + f(n, x(n - \sigma)), \quad n \in \mathbb{Z}, \tag{45}
\]
where \( \tau \) and \( r \) are positive integers,
\[
A(n, x) = \left( \frac{(-1)^n}{8} \exp \left( -x_1^2 - x_2^2 \right) \right), \quad n \in \mathbb{Z},
\]
and
\[
f(n, x) = \left( \frac{(-1)^n}{8} \exp \left( -x_1^2 - x_2^2 \right) \right), \quad n \in \mathbb{Z}.
\]
It is easy to see that \( |a_{ii}(n, x)| = \frac{1}{4} < 1 \) for \( i = 1, 2 \), \( \mu_\infty(A(n, x)) \leq -\frac{1}{8} \) and \( \sup_{0 \leq n \leq 1, |x| \leq M} |f(n, x)|_\infty \leq 1/4 \). If we let \( \alpha(n) = -\frac{1}{8} \) and \( M = 16 \), then \( \beta = \exp \left( \bigoplus_{i=0}^{\omega-1} \alpha(i) \right) = e^{-\frac{4}{\omega}} \) and \( L = \sup_{|x| \leq M} \| A(n, x) \|_\infty = \frac{1}{4}, b_0 = \sup_{0 \leq n \leq 1, |x| \leq M} |f(n, x)|_1 \leq \frac{1}{4} \). In view of these calculations, we may see that the conditions of Corollary 1 are satisfied. Hence (45) has a 2-periodic solution. This solution is also nontrivial, since \( f(n, 0) \neq 0 \).

References