A note on iterative solutions for a nonlinear fourth order ode *

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ABSTRACT: This work is concerned with the existence of iterative solutions for a class of fourth order differential equations with nonlinear boundary conditions modeling beams on nonlinear elastic foundations. Some numerical simulations are also considered.

Key Words: Beam equation, nonlinear boundary, numerical solutions.

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1. Introduction

In this work we are concerned with the boundary value problem

\[ u^{(iv)}(t) = f(t, u, u'), \quad 0 < t < L \] (1)

\[ u(0) = 0, \quad u(L) = 0 \] (2)

\[ u''(0) = g(u'(0)), \quad u''(L) = h(u'(L)), \] (3)

which models bending equilibrium of elastic beams on nonlinear supports. Following Ginsberg [7] or Grossinho and Tersian [8], \( u \) represents the configuration of an elastic beam of length \( L \), subject to a force \( f \) exerted by the foundation. Both ends are attached to fixed torsional springs represented by the functions \( g \) and \( h \).

Our objective is to show the existence of iterative solutions under local conditions on the functions \( f, g, h \). Some numerical simulations are also presented. We refer the reader to [2,3,4,5,8,9] for other related works.

2. Iterative Solutions

Our existence result is the following.

**Theorem 2.1** Suppose that \( f, g, h \) are continuous functions and there exist constants \( A, B, C > 0 \) such that

\[ |f(t, u, v)| \leq A, \quad \forall (t, u, v) \in [0, T] \times \left[-\frac{L}{2}R, \frac{L}{2}R\right] \times [-R, R], \] (4)

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\[ |g(z)| \leq B, \quad \forall z \in [-R, R], \quad (5) \]

and

\[ |h(z)| \leq C, \quad \forall z \in [-R, R]. \quad (6) \]

Then if

\[ \frac{L^3}{16} A + \frac{L}{2} (B + C) \leq R, \quad (7) \]

problem (1)-(3) has at least a solution.

**Theorem 2.2** Suppose the assumptions of Theorem 2.1 hold. Suppose further that there exist constants \( \lambda_f, \lambda_g, \lambda_h > 0 \) such that

\[ |f(t, u, u') - f(t, v, v')| \leq \lambda_f \max\{|u - v|, |u' - v'|\}, \quad (8) \]

for all \((t, u, u'), (t, v, v') \in [0, L] \times [-\frac{L}{2} R, \frac{L}{2} R] \times [-R, R],\)

\[ |g(u) - g(v)| \leq \lambda_g |u - v|, \quad \forall u, v \in [-R, R], \quad (9) \]

and

\[ |h(u) - h(v)| \leq \lambda_h |u - v|, \quad \forall u, v \in [-R, R]. \quad (10) \]

Then if

\[ \frac{L^3}{16} \max\{\frac{L}{2}, 1\} \lambda_f + \frac{L}{2} (\lambda_g + \lambda_h) < 1, \quad (11) \]

problem (1)-(3) has an iterative solution \( u \) with \( \|u'\|_\infty \leq R. \)

The proofs rely on fixed point theorems. We begin by rewriting problem (1)-(3) into a second order system. If \( v = u'' \) then we have

\[ \begin{cases} 
  u'' = v, & 0 < t < L \\
  u(0) = 0, & u(L) = 0 
\end{cases} \quad (12) \]

and

\[ \begin{cases} 
  v''(t) = f(t, u, u') \\
  v(0) = g(u'(0)), & v(L) = h(u'(L)). 
\end{cases} \quad (13) \]

The Green’s function associated to the second order problem (12) is precisely

\[ G(x, t) = \begin{cases} 
  \frac{x(L-t)}{L}, & \text{if } x \leq t \leq L \\
  \frac{L-t}{t(L-x)}, & \text{if } t \leq x \leq L, 
\end{cases} \]

and gives

\[ u(x) = \int_0^L -G(x, t)v(t)dt. \]

Analogously, from (13) we have

\[ v(t) = \int_0^L -G(t, s)f(s, u(s), u'(s))ds + \frac{L-t}{L} g(u'(0)) + \frac{t}{L} h(u'(L)). \]
Then, combining the above identities we get

\[ u(x) = \int_0^L \int_0^L G(x, t) G(t, s) f(s, u(s), u'(s)) ds dt \]

\[ - \int_0^L G(x, t) \left[ \frac{(L-t)}{L} g(u'(0)) + \frac{t}{L} h(u'(L)) \right] dt. \quad (14) \]

We can see that \( u \) is a solution of (1)-(3) if and only if it is a solution of (14). Next we apply fixed point arguments to solve (14). In view of (2) we apply fixed point theorems on the Banach space

\[ E = \{ u \in C^1([0, L]) \mid u(0) = u(L) = 0 \}. \]

Because \( u(0) = u(L) = 0 \), we see that

\[ \| u \|_\infty \leq \frac{L}{2} \| u' \|_\infty, \quad \forall u \in E, \]

and, in particular, the usual norm \( \| u \|_{C^1} = \max\{ \| u \|_\infty, \| u' \|_\infty \} \) is equivalent to

\[ \| u \|_E = \| u' \|_\infty, \quad (15) \]

which will be adopted here. Then we note that \( \| u \|_E \leq R \) implies \( |u'(x)| \leq R \) and \( |u(x)| \leq \frac{L}{2} R \), for all \( x \in [0, L] \).

**Proof of Theorem 2.1** Let us define the operator \( T : E \to E \) with \( (Tu)(x) \) equal to the right hand side of (14). Then fixed points of \( T \) are solutions of problem (1)-(3). Next we show that \( T \) maps the closed ball \( B[0, R] \) of \( E \) into itself. Indeed, noting that

\[ \int_0^L |G(x, t)| dt \leq \frac{L^2}{8} \quad \text{and} \quad \int_0^L |G_x(x, t)| dt \leq \frac{L}{2}, \]

we have from

\[ (Tu)'(x) = \int_0^L G_x(x, t) \left[ \int_0^L G(t, s) f(s, u(s), u'(s)) ds \right] dt \]

\[ - \int_0^L G_x(x, t) \left[ \frac{(L-t)}{L} g(u'(0)) + \frac{t}{L} h(u'(L)) \right] dt, \]

that for \( u \in B[0, R] \) and using (4)-(7),

\[ \| (Tu)' \|_\infty \leq \frac{L^3}{16} \max |f(t, u, u')| + \frac{L}{2} |(g(u'(L))| + |h(u'(L))|) \]

\[ \leq \frac{L^3}{16} A + \frac{L}{2} (B + C) \leq R. \]
Therefore with respect to the norm (15), \( T(B[0, R]) \subset B[0, R] \). To conclude the proof we note that \( T \) is completely continuous on \( B[0, R] \) (by Arzela-Ascoli theorem) and therefore it has a fixed point by the Schauder’s fixed point theorem (e.g. [1]). □

**Proof of Theorem 2.2** Let \( u, v \in B[0, R] \). Then as before, but using (8)-(10),

\[
\|(Tu - Tv)'\|_\infty \leq \frac{L^3}{16} \max \{ |f(t, u', u') - f(t, v', v')| \\
+ \frac{L^2}{2} |g(u'(L)) - g(v'(L))| + \frac{L}{2} |h(u'(L)) - h(v'(L))| \}
\]

\[
\leq \frac{L^3}{16} \max \lambda_f \{ |u - v|, |u' - v'| \} + \frac{L^2}{2} (\lambda_g + \lambda_h) |u' - v'|
\]

\[
\leq \frac{L^3}{16} \max \{ \frac{L^3}{2}, 1 \} \lambda_f \|u' - v'\|_\infty + \frac{L^2}{2} (\lambda_g + \lambda_h) \|u' - v'\|_\infty.
\]

Therefore

\[
\|(Tu - Tv)'\|_E \leq \left( \frac{L^3}{16} \max \{ \frac{L^3}{2}, 1 \} \lambda_f + \frac{L}{2} (\lambda_g + \lambda_h) \right) \|u - v\|_E.
\]

From (11) we see that \( T \) is a contraction on \( B[0, R] \) and then it has a fixed point from the Banach’s fixed point theorem (e.g. [1]). □

3. Numerical Simulations

From Theorem 2.2 we obtain the iterative formulae \( u^{k+1} = Tu^k \), were

\[
u^{k+1}(x) = \int_0^L \int_0^L G(x, t)G(t, s)f(s, u^k(s), u^k'(s))dsdt
\]

\[- \int_0^L G(x, t) \left[ \frac{(L-t)}{L} g(u^k(t)) + \frac{t}{L} h(u^k(L)) \right] dt, \quad (16)
\]

which converges to a solution of (1)-(3) for any initial approximation \( u^0 \in B[0, R] \).

We show two numerical simulations to illustrate the use of (16). In both examples, \( L = 1, u^0 = 0 \) and mesh size is 0.1. The integrals are approximated by trapezoidal method.

**Example 1** First example we take

\[f(x, u, v) = x^5 - x^4 - x^3 + 121x - 24 - u,
\]

\[g(v) = 0 \quad \text{and} \quad h(v) = -2v.
\]

The exact solution in \([0, 1]\) is

\[u(x) = x^5 - x^4 - x^3 + x.
\]
After 10 iterations we get maximum error

\[ E = \| u - u^{10}\|_\infty = .303411 \times 10^{-2}. \]

Other values are shown in the Table 1.

Table 1: Errors for Example 1 using mesh size \( \Delta = 0.1 \).

<table>
<thead>
<tr>
<th>Iteration</th>
<th>( E^k )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>.135393e-0</td>
</tr>
<tr>
<td>2</td>
<td>.811294e-1</td>
</tr>
<tr>
<td>3</td>
<td>.437432e-1</td>
</tr>
<tr>
<td>10</td>
<td>.303411e-2</td>
</tr>
<tr>
<td>20</td>
<td>.208102e-2</td>
</tr>
<tr>
<td>30</td>
<td>.207709e-2</td>
</tr>
</tbody>
</table>

Example 2 In this example we take

\[ f(x, u, v) = 4\pi^4 \sin(\pi x) \cos(\pi x) - \frac{1}{16} \sin^2(\pi x) \cos^2(\pi x) + u^2, \]

\[ g(v) = h(v) = \frac{v}{2} - \frac{\pi}{8}. \]

Then the exact solution in \([0, 1]\) is

\[ u(x) = \frac{1}{4} \sin(\pi x) \cos(\pi x). \]

After 4 iterations we get maximum error

\[ E = \| u - u^4\|_\infty = .812751 \times 10^{-2}. \]

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