A class of global weak solutions to the axisymmetric isentropic Euler equations of perfect gases in two space dimensions

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We consider the compressible isentropic Euler equations for a perfect gas \((t > 0, x \in \mathbb{R}^N)\):

\[
\partial_t \rho + \sum_{1 \leq k \leq N} \partial_k (\rho u_k) = 0 \quad (1)
\]

(conservation of mass),

\[
\partial_t (\rho u_i) + \sum_{1 \leq k \leq N} \partial_k (\rho u_k u_i) + \partial_i p = 0, \quad (2_i)
\]

\(1 \leq i \leq N\) (conservation of momentum), where \(\rho\) is the density, \(u = \begin{pmatrix} u_1 \\ \vdots \\ u_N \end{pmatrix}\) the velocity, and \(p(\rho)\) the pressure. We assume that \(p(\rho) = a \rho^\gamma, \ a > 0, 1 < \gamma \leq 1 + \frac{2}{N}\).

We impose the initial conditions

\[
u(x, 0) = u_0(x), \ \rho(x, 0) = \rho_0(x). \quad (3)
\]

One has the following results.

(I) If \(\rho_0 = \bar{\rho} + \rho_1\), where \(\bar{\rho} > 0\) is a constant, \(\rho_1\) and \(u_0 \in H^s(\mathbb{R}^N)\) with \(s\) an integer \(> \frac{N}{2} + 1\) and \(\inf \rho_0 > 0\), one can find a solution to (1), (2), (3) for \(t\) small (see \([7]\)).

(II) If \(\rho_0^{\frac{N+1}{2}}\) and \(u_0 \in H^s_{ul}(\mathbb{R}^N),\) \(s\) integer > \(\frac{N}{2} + 1\) and \(\rho_0 > 0\), one can find a solution to (1), (2), (3) for \(t\) small (Chemin \([2]\)). Here \(H^s_{ul}(\mathbb{R}^N) = \{ v \in H^s_{loc}(\mathbb{R}^N), \sup_{x \in \mathbb{R}^N} ||\varphi x^e||_s < +\infty \) if \(\varphi \in C^\infty_0(\mathbb{R}^N)\}\), where \(\varphi x(y) = \varphi(x-y)\) and \(|| \ ||_s\) is the standard \(H^s\) norm.

In general, solutions to (1), (2), (3) are not global in \(t\) (Sideris \([10]\), Rammaha \([8]\)). In case (I), when \(N = 2\) and \(\rho_0, u_0\) are rotation invariant around \(0\) with \(\rho_1 = \varepsilon \tilde{\rho}_1, u_0 = \varepsilon u_0, \tilde{\rho}_1, \tilde{u}_0 \in C^\infty_0(\mathbb{R}^2)\) and \(|\tilde{\rho}_1| + |\text{div} \tilde{u}_0| \neq 0\), Alinhac \([1]\) has shown that the lifespan of solutions is \(\sim \frac{1}{\varepsilon^2}\) (\(\varepsilon\) small); see also Sideris \([11]\).
Grassin-Serre [5] and Grassin [4] have obtained global results (see also [9]) that we are going to describe now. If \( \rho \) never vanishes, it follows from (1), (2) that

\[
\frac{\partial_t u_i + \sum_{1 \leq k \leq N} u_k \partial_k u_i + \frac{\partial_i \rho}{\rho}}{\rho} = 0 \tag{2'}
\]

for \( 1 \leq i \leq N \). One can symmetrize (1), (2') (1 \( \leq i \leq N \)) by introducing \( \pi = C_1^{-1} \sqrt{\rho'(\rho)} \), \( C_1 = \frac{\gamma - 1}{2} \). (1), (2') (1 \( \leq i \leq N \)) become

\[
\frac{\partial_t \pi + \sum_{1 \leq k \leq N} u_k \partial_k \pi + C_1 \pi \sum_{1 \leq k \leq N} \partial_k u_k}{\rho} = 0, \tag{4}
\]

\[
\frac{\partial_t u_i + \sum_{1 \leq k \leq N} u_k \partial_k u_i + C_1 \pi \partial_i \pi}{\rho} = 0, \tag{5_i}
\]

1 \( \leq i \leq N \). This symmetrization has already been used by Chemin [2] for (II).

Consider the initial data

\[
u(x,0) = u_0(x), \quad \pi(x,0) = \pi_0(x). \tag{6}\]

Grassin-Serre and Grassin have introduced the following assumptions:

\[
\partial^\alpha u_0 \in L^\infty(\mathbb{R}^N) \text{ if } |\alpha| = 1, \quad \partial^\alpha u_0 \in H^{s-1}(\mathbb{R}^N) \text{ if } |\alpha| = 2, \quad \inf_{x \in \mathbb{R}^N} \text{dist}(sp \, du_0, \mathbb{R}^-) > 0, \quad \pi_0 \in H^s(\mathbb{R}^N) \text{ and } ||\pi_0||_s \text{ is small (s integer } > \frac{N}{2} + 1). \tag{7}\]

**Theorem 1 ([5], [4]).** If (7) is satisfied, (4), (5), (6) has a global solution when \( t > 0, \ x \in \mathbb{R}^N \).

This theorem is obtained by comparing \((\pi, u)\) with \((0, \bar{u})\), where \((0, \bar{u})\) is the solution to (4), (5_i), \(1 \leq N \), with initial data \((0, u_0)\).

The purpose of this talk is to describe a result of the same type (contained in [3]) for a class of non-smooth initial data.

We shall assume that \( N = 2 \) and consider initial data which are rotation invariant around 0, so \( u_0(Sx) = S u_0(x) \) and \( \pi_0(Sx) = \pi_0(x) \) for every rotation \( S \) with center 0. It follows that

\[
u_0(x) = A_0(r) \frac{x}{r} + B_0(r) \frac{x^+}{r} \text{ and } \pi_0(x) = \Pi_0(r) \text{ with } r = |x|,
\]

\[
x^+ = (-x_2, x_1).
\]

We start with \( \bar{u}_0(x) = A_0(r) \frac{x}{r} + B_0(r) \frac{x^+}{r} \), satisfying (7) with \( s = 3 \), and consider two small perturbations of \( \bar{u}_0 \), namely \( u_0^{(1)}, u_0^{(2)} \), rotation invariant around 0. We assume that

\[
\sum_{|\alpha| \leq 1} ||\partial^\alpha (u_0^{(1)} - \bar{u}_0)|| + \sum_{|\alpha| = 2} ||\partial^\alpha (u_0^{(2)} - \bar{u}_0)|| \leq \varepsilon.
\]
Consider \( \pi^{(j)}_0(x) \equiv \Pi^{(j)}_0(r) > 0, j = 1, 2 \), such that \( ||\pi^{(j)}_0||_3 \leq \varepsilon \) and \( \Pi^{(j)}_0(r) \geq C_0 \varepsilon \) if \( 0 \leq (-1)^j(r - 1) \leq C \varepsilon \) \((C > 0 \text{ large enough})\). Put

\[
(\pi_0, u_0) = \begin{cases} 
(\pi^{(1)}_0, u^{(1)}_0) & \text{if } r < 1, \\
(\pi^{(2)}_0, u^{(2)}_0) & \text{if } r > 1.
\end{cases}
\]

Write \( u_0(x) = A_0(r) \frac{x}{r} + B_0(r) \frac{x^\perp}{r}, \pi_0(x) = \Pi_0(r) \), and assume that

\[
0 < [A_0 \pm \Pi_0](1) \leq C_2 \varepsilon^{2+\theta},
\]

\[
0 < ||B^2_0(1)|| \leq C_3 \varepsilon,
\]

where \( C_2, C_3 \) are small and \( 0 < \theta < \frac{1}{2} \). Here \([F](1) = \lim_{r \to 1} F(r) - \lim_{r \to 1} F(r)\). Theorem 2 \((5)\). If \( \varepsilon \) is small, there exists a weak solution to (1), (2) which is rotation invariant around 0 and global in \( t > 0 \), such that \( \rho_{|t=0} = \tilde{C} \pi_0^{1/C_1} \), \( u_{|t=0} = u_0 \), where \( \tilde{C} = C_1^{1/C_1} (a \gamma)^{-1/2 C_1} \). This solution consists of two centered waves (in the \((r, t)\) variables) and one contact discontinuity.

Local existence is obtained by adapting results and ideas of \( [6] \). The global results can be proved by a continuation method.

References


3. P. GODIN, Global centered waves and contact discontinuities for the axisymmetric isentropic Euler equations of perfect gases in two space dimensions, preprint.


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