On a Transmission Problem for Dissipative Klein-Gordon-Shrödinger Equations

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ABSTRACT: In this paper we consider a transmission problem for the Cauchy problem of coupled dissipative Klein-Gordon-Shrödinger equations and we prove the existence of global solutions.

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1. Introduction

Let \( [0, L_3] \) be a bounded open interval of \( \mathbb{R} \) such that \( L_1, L_2 \in [0, L_3] \). We denote by \( \Omega \) the set \( [0, L_1] \cup [L_2, L_3] \).

In this work we prove the existence of strong and weak solutions of a transmission problem for the coupled Klein-Gordon-Shrödinger equations with dissipative term, given by the following system:

\[
\begin{align*}
\psi_t + \psi_{xx} + i\alpha \psi + \phi \psi &= 0 & \text{in} & & \Omega \times [0, \infty[ \\
\phi_{tt} - \phi_{xx} + \phi + \beta \phi_t &= |\psi|^2 & \text{in} & & \Omega \times [0, \infty[ \\
\theta_{tt} - \theta_{xx} &= 0 & \text{in} & & [L_1, L_2] \times [0, \infty[ 
\end{align*}
\]

where \( \alpha \) and \( \beta \) are positive constants.

The system is subjected to the following boundary conditions.

\[
\begin{align*}
\psi(0, t) &= \psi(L_3, t) = \phi(0, t) = \phi(L_3, t) = 0 \\
\phi(L_i, t) &= \theta(L_i, t) ; \quad \phi_x(L_i, t) = \theta_x(L_i, t) ; \quad i = 1, 2 \\
\psi_x(L_i, t) &= 0 ; \quad i = 1, 2 
\end{align*}
\]

and initial conditions

\[
\begin{align*}
\psi(x, 0) &= \psi_0(x) ; \quad x \in \Omega \\
\phi(x, 0) &= \phi_0(x) ; \quad \phi_t(x, 0) = \phi_1(x) ; \quad x \in \Omega \\
\theta(x, 0) &= \theta_0(x) ; \quad \theta_t(x, 0) = \theta_1(x) ; \quad x \in [L_1, L_2] 
\end{align*}
\]

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Controllability for transmission problems has been studied by several authors, and we mention a few works. The transmission problem for the wave equation was studied by Lions [7], where he applied the Hilbert Uniqueness Method (HUM) to show exact controllability. Latter, Lagnese [6], also applying HUM, extended this result; he showed the exact controllability for a class of hyperbolic systems which include the transmission problem for homogeneous anisotropic materials. The exact controllability for the plate equation was proved by Liu and Williams [9]. Some results about existence, uniqueness and regularity for elliptic stationary transmission problem can be found in Athanasiadis and Stratis [1] and Ladyzhenskaya and Ural'tseva [5].

Concerning stability, Liu and Williams [8] studied a transmission problem for the wave equation and showed exponential decay of the energy provided a linear feedback velocity is applied at the boundary. Marzocchi et al. [10] proved that the solution of a semi-linear transmission problem between an elastic a thermoelastic material, decays exponentially to zero.

Let us mention some works related with the Klein-Gordon -Schrödinger equations. Fukuda and Tsutsumi [4] studied the initial-boundary value problem for the coupled Klein-Gordon -Schrödinger equations in three space dimensions. In the case of one space dimension, the existence of global smooth solutions has been established by the authors [3]. Boling and Yongsheng [2] considerer the Cauchy problem of coupled dissipative proved a existence Klein-Gordon -Schrödinger equations in $\mathbb{R}^3$ and prove the existence of the maximal attractor.

The objective of this paper is to prove the existence of strong and weak solutions to problem (1.1)-(1.9). The proof of the existence is based on the Galerkin method and employed techniques in [2].

2. Notation

For brevity, we denote the space of complex-valued functions and real-valued functions by the same symbols.

Let $L^p(\Omega)$ be the usual Lebesgue space of complex-valued or real-valued functions whose $p$-times powers are integrable with norm:

$$|u|_p = \left( \int_\Omega |u(x)|^p dx \right)^{1/p} < +\infty \quad (1 \leq p < +\infty).$$

$$|u|_\infty = \text{ess} \sup_{x \in \Omega} |u(x)| < +\infty \quad (p = +\infty).$$

In particular, $L^2(\Omega)$ is the Hilbert space with inner product and norm:

$$(u, v) = \int_\Omega u(x)v(x) dx, \quad |u|_2 = \|u\| = (u, u)^{1/2}.\quad (2.2)$$

$H^m(\Omega)$ ($m$ is an integer $\geq 1$) denote the complex or real Sobolev spaces whose distributional derivatives of order $\leq m$ lie in $L^2(\Omega)$ equipped with inner product and norm:

$$(u, v)_m = \sum_{j=0}^m \int_\Omega D^j u(x) D^j \overline{v}(x) dx, \quad \|u\|_m = (u, u)^{1/2}.\quad (2.3)$$
Let us define the subspace\[ H^1_1(\Omega) = \{ w \in H^1(\Omega); w(0) = w(L_0) = 0 \} \]

It follows that \( H^1_1(\Omega) \) is a Hilbert subspace of \( H^1(\Omega) \). We can prove that in \( H^1_1(\Omega) \) the norm\[ \|w\|^2 = \int_\Omega |w_x(x)|^2\,dx \quad (2.4) \]

and the \( H^1_1(\Omega) \) norm are equivalents. Consequently, we consider \( H^1_1(\Omega) \) equipped with the norm \( (2.4) \) and the scalar product\[ ((v, w)) = \int_\Omega v_x(x) \cdot w_x(x)\,dx \quad (2.5) \]

Also let us define the subspace\[ V = \{ \{u, v\} \in H^1_1(\Omega) \times H^1([L_1, L_2]) ; u(L_i) = v(L_i) , i = 1, 2 \} \]

Note that \( V \) is a closed subspace of \( H^1_1(\Omega) \times H^1([L_1, L_2]) \) which together with the norm\[ \|\{u, v\}\|^2_V = \int_\Omega |u_x(x)|^2\,dx + \int_{L_1}^{L_2} |v_x(x)|^2\,dx \quad (2.6) \]
is a Hilbert space.

### 3. Existence of solutions

In this section we establish existence and uniqueness results for problem \([1.1] - (1.9)\).

First of all, we define what we will understand for strong and weak solution of the problem \([1.1] - (1.9)\).

**Definition 3.1** We say that \((\psi, \phi, \theta)\) is a strong solution of \([1.1] - (1.9)\) when\[
\psi \in L_\infty^{\infty,0}(0, \infty; H^2(\Omega) \cap H^1_1(\Omega)) \\
\psi_t \in L_\infty^{\infty,0}(0, \infty; H^1_1(\Omega)) \\
\{\phi, \theta\} \in L_\infty^{\infty,0}(0, \infty; [H^2(\Omega) \times H^2([L_1, L_2])] \cap V) \\
\{\phi_t, \theta_t\} \in L_\infty^{\infty,0}(0, \infty; V) \\
\{\phi_{tt}, \theta_{tt}\} \in L_\infty^{\infty,0}(0, \infty; L^2([L_1, L_2]))
\]
satisfying the identities

\[ i\psi_t + \psi_{xx} + i\alpha \psi + \phi \psi = 0 \quad \text{in} \quad L^\infty_0(0, \infty; L^2(\Omega)) \]

\[ \phi_{tt} - \phi_{xx} + \phi + \beta \phi_t = |\psi|^2 \quad \text{in} \quad L^\infty_0(0, \infty; L^2(\Omega)) \]

\[ \theta_{tt} - \theta_{xx} = 0 \quad \text{in} \quad L^\infty_0(0, \infty; L^2([L_1, L_2])) \]

\[ \psi(0, t) = \psi(L_3, t) = \phi(0, t) = \phi(L_3, t) = 0 \quad ; \quad t > 0 \]

\[ \phi(L_i, t) = \theta(L_i, t) ; \phi_x(L_i, t) = \theta_x(L_i, t) \quad ; \quad t > 0 \quad , \quad (i = 1, 2) \]

\[ \psi_x(L_i, t) = 0 \quad ; \quad t > 0 \quad , \quad (i = 1, 2) \]

\[ \psi(x, 0) = \psi_0(x) ; \quad x \in \Omega \]

\[ \phi(x, 0) = \phi_0(x) \quad \epsilon \quad \phi_t(x, 0) = \phi_1(x) \quad ; \quad x \in \Omega \]

\[ \theta(x, 0) = \theta_0(x) \quad \epsilon \quad \theta_t(x, 0) = \theta_1(x) \quad ; \quad x \in [L_1, L_2[ \]

**Definition 3.2** Let \( T > 0 \) be real. We say that \((\psi, \phi, \theta)\) is a weak solution of

\[ (1.1) - (1.9) \]

when

\[ \psi \in L^\infty(0, T; H^1_0(\Omega)) \]

\[ \{\phi, \theta\} \in L^\infty(0, T; V) \quad , \quad \{\phi_1, \theta_1\} \in L^\infty(0, T; L^2(\Omega) \times L^2([L_1, L_2])) \]

satisfying the identities

\[ \int_0^T \int_\Omega [-i\psi \Psi_t - \psi_x \Psi_x + i\alpha \psi \Psi + \phi \Psi] \, dx \, dt = \int_\Omega i\psi_0(x)\Psi(x, 0) \, dx \]

\[ \int_0^T \int_{L_1}^{L_2} [\phi \Phi_{tt} + \phi_x \Phi_x + \phi \Phi - \beta \phi \Phi_t - |\psi|^2 \Phi] \, dx \, dt \]

\[ + \int_0^T \int_{L_1}^{L_2} [\theta \Theta_{tt} + \theta_t \Theta_x] \, dx \, dt \]

\[ = \int_\Omega \phi_1(x)\Phi(x, 0) \, dx - \int_\Omega \phi_0(x)\Phi_t(x, 0) \, dx + \beta \int_\Omega \phi_0(x)\Phi(x, 0) \, dx \]

\[ + \int_{L_1}^{L_2} \theta_1(x)\Theta(x, 0) \, dx + \int_{L_1}^{L_2} \theta_0(x)\Theta_t(x, 0) \, dx \]

for all \( \Psi \in C^1([0, T]; H^1_0(\Omega)) \), \( \{\Phi, \Theta\} \in C^2([0, T]; V) \) and a.e \( t \in [0, T] \) such that

\[ \Psi(T) = \Phi(T) = \Phi_t(T) = \Theta(T) = \Theta_t(T) = 0 \]

The existence of strong solution to system \[(1.1) - (1.9)\] is given in the following theorem:

**Theorem 1** Given

\[ \psi_0 \in H^2(\Omega) \cap H^1_0(\Omega) \]

\[ \{\phi_0, \theta_0\} \in [H^2(\Omega) \times H^2([L_1, L_2])] \cap V \]

\[ \{\phi_1, \theta_1\} \in V \]
with
\[
\psi_{0x}(L_i) = 0 \quad (i = 1, 2) \\
\phi_{0x}(L_i) = \theta_{0x}(L_i) \quad (i = 1, 2)
\]
there exists only a strong solution of \([1.1] - [1.3]\).

**Proof.** We follow a standard Faedo-Galerkin method and we divide the proof in four steps.

**Step 1 (Approximate System).** Let us denote by \(\{u_i; i \in \mathbb{N}\}\) a basis of \(H^2(\Omega) \cap H^1_t(\Omega)\) and by \(\{v_i, w_i; i \in \mathbb{N}\}\) a basis of \([H^2(\Omega) \times H^2([L_1, L_2])] \cap V\). We denote by
\[
H_\nu = \text{span}\{u_1, u_2, \ldots, u_\nu\} \\
V_\nu = \text{span}\{\{v_1, w_1\}, \{v_2, w_2\}, \ldots, \{v_\nu, w_\nu\}\}
\]
Let
\[
\psi^\nu(x, t) = \sum_{i=1}^\nu a_i(t)u_i \quad (a_i(t) : \text{Complex-valued})
\]
and
\[
\phi^\nu(x, t) = \sum_{i=1}^\nu b_i(t)v_i \quad (b_i(t) : \text{Real-valued})
\]
be solutions of the system \((j = 1, 2, \ldots, \nu)\) of ordinary differential equations
\[
\int_\Omega [i\psi^\nu \bar{u}_j - \psi^\nu x \bar{u}_j + i\alpha \psi^\nu \bar{u}_j + \phi^\nu \psi^\nu \bar{u}_j] \, dx = 0 \quad (3.1)
\]
\[
\int_\Omega [\phi^\nu \bar{v}_j + \phi^\nu x \bar{v}_j + \phi^\nu v_j + \beta \phi^\nu \bar{v}_j - |\psi^\nu|^2v_j] \, dx \\
+ \int_{L_2}^{L_2} [\theta^\nu \bar{w}_j + \theta^\nu_x \bar{w}_j] \, dx = 0 \quad (3.2)
\]
which satisfy the initial data
\[
\psi^\nu(0) = \psi_0 , \quad \phi^\nu(0) = \phi_0 , \quad \theta^\nu(0) = \theta_0 , \quad \phi^\nu_t(0) = \phi_1 , \theta^\nu_t(0) = \theta_1
\]
Standard theorems in the theory of ordinary differential equations ensure that this system has the solutions \(\{\psi^m, \phi^m, \psi^m\} (m = 1, 2, 3, \ldots)\) locally in time which are uniquely determined by initial data, for each \(m\).

**Step 2 (Estimate I).** Multiplying \((3.1)\) by \(a_{j\nu}(t)\), summing over \(j\) and taking imaginary parts, we have
\[
\frac{1}{2} \frac{d}{dt} \|\psi^\nu(t)\|^2 + \alpha \|\psi^\nu(t)\|^2 = 0
\]
It follows that
\[
\|\psi^\nu(t)\|^2 + \alpha \int_0^t \|\psi^\nu(s)\|^2 ds = \|\psi_0\|^2 \quad (3.3)
\]
From (3.3) it follows that:

\[ \psi^{\nu} \text{ is bounded in } L^\infty(0, \infty; L^2(\Omega)) \]  

(3.4)

**Step 3 (Estimate II).** Multiplying (3.1) by \(-a_j'(t)\) and summing over \(j\), we have

\[
-\overline{\psi}_t \psi(t) + (\psi_t(t), \psi^\nu_t(t)) - \int_\Omega \phi^\nu \overline{\psi}_t dx = 0
\]

(3.5)

Multiplying (3.1) by \(-a_j(t)\) and summing over \(j\), we have

\[
-\overline{\psi}_t \psi^\nu(t) + \alpha \|\psi^\nu_t(t)\|^2 + \alpha \int_\Omega \phi^\nu |\psi^\nu|^2 dx = 0
\]

(3.6)

Taking real parts in \[-\overline{\psi}_t \psi^\nu(t) + \alpha \|\psi^\nu_t(t)\|^2 - \alpha \int_\Omega \phi^\nu |\psi^\nu|^2 dx = 0\], we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi^\nu_t(t)\|^2 - \text{Re}(\overline{\alpha} \psi^\nu(t), \psi^\nu_t(t)) - \text{Re} \left( \int_\Omega \phi^\nu \overline{\psi}_t dx \right) = 0
\]

(3.7)

\[
-\text{Re}(\overline{\alpha} \psi^\nu_t(t)) + \alpha \|\psi^\nu_t(t)\|^2 - \alpha \int_\Omega \phi^\nu |\psi^\nu|^2 dx = 0
\]

(3.8)

Summing (3.7) and (3.8), we obtain

\[
\frac{1}{2} \frac{d}{dt} \|\psi^\nu_t(t)\|^2 + \alpha \|\psi^\nu_t(t)\|^2 - \text{Re} (\phi^\nu \psi^\nu, \psi^\nu_t(t)) = 0
\]

(3.9)

Noticing that

\[
-\text{Re} (\phi^\nu \psi^\nu, \psi^\nu_t(t)) = -\frac{1}{2} \frac{d}{dt} (\phi^\nu_t, |\psi^\nu|^2) + \frac{1}{2} (\phi^\nu_t, |\psi^\nu|^2)
\]

(3.10)

We infer from (3.9) that

\[
\frac{1}{2} \frac{d}{dt} \left( \|\psi^\nu_t(t)\|^2 - \int_\Omega \phi^\nu |\psi^\nu|^2 dx \right) + \alpha \|\psi^\nu_t(t)\|^2 + \frac{1}{2} \int_\Omega \phi^\nu_t |\psi^\nu|^2 dx - \alpha \int_\Omega \phi^\nu |\psi^\nu|^2 dx = 0
\]

(3.11)

or

\[
\frac{d}{dt} \left( 2\|\psi^\nu_t(t)\|^2 - 2\int_\Omega \phi^\nu |\psi^\nu|^2 dx \right) + 4\alpha \|\psi^\nu_t(t)\|^2 + 2 \int_\Omega \phi^\nu_t |\psi^\nu|^2 dx - 4\alpha \int_\Omega \phi^\nu |\psi^\nu|^2 dx = 0
\]

(3.12)
We introduce the transformations
\[ \eta^\nu(t) = \phi^\nu_t(t) + \delta \phi^\nu(t) \]
and
\[ \gamma^\nu(t) = \theta^\nu_t(t) + \delta \theta^\nu(t) \]
where \( \delta = \min\left(\frac{d}{2}, \frac{1}{2d}\right) \). Then (3.2) is equivalent to.

\[ (\eta^\nu_t(t), v_j) + (\beta - \delta)(\eta^\nu(t), v_j) + (1 - \delta(\beta - \delta))(\phi^\nu_t(t), v_j) + (\phi^\nu_t(t), v_{j,x}) + \int_{L_1} \gamma^\nu_t w_j dx + \delta^2 \int_{L_2} \gamma^\nu w_j dx + \int_{L_2} \theta^\nu_t w_j dx = \]

\[ + \int_{\Omega} |\gamma^\nu|^2 v_j dx + \delta \int_{L_1} \gamma^\nu w_j dx \]

Multiplying (3.13) by \( b_j(t) + \delta b_j(t) \) and summing over \( j \), we have.

\[ \frac{1}{2} \frac{d}{dt} \int_{L_2} [|\eta^\nu(t)|^2 + (1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + |\phi^\nu_t(t)|^2] \]

\[ + \frac{1}{2} \frac{d}{dt} \int_{L_1} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1} |\theta^\nu|^2 dx + \int_{L_1} |\theta^\nu_t|^2 dx \]

\[ + (\beta - \delta)|\eta^\nu(t)|^2 + \delta(1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 \]

\[ + \delta|\phi^\nu_t(t)|^2 + \delta^3 \int_{L_2} |\theta^\nu|^2 dx + \delta \int_{L_2} |\theta^\nu|^2 dx = \]

\[ + \int_{\Omega} |\gamma^\nu|^2 dx + \delta \int_{L_2} |\gamma^\nu|^2 dx = \]

\[ + \int_{\Omega} \phi^\nu_t |\psi^\nu|^2 dx + \delta \int_{L_1} \phi^\nu |\psi^\nu|^2 dx + \delta \int_{L_1} |\gamma^\nu|^2 dx \]

or

\[ \frac{d}{dt} \int_{L_2} [|\eta^\nu(t)|^2 + (1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + |\phi^\nu_t(t)|^2] \]

\[ + \frac{d}{dt} \int_{L_1} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1} |\theta^\nu|^2 dx + \int_{L_1} |\theta^\nu_t|^2 dx \]

\[ + 2(\beta - \delta)|\eta^\nu(t)|^2 + 2\delta(1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 \]

\[ + 2\delta|\phi^\nu_t(t)|^2 + 2\delta^3 \int_{L_1} |\theta^\nu|^2 dx + 2\delta \int_{L_1} |\theta^\nu|^2 dx = \]

\[ + 2 \int_{\Omega} \eta^\nu |\psi^\nu|^2 dx + 2\delta \int_{L_2} |\gamma^\nu|^2 dx = \]

\[ + 2 \int_{\Omega} \phi^\nu_t |\psi^\nu|^2 dx + 2\delta \int_{L_1} \phi^\nu |\psi^\nu|^2 dx + 2\delta \int_{L_1} |\gamma^\nu|^2 dx \]

then (3.12) + (3.14) implies that

\[ \frac{d}{dt} H^\nu(t) + I^\nu(t) = 0 \]
where

\[
H^\nu(t) = 2\|\psi^\nu(t)\|^2 - 2\int_\Omega \phi^\nu|\psi^\nu|^2 dx + \|\eta^\nu(t)\|^2 \\
+ (1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + |\phi^\nu_x(t)|^2 \\
+ \int_{L_1} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1} |\theta^\nu|^2 dx + \int_{L_1} |\theta^\nu_x|^2 dx,
\]

(3.16)

\[
I^\nu(t) = 4\alpha\|\psi^\nu_x\|^2 - 2(2\alpha + \delta)\int_\Omega \phi^\nu|\psi^\nu|^2 dx + 2(\beta - \delta)|\eta^\nu(t)|^2 \\
+ 2\delta(1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + 2\delta|\phi^\nu_x(t)|^2 \\
+ 2\delta^3 \int_{L_1} |\theta^\nu|^2 dx + 2\delta \int_{L_1} |\theta^\nu_x|^2 dx - 2\delta \int_{L_1} |\gamma^\nu|^2 dx
\]

(3.17)

For arbitrary \( \epsilon_1, \epsilon_2 > 0 \),

\[
\left| \int_\Omega \phi^\nu|\psi^\nu|^2 dx \right| \leq \epsilon_1\|\psi^\nu_x(t)\|^2 + \epsilon_2\|\phi^\nu_x(t)\|^2 + c(\epsilon_1, \epsilon_2)|\psi^\nu(t)|^6
\]

(3.18)

Taking \( \epsilon_1 = \frac{1}{2}, \epsilon_2 = \frac{1}{4} \) in (3.18), we deduce that

\[
H^\nu(t) \geq \|\psi^\nu_x(t)\|^2 + \|\eta^\nu(t)\|^2 + (1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + \frac{1}{2}|\phi^\nu_x(t)|^2 \\
+ \int_{L_1} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1} |\theta^\nu|^2 dx + \int_{L_1} |\theta^\nu_x|^2 dx - c|\psi^\nu(t)|^6,
\]

(3.19)

\[
H^\nu(t) \leq 3\|\psi^\nu_x(t)\|^2 + \|\eta^\nu(t)\|^2 + (1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 + \frac{3}{2}|\phi^\nu_x(t)|^2 \\
+ c\|\psi^\nu(t)\|^6 + \int_{L_1} |\gamma^\nu|^2 dx + \delta^2 \int_{L_1} |\theta^\nu|^2 dx + \int_{L_1} |\theta^\nu_x|^2 dx
\]

(3.20)

Taking \( \epsilon_1 = \frac{\alpha}{2\alpha + \delta}, \epsilon_2 = \frac{\delta}{2(2\alpha + \delta)} \) in (3.18), we see that

\[
I^\nu(t) \geq +2\alpha\|\psi^\nu_x\|^2 + 2(\beta - \alpha)|\eta^\nu(t)|^2 + 2\delta(1 - \delta(\beta - \delta))|\phi^\nu(t)|^2 \\
+ \delta|\phi^\nu_x(t)|^2 - c\|\psi^\nu(t)\|^6 + 2\delta^3 \int_{L_1} |\theta^\nu|^2 dx \\
+ 2\delta \int_{L_1} |\theta^\nu_x|^2 dx - 2\delta \int_{L_1} |\gamma^\nu|^2 dx
\]

(3.21)

Thus from (3.20) and (3.21) we find a \( \beta_1 > 0 \) such that

\[
\beta_1 H^\nu(t) \leq I^\nu(t) + C\|\psi^\nu(t)\|^6 + C \int_{L_1} |\gamma^\nu|^2 dx.
\]

(3.22)
Therefore we derive from (3.15) and (3.22) that
\[
\frac{d}{dt} H^\nu(t) + \beta_1 H^\nu(t) \leq C\|\psi^\nu(t)\|^6 + C \int_{L_1}^{L_2} |\gamma^\nu|^2 dx. \tag{3.23}
\]
From (3.14) and (3.23) we obtain
\[
\frac{d}{dt} H^\nu(t) + \beta_1 H^\nu(t) \leq C + C \int_{L_1}^{L_2} |\gamma^\nu|^2 dx. \tag{3.24}
\]
It follows that
\[
H^\nu(t) \leq C|H^\nu(0)| + C \int_{L_1}^{L_2} |\gamma|^2 dx. \tag{3.25}
\]
From (3.25) and observing that \( |H^\nu(0)| \) is bounded, we have
\[
H^\nu(t) \leq C + C \int_{L_1}^{L_2} |\gamma|^2 dx. \tag{3.26}
\]
From (3.19), (3.26) and using Gronwall inequality we obtain
\[
\|\psi^\nu(t)\|^2 + \|\eta^\nu(t)\|^2 + \|\phi^\nu(t)\|^2 + \|\phi^\nu_{xx}(t)\|^2
+ \int_{L_1}^{L_2} |\gamma|^2 dx + \int_{L_1}^{L_2} |\theta|^2 dx + \int_{L_1}^{L_2} |\theta_{xx}|^2 dx \leq C(T). \tag{3.27}
\]
From (3.27) it follows that:
\[
\psi^\nu \text{ is bounded in } L^\infty(0, T; H^1_1(\Omega)) \tag{3.28}
\]
\[
(\phi^\nu, \theta^\nu) \text{ is bounded in } L^\infty(0, T; V) \tag{3.29}
\]
\[
(\phi^\nu_{xx}, \theta^\nu_{xx}) \text{ is bounded in } L^\infty(0, T; L^2(\Omega) \times L^2(\Omega)) \tag{3.30}
\]
**Step 4 (Estimate III)** First, we are going to estimate \( \|\psi^\nu_t(0)\|, \|\phi^\nu_{tt}(0)\| \) and \( \|\theta^\nu_{tt}(0)\| \). Indeed, from (16)-(17) and observing that
\[
\psi_0(L_i) = 0; \quad (i = 1, 2)
\]
\[
\phi_0(L_i) = \theta_0(L_i); \quad (i = 1, 2)
\]
we have
\[
\|\psi^\nu_t(0)\|^2 + \|\phi^\nu_{tt}(0)\|^2 + \|\theta^\nu_{tt}(0)\|^2 = i(\psi^\nu_{0xx}, \psi^\nu_t(0)) - \alpha(\phi_0, \psi^\nu_t(0))
+ i(\phi_0 \psi_0, \psi^\nu_t(0)) + (\phi_{0xx}, \phi^\nu_{tt}(0))
- (\psi_0, \phi^\nu_{tt}(0)) - \beta(\phi_1, \phi^\nu_{tt}(0))
+ (\psi_0^2, \phi^\nu_{tt}(0)) + (\theta_{0xx}, \theta^\nu_{tt}(0)) \tag{3.31}
\]
If follows that
\[
\|\psi_{t}^{*}(0)\| + \|\phi_{t}^{*}(0)\| + \|\theta_{t}^{*}(0)\| \leq C \quad \forall \, \nu \in \mathbb{N}
\] (3.32)

Now, taking the derivative of (3.1) and (3.2) with respect to \(t\) and, using arguments of step 3, we get that
\[
\frac{1}{2} \frac{d}{dt} \left[ \|\psi_{t}^{*}(t)\|^2 + \|\phi_{x,t}^{*}(t)\|^2 + \|\theta_{x,t}^{*}(t)\|^2 + \|\phi_{x}^{*}(t)\|^2 \right] \\
+ \frac{1}{2} \frac{d}{dt} \int_{L_1}^{L_2} \|\theta_{x}^{*}\|^2 dx + \int_{L_1}^{L_2} \|\theta_{x,t}^{*}\|^2 dx + \alpha \|\psi_{t}^{*}\|^2 + \beta \|\phi_{x,t}^{*}\|^2
\]

\[
= -Im \int_{\Omega} (\phi_{x}^{*} \psi^{*} \nu_t^{*}) dx + 2 \int_{\Omega} \psi^{*} \nu_t^{*} \phi^{*} dx
\]
\[
\leq C[\|\psi_{t}^{*}(t)\|^2 + \|\phi_{x,t}^{*}(t)\|^2 + \|\theta_{x,t}^{*}(t)\|^2]
\] (3.33)

Integration (3.33) from zero to \(t\), for \(0 \leq t \leq T\), \(T > 0\) any real number and observing the estimate (3.32), we have
\[
\|\psi_{t}^{*}(t)\|^2 + \|\phi_{x,t}^{*}(t)\|^2 + \|\theta_{x,t}^{*}(t)\|^2 + \|\phi_{x}^{*}(t)\|^2
\]

\[
+ \int_{L_1}^{L_2} \|\theta_{x}^{*}\|^2 dx + \int_{L_1}^{L_2} \|\theta_{x,t}^{*}\|^2 dx
\]
\[
\leq C + C \int_{0}^{T} \left[ \|\psi_{t}^{*}(s)\|^2 + \|\phi_{x,t}^{*}(s)\|^2 + \|\theta_{x,t}^{*}(s)\|^2 \right] ds
\] (3.34)

Applying Gronwall inequality to (3.34), we obtain:
\[
\|\psi_{t}^{*}(t)\|^2 + \|\phi_{x,t}^{*}(t)\|^2 + \|\theta_{x,t}^{*}(t)\|^2
\]
\[
+ \|\phi_{x}^{*}(t)\|^2 + \int_{L_1}^{L_2} \|\theta_{x}^{*}\|^2 dx + \int_{L_1}^{L_2} \|\theta_{x,t}^{*}\|^2 dx \leq C
\] (3.35)

independent of \(\nu\), for all \(t\) in \([0, T]\).

From (3.35) it follows that:
\[
\psi_{t}^{*} \text{ is bounded in } L^\infty(0, T; L^2(\Omega))
\] (3.36)
\[
(\phi_{x}^{*}, \theta_{t}^{*}) \text{ is bounded in } L^\infty(0, T; V)
\] (3.37)
\[
(\phi_{x,t}^{*}, \theta_{x,t}^{*}) \text{ is bounded in } L^\infty(0, T; L^2(\Omega) \times L^2([L_1, L_2]))
\] (3.38)

The rest of the proof of the existence of strong solution is a matter routine.

The existence of weak solution to system \([1.1] - [1.9]\) is given in the following theorem:

**Theorem 2** Given
\[
\psi_0 \in H_1^2(\Omega), \quad \{\phi_0, \theta_0\} \in V \quad \text{and} \quad \{\phi_1, \theta_1\} \in L^2(\Omega) \times L^2([L_1, L_2])
\]

there exists only a weak solution of \([1.1] - [1.9]\).
Proof. Given \( \psi_0 \in H^1_0(\Omega), \{\phi_0, \theta_0\} \in V \) and \( \{\phi_1, \theta_1\} \in L^2(\Omega) \), there exists \( \psi'_0 \in H^2(\Omega) \cap H^1_0(\Omega), \{\phi'_0, \theta'_0\} \in [H^2(\Omega) \times H^2([L_1, L_2])] \cap V \) and \( \{\phi_1, \theta_1\} \in V \) such that

\[
\psi'_0 \quad \rightarrow \quad \psi_0 \quad \text{strongly in } H^1_0(\Omega) \\
\{\phi'_0, \theta'_0\} \quad \rightarrow \quad \{\phi_0, \theta_0\} \quad \text{strongly in } V \\
\{\phi'_1, \theta'_1\} \quad \rightarrow \quad \{\phi_1, \theta_1\} \quad \text{strongly in } L^2(\Omega) \times L^2([L_1, L_2])
\]

and

\[
\psi_{0x}(L_i) = 0 \quad (i = 1, 2) \\
\phi_{0x}(L_i) = \theta_{0x}(L_i) \quad (i = 1, 2)
\]

With \( \psi'_0, \{\phi'_0, \theta'_0\} \) and \( \{\phi'_1, \theta'_1\} \), above defined, we determine an unique strong solution \( \{\psi, \phi', \theta'\} \) satisfying all conditions of Theorem (I).

Using similar arguments of step 3 of Theorem (I), we have

\[
\psi'^{\nu} \quad \text{is bounded in } L^\infty(0, T, H^1_0(\Omega)) \\
\{\phi'^{\nu}, \theta'^{\nu}\} \quad \text{is bounded in } L^\infty(0, T, V) \\
\{\phi'^{\sigma}, \theta'^{\sigma}\} \quad \text{is bounded in } L^\infty(0, T, L^2(\Omega) \times L^2([L_1, L_2]))
\]

If follows that

\[
\psi^{\nu} \quad \overset{\Delta}{=} \quad \psi \quad \text{in } L^\infty(0, T, H^1_0(\Omega)) \\
\{\phi^{\nu}, \theta^{\nu}\} \quad \overset{\Delta}{=} \quad \{\phi, \theta\} \quad \text{in } L^\infty(0, T, V) \\
\{\phi^{\sigma}, \theta^{\sigma}\} \quad \overset{\Delta}{=} \quad \{\phi_1, \theta_1\} \quad \text{in } L^\infty(0, T, L^2(\Omega) \times L^2([L_1, L_2]))
\]

We suppose that \( \{\psi'^{\nu}, \phi'^{\nu}, \theta'^{\nu}\} \) and \( \{\psi'^{\sigma}, \phi'^{\sigma}, \theta'^{\sigma}\} \) are two strong solutions of \([1.1] - (1.9)\) with initial data

\[
\{\psi'_0, \phi'_0, \theta'_0\} \quad \text{and } \quad \{\psi'^{\sigma}, \phi'^{\sigma}, \theta'^{\sigma}\}
\]

After direct calculations, we have

\[
\frac{1}{2} \frac{d}{dt} E^{\nu}(t) + \alpha \|\psi'^{\nu} - \psi'^{\sigma}\|^2 + \beta \|\phi'^{\nu} - \phi'^{\sigma}\|^2 \\
\leq C \left[ \|\psi'^{\nu}(t) - \psi'^{\sigma}(t)\|^2 + \|\phi'^{\nu}(t) - \phi'^{\sigma}(t)\|^2 + \|\phi'^{\nu}(t) - \phi'^{\sigma}(t)\|^2 \right]
\]

where

\[
E^{\nu}(t) = \|\psi'^{\nu}(t) - \psi'^{\sigma}(t)\|^2 + \|\phi'^{\nu}(t) - \phi'^{\sigma}(t)\|^2 + \|\phi'^{\nu}(t) - \phi'^{\sigma}(t)\|^2 \\
+ \|\phi'^{\nu}(t) - \phi'^{\sigma}(t)\|^2 + \int_{L_1}^{L_2} |\theta'^{\nu}_x - \theta'^{\nu}_x|^2 dx + \int_{L_1}^{L_2} |\theta'^{\nu}_y - \theta'^{\nu}_y|^2 dx
\]

By Gronwall inequality, we have

\[
E^{\nu}(t) \leq C(T) E^{\nu}(t)
\]
From (3.39) and (3.42), we obtain

\[
\begin{align*}
\psi' & \rightarrow \psi \quad \text{in} \quad C([0,T];L^2(\Omega)) \\
\{\phi'_\nu, \theta'_\nu\} & \rightarrow \{\phi, \theta\} \quad \text{in} \quad C([0,T];V) \\
\{\phi^\nu_t, \theta^\nu_t\} & \rightarrow \{\phi_t, \theta_t\} \quad \text{in} \quad C([0,T];L^2(\Omega) \times L^2([L_1,L_2]))
\end{align*}
\]

(3.43)

The rest of the proof of the existence of weak solution is a matter routine.

REFERENCES


