Existence of Solutions for Some Strongly Nonlinear Parabolic Problems Involving Lower Order Terms in Divergence Form in Musielak-Orlicz Spaces

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ABSTRACT: In this paper, we study an existence of solutions for a class of nonlinear parabolic problems with two lower order terms and $L^1$-data in the context of Musielak-Orlicz spaces.

Key Words: Parabolic problems, Inhomogeneous Musielak-Orlicz-Sobolev space, Lower order term, Truncation.

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1. Introduction:

Let $\Omega$ be a bounded open set of $\mathbb{R}^N$, $T$ is a positive real number, and $Q = \Omega \times (0, T]$. We deal with boundary value problem:

$$\begin{cases}
\frac{\partial u}{\partial t} - \text{div} \left( a(x, t, u, \nabla u) + \Phi(x, t, u) \right) + g(x, t, u, \nabla u) = f & \text{in } Q, \\
u = 0 & \text{in } \partial \Omega \times (0, T), \\
u(x, 0) = u_0 & \text{in } \Omega,
\end{cases} \quad (P)$$

where $A(u) = -\text{div}(a(x, t, u, \nabla u))$ is a Leray-Lions Operator defined on $D(A) \subset W^{1,\varphi}_0(Q) \hookrightarrow W^{-1,\psi}_0(Q)$ where $\varphi$ and $\psi$ are two complementary Musielak-Orlicz functions. The lower order term $\Phi : \Omega \times (0, T) \times \mathbb{R} \rightarrow \mathbb{R}$ is a Carathéodory function satisfies the following growth condition for a.e. $(x, t) \in Q$ and for all $s \in \mathbb{R}$,

$$|\Phi(x, t, s)| \leq P(x, t) \gamma_{\varphi}(|s|).$$

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where $P(x,t) \in L^\infty(Q)$ and $\gamma$ is a Musielak-Orlicz function such that $\gamma \ll \varphi$ means that $\gamma$ grows essentially less rapidly than $\varphi$ (see Preliminaries). $g$ is a non-linearity with sign condition and satisfying, for all $s \in \mathbb{R}$, $\xi \in \mathbb{R}^N$ and almost all $(x,t) \in Q$ the following natural growth condition:

$$|g(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)),$$

where $c_2(x,t) \in L^1(Q)$ and $b : \mathbb{R}^+ \rightarrow \mathbb{R}$ is a continuous and nondecreasing function. The right-hand side $f$ is assumed to belong to $L^1(Q)$.

On Orlicz-Sobolev spaces, Elmahi had studied in [6] the problem $(P)$ for $\Phi \equiv 0$, without assuming any restriction on the N-function $M$. In the case where $u \equiv b(x,u)$ and $g \equiv 0$, the existence of solution has been proved in [10] by Hadj Nassar, Moussa and Rhoudaf.

In the framework of variable exponent Sobolev spaces, Azroul, Benboubker, Redwane and Yazough in [2] have proved the existence result of solutions for the problem $(P)$ without sign condition involving nonstandard growth and where $u = b(x,u)$ and $\Phi \equiv 0$. Fu and Pan have treated in [8] the existence of solutions for the problem $(P)$ where $\Phi \equiv g \equiv 0$ and the second member $f$ is in $W^{-1,x}_{L^p(x)}(Q)$.

In the setting of Musielak spaces and in variational case, the existence of a weak solution for the problem $(P)$ has been proved by M. L. Ahmed Oubeid, A. Benkirane and M. Sidi El Vally in [1] where $\Phi \equiv 0$, the existence of solutions for the problem $(P)$ has been studied by A. Talha, A. Benkirane, and M.S.B. Elemine vall in [16] when $\Phi \equiv 0$ and the right hand side is a measure data. A large number of papers was devoted to the study the existence solutions of elliptic and parabolic problems under various assumptions and in different contexts for a review on classical results see [1,3,4,11,13,15].

Our main goal in this paper is to study the problem $(P)$ in the context of Musielak-Orlicz spaces without assuming the $\Delta_2$ condition, neither on the Musielak function $\varphi$ nor on its complementary $\psi$. The main difficulty in our study is due to the fact that the second member is in $L^1$ and the fact that no hypothesis of coercivity is assumed on $\Phi$. Our result generalizes that of Elmahi and Meskine [7] and that of Ahmed Oubeid, Benkirane, and Sidi El Vally [1].

This research is divided into several parts. In Section 2 we recall some well-know preliminaries, properties and results of Musielak-Orlicz-Sobolev Spaces. Section 3 is devoted to specify the assumptions on $a, \Phi, g, f$ and $u_0$. Section 4 is devoted to some technical lemmas where be used to our results. Final section 5 consecrate to prove the existence of solution of $(P)$.

2. Preliminaries

2.1. Musielak-Orlicz functions

Let $\Omega$ be an open set in $\mathbb{R}^N$ and let $\varphi$ be a real-valued function defined in $\Omega \times \mathbb{R}_+$ and satisfying the following conditions:

(a) $\varphi(x,.)$ is an N-function for all $x \in \Omega$ (i.e. convex, strictly increasing, continuous,
\[ \varphi(x,0) = 0, \varphi(x,t) > 0, \text{ for all } t > 0, \]
\[ \lim_{t \to 0} \sup_{x \in \Omega} \frac{\varphi(x,t)}{t} = 0 \]
and
\[ \lim_{t \to \infty} \inf_{x \in \Omega} \frac{\varphi(x,t)}{t} = \infty. \]

(b) \( \varphi(\cdot, t) \) is a measurable function.

The function \( \varphi \) is called a Musielak-Orlicz function.

For a Musielak-orlicz function \( \varphi \) we put \( \varphi_x(t) = \varphi(x,t) \) and we associate its nonnegative reciprocal function \( \varphi_x^{-1} \), with respect to \( t \), that is
\[ \varphi_x^{-1}(\varphi(x,t)) = \varphi(x,\varphi_x^{-1}(t)) = t. \]

The Musielak-orlicz function \( \varphi \) is said to satisfy the \( \Delta_2 \)-condition if for some \( k > 0 \), and a non negative function \( h \), integrable in \( \Omega \), we have
\[ \varphi(x,2t) \leq k \varphi(x,t) + h(x) \text{ for all } x \in \Omega \text{ and } t \geq 0. \quad (2.1) \]

When (2.1) holds only for \( t \geq t_0 > 0 \), then \( \varphi \) is said to satisfy the \( \Delta_2 \)-condition near infinity.

Let \( \varphi \) and \( \gamma \) be two Musielak-orlicz functions, we say that \( \varphi \) dominate \( \gamma \) and we write \( \gamma \prec \varphi \), near infinity (resp. globally) if there exist two positive constants \( c \) and \( t_0 \) such that for almost all \( x \in \Omega \)
\[ \gamma(x,t) \leq \varphi(x,ct) \text{ for all } t \geq t_0, \quad (\text{resp. for all } t \geq 0 \text{ i.e. } t_0 = 0). \]

We say that \( \gamma \) grows essentially less rapidly than \( \varphi \) at 0 (resp. near infinity) and we write \( \gamma \ll \varphi \) if for every positive constant \( c \) we have
\[ \lim_{t \to 0} \left( \sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0, \quad (\text{resp. } \lim_{t \to \infty} \left( \sup_{x \in \Omega} \frac{\gamma(x,ct)}{\varphi(x,t)} \right) = 0). \]

**Remark 2.1.** [4] If \( \gamma \ll \varphi \) near infinity, then \( \forall \epsilon > 0 \) there exist \( k(\epsilon) > 0 \) such that for almost all \( x \in \Omega \) we have
\[ \gamma(x,t) \leq k(\epsilon) \varphi(x,\epsilon t), \quad \text{for all } t \geq 0. \quad (2.2) \]

**2.2. Musielak-Orlicz spaces**

For a Musielak-Orlicz function \( \varphi \) and a measurable function \( u : \Omega \to \mathbb{R} \), we define the functional
\[ \rho_{\varphi} : \Omega \to \mathbb{R} \]
\[ \rho_{\varphi}(u) = \int_{\Omega} \varphi(x,|u(x)|) \, dx. \]

The set \( K_\varphi(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable} / \rho_{\varphi}(u) < \infty \right\} \) is called the Musielak-Orlicz class (or generalized Orlicz class). The Musielak-Orlicz space (the
generalized Orlicz spaces) $L_\varphi(\Omega)$ is the vector space generated by $K_\varphi(\Omega)$, that is, $L_\varphi(\Omega)$ is the smallest linear space containing the set $K_\varphi(\Omega)$. Equivalently

$$L_\varphi(\Omega) = \{ u : \Omega \rightarrow \mathbb{R} \text{ measurable } / \rho_\varphi,\Omega \left( \frac{u}{\lambda} \right) < \infty, \text{ for some } \lambda > 0 \}.$$  

For a Musielak-Orlicz function $\varphi$ we put: $\psi(x,s) = \sup_{t \geq 0} \{ st - \varphi(x,t) \}$, $\psi$ is the Musielak-Orlicz function complementary to $\varphi$ (or conjugate of $\varphi$) in the sense of Young with respect to the variable $s$.

In the space $L_\varphi(\Omega)$ we define the following two norms:

$$\| u \|_{\varphi,\Omega} = \inf \{ \lambda > 0 / \int_{\Omega} \varphi \left( x, \frac{|u(x)|}{\lambda} \right) dx \leq 1 \}.$$  

which is called the Luxemburg norm and the so-called Orlicz norm by:

$$\| | u | \|_{\varphi,\Omega} = \sup_{\| v \|_{\psi} \leq 1} \int_{\Omega} |u(x)v(x)| dx,$$  

where $\psi$ is the Musielak Orlicz function complementary to $\varphi$. These two norms are equivalent $\cite{12}$.

The closure in $L_\varphi(\Omega)$ of the bounded measurable functions with compact support in $\Omega$ is denoted by $E_\varphi(\Omega)$.

A Musielak function $\varphi$ is called locally integrable on $\Omega$ if $\rho_\varphi(t\chi_D) < \infty$ for all $t > 0$ and all measurable $D \subset \Omega$ with $\text{meas}(D) < \infty$.

Let $\varphi$ a Musielak function which is locally integrable. Then $E_\varphi(\Omega)$ is separable ($\cite{12}$, Theorem 7.10.)

We say that sequence of functions $u_n \in L_\varphi(\Omega)$ is modular convergent to $u \in L_\varphi(\Omega)$ if there exists a constant $\lambda > 0$ such that

$$\lim_{n \rightarrow \infty} \rho_\varphi,\Omega \left( \frac{u_n - u}{\lambda} \right) = 0.$$  

For any fixed nonnegative integer $m$ we define

$$W^mL_\varphi(\Omega) = \{ u \in L_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in L_\varphi(\Omega) \}.$$  

and

$$W^mE_\varphi(\Omega) = \{ u \in E_\varphi(\Omega) : \forall |\alpha| \leq m, \ D^\alpha u \in E_\varphi(\Omega) \}.$$  

where $\alpha = (\alpha_1, ..., \alpha_n)$ with nonnegative integers $\alpha_i$, $|\alpha| = |\alpha_1| + ... + |\alpha_n|$ and $D^\alpha u$ denote the distributional derivatives. The space $W^mL_\varphi(\Omega)$ is called the Musielak Orlicz Sobolev space.

Let

$$\overline{\overline{\rho}},\Omega(u) = \sum_{|\alpha| \leq m} \rho_\varphi,\Omega \left( D^\alpha u \right) \text{ and } \| u \|_{\overline{\overline{\rho}},\Omega} = \inf \left\{ \lambda > 0 : \overline{\overline{\rho}},\Omega \left( \frac{u}{\lambda} \right) \leq 1 \right\}$$  

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for $u \in W^m L_\varphi(\Omega)$. These functionals are a convex modular and a norm on $W^m L_\varphi(\Omega)$, respectively, and the pair $(W^m L_\varphi(\Omega), \|\cdot\|_{\varphi, \Omega})$ is a Banach space if $\varphi$ satisfies the following condition [12]:

$$\inf_{x \in \Omega} \varphi(x, 1) \geq c_0.$$  \hspace{1cm} (2.3)

The space $W^m L_\varphi(\Omega)$ will always be identified to a subspace of the product $\prod_{|\alpha| \leq m} L_\varphi(\Omega) = \prod L_\varphi$, this subspace is $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closed.

The space $W^m_0 L_\varphi(\Omega)$ is defined as the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$, and the space $W^m_0 E_\psi(\Omega)$ as the (norm) closure of the Schwartz space $\mathcal{D}(\Omega)$ in $W^m L_\varphi(\Omega)$.

Let $W^m_0 L_\varphi(\Omega)$ be the $\sigma(\Pi L_\varphi, \Pi E_\psi)$ closure of $D(\Omega)$ in $W^m L_\varphi(\Omega)$.

The following spaces of distributions will also be used:

$$W^{-m} L_\varphi(\Omega) = \{ f \in D'(\Omega); \ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in L_\varphi(\Omega) \}.$$ and

$$W^{-m} E_\psi(\Omega) = \{ f \in D'(\Omega); \ f = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha f_\alpha \text{ with } f_\alpha \in E_\psi(\Omega) \}.$$

We say that a sequence of functions $u_n \in W^m L_\varphi(\Omega)$ is modular convergent to $u \in W^m L_\varphi(\Omega)$ if there exists a constant $k > 0$ such that

$$\lim_{n \to \infty} \overline{p}_{\varphi, \Omega}(\frac{u_n - u}{k}) = 0.$$  

For $\varphi$ and her complementary function $\psi$, the following inequality is called the Young inequality [12]:

$$ts \leq \varphi(x, t) + \psi(x, s), \ \forall t, s \geq 0, x \in \Omega.$$  \hspace{1cm} (2.4)

This inequality implies that

$$\|u\|_{\varphi, \Omega} \leq \rho_{\varphi, \Omega}(u) + 1.$$  \hspace{1cm} (2.5)

In $L_\varphi(\Omega)$ we have the relation between the norm and the modular

$$\|u\|_{\varphi, \Omega} \leq \rho_{\varphi, \Omega}(u) \text{ if } \|u\|_{\varphi, \Omega} > 1.$$  \hspace{1cm} (2.6)

$$\|u\|_{\varphi, \Omega} \geq \rho_{\varphi, \Omega}(u) \text{ if } \|u\|_{\varphi, \Omega} \leq 1.$$  \hspace{1cm} (2.7)

For two complementary Musielak Orlicz functions $\varphi$ and $\psi$, let $u \in L_\varphi(\Omega)$ and $v \in L_\psi(\Omega)$, then we have the Hölder inequality [12]:

$$\left| \int_{\Omega} u(x)v(x) \, dx \right| \leq \|u\|_{\varphi, \Omega}\|v\|_{\psi, \Omega}.$$  \hspace{1cm} (2.8)
2.3. Inhomogeneous Musielak-Orlicz-Sobolev spaces

Let $\Omega$ a bounded open subset of $\mathbb{R}^N$ and let $Q = \Omega \times [0,T]$ with some given $T > 0$. Let $\varphi$ and $\psi$ be two complementary Musielak-Orlicz functions. For each $\alpha \in \mathbb{N}^N$ denote by $D_\alpha$ the distributional derivative on $Q$ of order $\alpha$ with respect to the variable $x \in \mathbb{R}^N$. The inhomogeneous Musielak-Orlicz-Sobolev spaces of order 1 are defined as follows.

$$W^{1,x}_0(Q) = \{ u \in L_\varphi(Q) : \forall |\alpha| \leq 1 \ D_\alpha u \in L_\varphi(Q) \}$$

et

$$W^{1,x}_0 E_\varphi(Q) = \{ u \in E_\varphi(Q) : \forall |\alpha| \leq 1 \ D_\alpha u \in E_\varphi(Q) \}.$$

This second space is a subspace of the first one, and both are Banach spaces under the norm

$$\| u \| = \sum_{|\alpha| \leq 1} \| D_\alpha u \|_{\varphi,Q}.$$

These spaces constitute a complementary system since $\Omega$ satisfies the segment property. These spaces are considered as subspaces of the product space $\Pi L_\varphi(Q)$ which has $(N+1)$ copies.

We shall also consider the weak topologies $\sigma(\Pi L_\varphi, \Pi E_\psi)$ and $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

If $u \in W^{1,x}_0 L_\varphi(Q)$ then the function $t \rightarrow u(t) = u(\cdot , t)$ is defined on $[0,T]$ with values in $W^{1,1}_0 L_\varphi(\Omega)$. If $u \in W^{1,x}_0 E_\varphi(Q)$, then $u \in W^{1,1}_0 E_\varphi(\Omega)$ and it is strongly measurable. Furthermore, the imbedding $W^{1,x}_0 E_\varphi(Q) \subset L^1(0,T,W^{1,1}_0 E_\varphi(\Omega))$ holds. The space $W^{1,x}_0 L_\varphi(Q)$ is not in general separable, for $u \in W^{1,x}_0 L_\varphi(Q)$ we cannot conclude that the function $u(t)$ is measurable on $[0,T]$.

However, the scalar function $t \rightarrow \| u(t) \|_{\varphi, \Omega}$ is in $L^1(0,T)$. The space $W^{1,x}_0 E_\varphi(Q)$ is defined as the norm closure of $\mathcal{D}(Q)$ in $W^{1,1}_0 E_\varphi(Q)$. We can easily show as in [9] that when $\Omega$ has the segment property, then each element $u$ of the closure of $\mathcal{D}(Q)$ with respect of the weak* topology $\sigma(\Pi L_\varphi, \Pi E_\psi)$ is a limit in $W^{1,x}_0 L_\varphi(Q)$ of some subsequence $(v_j) \in \mathcal{D}(Q)$ for the modular convergence, i.e. there exists $\lambda > 0$ such that for all $|\alpha| \leq 1$,

$$\int_Q \varphi(x, (D^2 v_j - D^2 u)/\lambda) \, dx \, dt \to 0 \text{ as } j \to \infty,$$

this implies that $(v_j)$ converges to $u$ in $W^{1,x}_0 L_\varphi(Q)$ for the weak topology $\sigma(\Pi L_\varphi, \Pi L_\psi)$.

Consequently

$$\overline{\mathcal{D}(Q)}^{\sigma(\Pi L_\varphi, \Pi E_\psi)} = \mathcal{D}(Q)^{\sigma(\Pi L_\varphi, \Pi L_\psi)}.$$

The space of functions satisfying such a property will be denoted by $W^{1,x}_0 L_\varphi(Q)$. Furthermore, $W^{1,x}_0 E_\varphi(Q) = W^{1,x}_0 L_\varphi(Q) \cap \Pi E_\varphi(Q)$. Thus, both sides of the last inequality are equivalent norms on $W^{1,x}_0 L_\varphi(Q)$. We then have the following complementary system:

$$\begin{pmatrix} W^{1,x}_0 L_\varphi(Q) & F_0 \\ W^{1,x}_0 E_\varphi(Q) & F_0 \end{pmatrix}.$$
where \( F \) states for the dual space of \( W^{1,x}_0 L^\psi(Q) \). and can be defined, except for an isomorphism, as the quotient of \( II L^\psi \) by the polar set \( W^{1,x}_0 L^\psi(Q)^\perp \). It will be denoted by \( F = W^{-1,x} L^\psi(Q) \), where

\[
W^{-1,x} L^\psi(Q) = \{ f = \sum_{|\alpha| \leq 1} D^\alpha f_\alpha : f_\alpha \in L^\psi(Q) \}.
\]

This space will be equipped with the usual quotient norm

\[
\| f \| = \inf \sum_{|\alpha| \leq 1} \| f_\alpha \|_{\psi, Q},
\]

where the infimum is taken over all possible decompositions

\[
f = \sum_{|\alpha| \leq 1} D^\alpha f_\alpha, \quad f_\alpha \in L^\psi(Q).
\]

The space \( F_0 \) is then given by

\[
F_0 = \{ f = \sum_{|\alpha| \leq 1} D^\alpha f_\alpha : f_\alpha \in E^\psi(Q) \},
\]

and is denoted by \( F_0 = W^{-1,x} E^\psi(Q) \), see [1].

3. Essential Assumptions

Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) and \( T > 0 \), we denote \( \Omega \times [0,T] \), and let \( \varphi \) and \( \gamma \) be two Musielak-Orlicz functions such that \( \varphi \) is locally integrable and \( \gamma \prec \prec \varphi \).

Let \( A : D(A) \subset W^{1,x}_0 L^\varphi(Q) \to W^{-1,x} L^\psi(Q) \) be a mapping given by

\[
A(u) = -\text{div}(a(x,t,u,\nabla u)),
\]

where \( a : a(x,t,s,\xi) : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) is a Carathéodory function satisfying,

for a.e. \( (x,t) \in Q \) and for all \( s \in \mathbb{R} \) and all \( \xi, \xi' \in \mathbb{R}^N, \xi \neq \xi' \):

\[
|a(x,t,s,\xi)| \leq \beta \left( c(x,t) + \psi_\varphi^{-1} \gamma(x,\nu|s|) + \psi_\varphi^{-1} \varphi(x,\nu|s|) \right), \tag{3.1}
\]

\[
a(x,t,s,\xi) - a(x,t,s,\xi') (\xi - \xi') > 0, \tag{3.2}
\]

\[
a(x,t,s,\xi) \xi \geq \alpha \varphi(x,|\xi|). \tag{3.3}
\]

where \( c(x,t) \) a positive function, \( c(x,t) \in E^\psi(Q) \) and positive constants \( \nu, \beta, \alpha \).

Let \( g : \Omega \times [0,t] \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}^N \) be a Carathéodory function satisfying for a.e. \( (x,t) \in \Omega \times [0,t] \) and all \( s \in \mathbb{R}, \xi \in \mathbb{R}^N : \)

\[
|g(x,t,s,\xi)| \leq b(|s|)(c_2(x,t) + \varphi(x,|\xi|)), \tag{3.4}
\]
where \( c_2(x, t) \in L^1(Q) \) and \( b : \mathbb{R}^+ \rightarrow \mathbb{R} \) is a continuous and nondecreasing function.

Furthermore the function \( \Phi \) is a Carathéodory function which satisfies the following growth condition for a.e. \( (x, t) \in Q \) and for all \( \forall s \in \mathbb{R} \),

\[
|\Phi(x, t, s)| \leq P(x, t) \gamma_x(|s|).
\]

where \( P(x, t) \in L^\infty(Q) \).

\( f \) is an element of \( L^1(Q) \), \( u_0 \) is an element of \( L^1(\Omega) \).

Let us give the following lemma which will be needed later.

4. Some technical Lemmas

Lemma 4.1. \([3]\). Let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \) and let \( \varphi \) and \( \psi \) be two complementary Musielak-Orlicz functions which satisfy the following conditions:

i) There exist a constant \( c > 0 \) such that \( \inf_{x \in \Omega} \varphi(x, 1) \geq c \),

ii) There exist a constant \( A > 0 \) such that for all \( x, y \in \Omega \) with \( |x - y| \leq \frac{1}{2} \) we have

\[
\frac{\varphi(x, t)}{\varphi(y, t)} \leq \left( \frac{1}{\log(1/|x - y|)} \right), \quad \forall t \geq 1.
\]

iii) If \( D \subset \Omega \) is a bounded measurable set, then \( \int_D \varphi(x, 1)dx < \infty \).

iv) There exist a constant \( C > 0 \) such that \( \psi(x, 1) \leq C \) a.e in \( \Omega \).

Under this assumptions, \( D(\Omega) \) is dense in \( L_\varphi(\Omega) \) with respect to the modular topology, \( D(\Omega) \) is dense in \( W^{1,1}_\varphi(\Omega) \) for the modular convergence and \( D(\Omega) \) is dense in \( W^{1}_\varphi(\Omega) \) the modular convergence.

Consequently, the action of a distribution \( S \) in \( W^{-1}_\psi(\Omega) \) on an element \( u \) of \( W^{1}_\varphi(\Omega) \) is well defined. It will be denoted by \( \langle S, u \rangle \).

Lemma 4.2. \([13]\). Let \( F : \mathbb{R} \rightarrow \mathbb{R} \) be uniformly Lipschitzian, with \( F(0) = 0 \). Let \( \varphi \) be a Musielak-Orlicz function and let \( u \in W^{1}_\varphi(\Omega) \). Then \( F(u) \in W^{1,1}_\varphi(\Omega) \). Moreover, if the set \( D \) of discontinuity points of \( F' \) is finite, we have

\[
\frac{\partial}{\partial x_i} F(u) = \begin{cases} F'(u) \frac{\partial u}{\partial x_i} & \text{a.e in } \{x \in \Omega : u(x) \in D\}, \\ 0 & \text{a.e in } \{x \in \Omega : u(x) \notin D\}. \end{cases}
\]
Lemma 4.3 (Poincaré inequality). [15] Let \( \varphi \) a Musielak Orlicz function which satisfies the assumptions of lemma 4.1, suppose that \( \varphi(x,t) \) decreases with respect of one of coordinate of \( x \).

Then, that exists a constant \( c > 0 \) depends only of \( \Omega \) such that

\[
\int_{\Omega} \varphi(x,|u(x)|)dx \leq \int_{\Omega} \varphi(x,|\nabla u(x)|)dx, \quad \forall u \in W_0^1 L_\varphi(\Omega). \tag{4.3}
\]

Lemma 4.4. [3] Suppose that \( \Omega \) satisfies the segment property and let \( u \in W_0^1 L_\varphi(\Omega) \). Then, there exists a sequence \( (u_n) \subset \mathcal{D}(\Omega) \) such that

\[
u_n \rightharpoonup u \quad \text{for modular convergence in} \quad W_0^1 L_\varphi(\Omega)
\]

Furthermore, if \( u \in W_0^1 L_\varphi(\Omega) \cap L^\infty(\Omega) \) then \( ||u||_\infty \leq (N+1)||u||_\infty \).

Lemma 4.5 (The Nemytskii Operator). Let \( \Omega \) be an open subset of \( \mathbb{R}^N \) with finite measure and let \( \varphi \) and \( \psi \) be two Musielak Orlicz functions. Let \( f : \Omega \times \mathbb{R}^p \rightarrow \mathbb{R}^q \) be a Carathéodory function such that for a.e. \( x \in \Omega \) and all \( s \in \mathbb{R}^p : \)

\[
|f(x,s)| \leq c(x) + k_1 \psi^{-1}_x \varphi(x,k_2|s|). \tag{4.4}
\]

where \( k_1 \) and \( k_2 \) are real positives constants and \( c(.) \in E_\psi(\Omega) \).

Then the Nemytskii Operator \( N_f \) defined by \( N_f(u)(x) = f(x,u(x)) \) is continuous from

\[
\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}) = \prod \left\{ u \in L_\varphi(\Omega) : d(u,E_\varphi(\Omega)) < \frac{1}{k_2} \right\}
\]

into \( (L_\psi(\Omega))^q \) for the modular convergence.

Furthermore if \( c(.) \in E_\psi(\Omega) \) and \( \gamma \prec \prec \psi \) then \( N_f \) is strongly continuously from

\[
\mathcal{P}(E_\varphi(\Omega), \frac{1}{k_2}) \quad \text{to} \quad (E_\psi(\Omega))^q
\]

Theorem 4.6. [1] Let \( \varphi \) be an Musielak-Orlicz function which satisfies the assumption (4.1). If \( u \in W^{1,x}_0 L_\varphi(Q) \cap L^2(Q) \) (respectively \( u \in W^{1,x}_0 L_\varphi(Q) \cap L^2(Q) \)) and \( \frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(Q) + L^2(Q) \), then there exists a sequence \( (v_j) \in \mathcal{D}'(Q) \) (respectively \( \mathcal{D}'(Q) \)) such that \( v_j \rightharpoonup u \) in \( W^{1,x} L_\varphi(Q) \cap L^2(Q) \) and \( \frac{\partial v_j}{\partial t} \rightharpoonup \frac{\partial u}{\partial t} \) in \( W^{-1,x} L_\psi(Q) + L^2(Q) \) for the modular convergence.

Lemma 4.7. [1] Let \( a < b \in \mathbb{R} \) and let \( \Omega \) be a bounded Lipschitz domain in \( \mathbb{R}^N \).

Then

\[
\left\{ u \in W^{1,x}_0 L_\varphi(\Omega \times [a,b]) : \frac{\partial u}{\partial t} \in W^{-1,x} L_\psi(\Omega \times [a,b]) + L^1(\Omega \times [a,b]) \right\}
\]

is a subset of \( C([a,b], L^1(\Omega)) \).

Lemma 4.8. Let \( \varphi \) be a Musielak function. Let \( (u_n) \) be a sequence of \( W^{1,x} L_\varphi(Q) \) such that

\[
u_n \rightharpoonup u \quad \text{weakly in} \quad W^{1,x} L_\varphi(Q) \quad \text{for} \quad \sigma(\Pi L_\varphi, \Pi L_\varphi)
\]
and
\[ \frac{\partial u_n}{\partial t} = h_n + k_n \text{ in } \mathcal{D}'(Q) \]

with \((h_n)_n\) is bounded in \(W^{-1, \infty}(Q)\) and \((k_n)_n\) bounded in the space \(L^1(Q)\). Then \(u_n \rightharpoonup u\) strongly in \(L^1_{\text{loc}}(Q)\). If further \(u_n \in W^{1, \infty}_0(Q)\) then \(u_n \rightharpoonup u\) strongly in \(L^1(Q)\).

**Proof:** It is easily adapted from that given in [5] by using Theorem 4.4 and Remark 4.3 instead of Lemma 8 of [14]. \(\blacksquare\)

5. Main results

For \(k > 0\) we define the truncation at height \(k\): \(T_k : \mathbb{R} \rightarrow \mathbb{R}\) by:

\[ T_k(s) = \begin{cases} 
  s & \text{if } |s| \leq k, \\
  k & \text{if } |s| > k.
\end{cases} \] (5.1)

We note also

\[ S_k(r) = \int_0^r T_k(\sigma)d\sigma = \begin{cases} 
  \frac{r^2}{2} & \text{if } |r| \leq k, \\
  k|r| - \frac{r^2}{2} & \text{if } |r| > k.
\end{cases} \] (5.2)

We define

\[ T_0^{1, \psi}(Q) = \left\{ u : \Omega \rightarrow \mathbb{R} \text{ measurable such that } T_k(u) \in W^{1, \infty}_0(Q) \forall k > 0 \right\}. \]

We consider the following boundary value problem:

\[
\begin{cases} 
  \frac{\partial u}{\partial t} + \text{div} \left( a(x, t, u, \nabla u) + \Phi(x, t, u) \right) + g(x, t, u, \nabla u) = f & \text{in } Q, \\
  u \equiv 0 & \text{on } \partial Q = \partial \Omega \times [0, T], \\
  u(\cdot, 0) = u_0 & \text{in } \Omega.
\end{cases} \] (P)

Our goal now is to show the following existence theorem.

**Theorem 5.1.** Let \(\Omega\) be a bounded Lipschitz domain in \(\mathbb{R}^N\), \(\varphi\) and \(\psi\) be two complementary Musielak-Orlicz functions satisfying the assumptions of Lemma 4.1 and \(\varphi(x, t)\) decreases with respect to one of coordinate of \(x\), we assume also that (3.1)–(3.7) are fulfilled, then there exists at least one solution of (P) in the following
sense
\[
\begin{aligned}
    u &\in T_{\alpha}^{1,\varphi}(Q), \quad S_k(u) \in L^1(Q), \quad g(\ldots, u, \nabla u) \in L^1(Q) \\
    \int_{\Omega} S_k(u(T) - v(T)) dx + \left( \frac{\partial u}{\partial t}, T_k(u - v) \right) &+ \int_Q a(x, t, u, \nabla u) \cdot \nabla T_k(u - v) dx dt \\
    + \int_Q \Phi(x, t, u, \nabla u) \cdot \nabla T_k(u - v) dx dt &+ \int_Q g(x, t, u, \nabla u) T_k(u - v) dx dt \\
    \leq &\int_Q f(x, t, u, \nabla u) - T_k(u - v) dx dt + \int_{\Omega} S_k(u_0 - v(0)) dx
\end{aligned}
\]

and
\[
    u(x, 0) = u_0(x) \text{ for a.e. } x \in \Omega,
\]

\forall v \in W_{0,1}^{1,\varphi}(Q) \cap L^\infty(Q) \text{ such that } \frac{\partial u}{\partial t} \in W^{-1,\varphi}(Q) + L^1(Q).
\]

(5.3)

The following remarks are concerned with a few comments on Theorem 5.1.

**Remark 5.2.** Equation (5.3) is formally obtained through pointwise multiplication of the problem (P) by \(T_k(u - v)\). Note that each term in (5.3) has a meaning since \(T_k(u - v) \in W_{0,1}^{1,\varphi}(Q) \cap L^\infty(Q)\). In addition by Lemma 4.7, we have \(v \in C([0, T]; L^1(\Omega))\) and then the first and last terms of Eq. (5.3) are well defined.

**Proof:** The proof of Theorem 5.1 is done in 6 steps.

**Step 1: Approximate problem.**

Let us introduce the following regularization of the data:

\[
a(x, t, r, \xi) = a(x, t, T_n(r, \xi)) \text{ a.e } (x, t) \in Q, \quad \forall r \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,
\]

(5.4)

\[
g_n(x, t, r, \xi) = g(x, t, T_n(r, \xi)) \text{ a.e } (x, t) \in Q, \quad \forall r \in \mathbb{R}, \quad \forall \xi \in \mathbb{R}^N,
\]

(5.5)

\[
\Phi_n(x, t, r) = \Phi(x, t, T_n(r)) \quad \text{a.e } (x, t) \in Q, \quad \forall r \in \mathbb{R},
\]

(5.6)

\[
f_n \in C_0^\infty(Q) : \|f_n\|_{L^1} \leq \|f\|_{L^1} \text{ and } f_n \rightharpoonup f \text{ in } L^1(Q) \text{ as } n \text{ tends to } +\infty,
\]

(5.7)

\[
u_{0n} \in C_0^\infty(\Omega) : \|u_{0n}\|_{L^1} \leq \|u\|_{L^1} \text{ and } u_{0n} \rightharpoonup u_0 \text{ in } L^1(\Omega) \text{ as } n \text{ tends to } +\infty.
\]

(5.8)

Let us now consider the following regularized problem:

\[
(P_n) \begin{cases}
    \frac{\partial u_n}{\partial t} - \text{div} \left(a(x, t, u_n, \nabla u_n) + \Phi_n(x, t, u_n)\right) + g_n(x, t, u_n, \nabla u_n) = f_n & \text{in } Q, \\
    u_n = 0 & \text{on } \partial \Omega \times (0, T), \\
    u_n(x, t = 0) = u_{0n} & \text{in } \Omega.
\end{cases}
\]

Since \(g_n\) is bounded for any fixed \(n\), as a consequence, proving of a weak solution \(u_n \in W_{0,1}^{1,\varphi}(Q)\) of (\(P_n\)) is an easy task (see e.g. [1,11]).
Step 2: A priori estimates.

The estimates derived in this step rely on usual techniques for problems of the type ($P_n$).

We take $T_k(u_n)\chi_{(0,\tau)}$ as test function in $(P_n)$, we get for every $\tau \in (0,T)$

$$
\langle \frac{\partial u_n}{\partial t}, T_k(u_n)\chi_{(0,\tau)} \rangle + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt \\
+ \int_{Q_\tau} \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n) \, dx \, dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n) \, dx \, dt
$$

(5.9)

which implies that

$$
\int_\Omega S_k(u_n)(\tau)dx + \int_{Q_\tau} a(x,t,T_k(u_n),\nabla T_k(u_n)) \cdot \nabla T_k(u_n) \, dx \, dt \\
+ \int_{Q_\tau} \Phi_n(x,t,u_n) \cdot \nabla T_k(u_n) \, dx \, dt + \int_{Q_\tau} g_n(x,t,u_n,\nabla u_n)T_k(u_n) \, dx \, dt
$$

(5.10)

while $\gamma \ll \varphi$, we have, for all $\varepsilon > 0$ there exists a constant $d_\varepsilon > 0$ depending on $\varepsilon > 0$ such that for almost all $x \in \Omega$

$$
\gamma(x,t) \leq \varphi(x,\varepsilon t) + d_\varepsilon, \quad \text{for all } t \geq 0.
$$

(5.11)

Without loss of generality, we can assume that $\varepsilon = \frac{\gamma}{(\alpha + C_p)(\lambda + 1)}$, (with $\alpha$ is the constant of (3.3)).

Using (3.6) we get

$$
\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n) \, dx \, dt \leq \int_{Q_\tau} P(x,t)\frac{1}{\gamma_x} \gamma_x(|T_k(u_n)|) \nabla T_k(u_n) \, dx \, dt. 
$$

(5.12)

Recall that $\gamma \ll \varphi \iff \overline{\varphi} = \psi \ll \overline{\psi}$ then, with Young inequality and bearing in mind that $P \in L^\infty(Q_\tau)$, we obtain

$$
\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n) \, dx \, dt \leq C_p \int_{Q_\tau} \varphi \left( x, \frac{\varepsilon \lambda |T_k(u_n)|}{\lambda} \right) + 2d_\varepsilon \text{meas}(Q_\tau)
$$

$$
+ \varepsilon C_p \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|) \, dx \, dt,
$$

(5.13)

by Lemma 4.3 and the convexity of $\varphi$ with $\lambda \varepsilon \leq 1$, we get

$$
\int_{Q_\tau} \Phi_n(x,t,u_n)\nabla T_k(u_n) \, dx \, dt \leq (\varepsilon C_p + \varepsilon \lambda C_p) \int_{Q_\tau} \varphi(x,|\nabla T_k(u_n)|) \, dx \, dt
$$

$$
+ 2d_\varepsilon \text{meas}(Q_\tau).
$$

(5.14)
By using (3.5), (5.7), (5.8), (5.14), and the fact that $S_k(u_n)(\tau) \geq 0$, $S_k(u_0n) \leq k|u_0n|$, permit to deduce from (5.10) that
\[
\int_{Q_\tau} a(x, t, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \leq (\varepsilon C_p + \varepsilon \lambda C_p) \int_{\Omega} \varphi(x, |\nabla T_k(u_n)|) \, dx \, dt \\
+ 2d_c \text{meas}(Q_\tau) \\
+ k \left( \|f\|_{L^1(Q_\tau)} + \|u_0\|_{L^1(Q_\tau)} \right),
\]
(5.15)
by (3.3) and since $\left( \alpha - \varepsilon C_p(1 + \lambda) \right) > 0$, then
\[
\int_{Q_\tau} \varphi(x, |\nabla T_k(u_n)|) \, dx \, dt \leq \frac{1}{\alpha} \int_{\Omega} a(x, u_n, \nabla u_n) \nabla T_k(u_n) \, dx \, dt \leq kC_1.
\]
(5.16)
where $C_1$ is a constant independently of $n$.

Using Lemma 4.3, one has
\[
\int_{Q_\tau} \varphi(x, \frac{|T_k(u_n)|}{\lambda}) \, dx \, dt \leq kC_1.
\]
(5.17)
Then we deduce by using (5.17), that
\[
\text{meas}\{|u_n| > k\} \leq \frac{1}{\inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \int_{\{|u_n| > k\}} \varphi(x, \frac{k}{\lambda}) \, dx \, dt \\
\leq \frac{1}{\inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \int_{Q_\tau} \varphi(x, \frac{1}{\lambda}|T_k(u_n)|) \, dx \, dt \\
\leq \frac{C_1k}{\inf_{x \in \Omega} \varphi(x, \frac{k}{\lambda})} \quad \forall n, \forall k \geq 0.
\]
(5.18)
For every $\lambda > 0$ we have
\[
\text{meas}\{|u_n - u_m| > \lambda\} \leq \text{meas}\{|u_n| > k\} \\
+ \text{meas}\{|u_m| > k\} \\
+ \text{meas}\{|T_k(u_n) - T_k(u_m)| > \lambda\}.
\]
(5.19)
Consequently, by (5.17) we can assume that $(T_k(u_n))_n$ is a Cauchy sequence in measure in $Q$.

Let $\varepsilon > 0$, then by (5.19) there exists some $k = k(\varepsilon) > 0$ such that
\[
\text{meas}\{|u_n - u_m| > \lambda\} < \varepsilon, \quad \text{for all } n, m \geq h_0(k(\varepsilon), \lambda).
\]
Which means that $(u_n)_n$ is a Cauchy sequence in measure in $Q$, thus converge almost every where to some measurable functions $u$. 

\[\text{NONLINEAR PARABOLIC PROBLEMS IN MUSIELAK-ORLICZ SPACES} \]
We have from (5.17) that $T_k(u_n)$ is bounded in $W^{1, \infty}_0 L_{\psi} (Q)$ for every $k > 0$. Consider now a $C^2(\mathbb{R})$ nondecreasing function $\zeta_k(s) = s$ for $|s| \leq \frac{k}{2}$ and $\zeta_k(s) = k \operatorname{sign}(s)$.

Multiplying the approximating equation by $\zeta'_k(u_n)$, we obtain

$$\frac{\partial (\zeta'_k(u_n))}{\partial t} = \operatorname{div} \left(a(x, t, u_n, \nabla u_n) \zeta_k'(u_n)\right) - \zeta''_k(u_n) a(x, t, u_n, \nabla u_n) \cdot \nabla u_n$$

$$+ \operatorname{div} \left(\Phi_n(x, t, u_n) \zeta'_k(u_n)\right) - \zeta''_k(u_n) \Phi_n(x, t, u_n) \cdot \nabla u_n - g_n(x, t, u_n, \nabla u_n) \zeta'_k(u_n) + f_n \zeta'_k(u_n),$$

(5.20)

Due to (3.1), (3.4), (5.4), (5.5) and the fact that $T_k(u_n)$ is bounded in $W^{1, \infty}_0 L_{\psi} (Q)$,

$$\operatorname{div} \left(a(x, t, u_n, \nabla u_n) \zeta'_k(u_n)\right) - \zeta''_k(u_n) a(x, t, u_n, \nabla u_n) \cdot \nabla u_n - g_n(x, t, u_n, \nabla u_n) \zeta'_k(u_n) + f_n \zeta'_k(u_n),$$

is bounded in $L^1(Q) + W_0^{-1, \infty} L_{\psi} (Q)$, so $\zeta_n(u_n)$ is bounded in $L^1(Q) + W_0^{-1, \infty} L_{\psi} (Q)$.

Moreover since $\operatorname{supp}(\zeta'_k)$ and $\operatorname{supp}(\zeta''_k)$ are both included in $[-k, k]$ by (3.6) and (5.6) if follows that,

$$\left| \int_Q \zeta'_k(u_n) \Phi_n(x, t, u_n) \, dx \, dt \right| \leq \| \zeta''_k \|_{L^\infty} \int_Q \left( P(x, t) \frac{1}{\gamma_x^2} \gamma_x (|T_k(u_n)|) \right) \, dx \, dt.$$  

Furthermore, We have $P \in L^\infty(Q)$ and $\frac{1}{\gamma_x^2} \gamma_x$ is increasing function, hence

$$\left| \int_Q \zeta'_k(u_n) \Phi_n(x, t, u_n) \, dx \, dt \right| \leq C_2,$$

where $C_2$ is a positive constant independent of $n$.

In the same way, we get $\left| \int_Q \zeta''_k(u_n) \Phi_n(x, t, u_n) \, dx \, dt \right| \leq C_3$, where $C_3$ is a positive constant independent of $n$.

Then all above implies that

$$\frac{\partial (\zeta'_k(u_n))}{\partial t}$$

is bounded in $L^1(Q) + W_0^{-1, \infty} L_{\psi} (Q)$.

Hence by Lemma 4.8 and using the same technics in [13], we can see that there exists a measurable function $u \in L^\infty(0, T; L^1(\Omega))$ such that for every $k > 0$ and a subsequence, not relabeled,

$$u_n \rightarrow u \text{ a. e. in } Q,$$

(5.22)

and

$$T_k(u_n) \rightarrow T_k(u) \text{ weakly in } W^{1, \infty}_0 L_{\psi} (Q) \text{ for } \sigma(\Pi L_{\psi}, \Pi E_{\psi}),$$

(5.23)

strongly in $L^1(Q)$ and a. e. in $Q$. 


Step 3: Boundedness of $a(x,t,T_k(u_n),\nabla T_k(u_n))$ in $(L_\varphi(Q))^N$.

Now we shall prove the boundedness of $(a(x,t,T_k(u_n),\nabla T_k(u_n)))$ in $(L_\varphi(Q))^N$. Let $\phi \in (E_\varphi(Q))^N$ with $\|\phi\|_{\varphi,Q} = 1$. In view of the monotonicity of $a$ one easily has,

$$\left(a(x,t,T_k(u_n),\nabla T_k(u_n)) - a(x,t,T_k(u_n),\frac{w}{\nu})\right)(\nabla T_k(u_n) - \frac{w}{\nu}) > 0,$$

hence

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\frac{w}{\nu} \, dx \, dt \leq \int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \, dx \, dt - \int_Q a(x,t,T_k(u_n),\frac{w}{\nu})(\nabla T_k(u_n) - \frac{w}{\nu}) \, dx \, dt. \tag{5.24}$$

Thanks to (5.16), we have

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))\nabla T_k(u_n) \, dx \, dt \leq C_4,$$

where $C_4$ is a positive constant which is independent of $n$.

On the other hand, for $\lambda$ large enough ($\lambda > \beta$), we have by using (3.1).

$$\int_Q \psi_\lambda\left(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}\right) \, dx \, dt$$

$$\leq \int_Q \psi_\lambda\left(\frac{\beta(c(x,t) + \psi_x^{-1}(\gamma(x,|T_k(u_n)|) + \psi_x^{-1}(\varphi(x,|w|)))}{3\lambda}\right) \, dx \, dt$$

$$\leq \frac{\beta}{\lambda} \int_Q \psi_\lambda\left(\frac{c(x,t) + \psi_x^{-1}(\gamma(x,|T_k(u_n)|) + \psi_x^{-1}(\varphi(x,|w|)))}{3}\right) \, dx \, dt$$

$$\leq \frac{\beta}{3\lambda} \left(\int_Q \psi_\lambda(c(x,t)) \, dx \, dt + \int_Q \gamma(x,|T_k(u_n)|) \, dx \, dt + \int_Q \varphi(x,|w|) \, dx \, dt\right)$$

$$\leq \frac{\beta}{3\lambda} \left(\int_Q \psi_\lambda(c(x,t)) \, dx \, dt + \int_Q \gamma(x,|T_k(u_n)|) \, dx \, dt + \int_Q \varphi(x,|w|) \, dx \, dt\right).$$

Now, since $\gamma$ grows essentially less rapidly than $\varphi$ near infinity and by using the Remark 2.1, there exists $r(\varepsilon) > 0$ such that $\gamma(x,|T_k(u_n)|) \leq r(\varepsilon)\varphi(x,\varepsilon|T_k(u_n)|)$ and so we have

$$\int_Q \psi_\lambda\left(\frac{a(x,t,T_k(u_n),\frac{w}{\nu})}{3\lambda}\right) \, dx \, dt \leq \frac{\beta}{3\lambda} \left(\int_Q \psi_\lambda(c(x,t)) \, dx \, dt + r(k) \int_Q \varphi(x,\varepsilon|T_k(u_n)|) \, dx \, dt + \int_Q \varphi(x,|w|) \, dx \, dt\right).$$

hence $a(x,t,T_k(u_n),\frac{w}{\nu})$ is bounded in $(L_\varphi(Q))^N$. Which implies that second term of the right hand side of (5.24) is bounded, consequently we obtain

$$\int_Q a(x,t,T_k(u_n),\nabla T_k(u_n))w \, dx \, dt \leq C_5, \quad \text{for all } w \in (E_\varphi(Q))^N \text{ with } \|w\|_{\varphi,Q} \leq 1,$$
where \( C_5 \) is a positive constant which is independent of \( n \).

Hence, thanks the Banach-Steinhaus Theorem, the sequence

\[
(a(x, t, T_k(u_n)), \nabla T_k(u_n))_n
\]

is a bounded sequence in \((L_\psi(Q))^N\), thus up to a subsequence

\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \rightharpoonup \phi_k \text{ weakly star in } (L_\psi(Q))^N \text{ for } \sigma(\Pi L_\psi, \Pi E_\phi) \tag{5.25}
\]

for some \( \phi_k \in (L_\psi(Q))^N \).

**Step 4: Almost everywhere convergence of the gradients.**

Fix \( k > 0 \) and let \( \phi(s) = s \exp(\delta s^2), \delta > 0 \). It is well known that when \( \delta \geq \left( \frac{b(k)}{2\alpha} \right)^2 \) one has

\[
\phi'(s) - \frac{b(k)}{\alpha} |\phi(s)| \geq \frac{1}{2} \text{ for all } s \in \mathbb{R}. \tag{5.26}
\]

Let \( v_j \in \mathcal{D}(Q) \) be a sequence such that

\[
v_j \to u \text{ for the modular convergence in } W_0^{1,\infty}L_\psi(Q). \tag{5.27}
\]

and let \( \omega_i \in \mathcal{D}(Q) \) be a sequence which converges strongly to \( u_0 \) in \( L^2(\Omega) \).

Set \( \omega_{i,j}^\mu = T_k(v_j)_\mu + \exp(-\mu t)T_k(w_i) \) where \( T_k(v_j)_\mu \) is the mollification with respect to time of \( T_k(v_j) \), see [4].

Note that \( \omega_{i,j}^\mu \) is a smooth function having the following properties

\[
\frac{\partial}{\partial t} (\omega_{i,j}^\mu) = \mu(T_k(v_j) - \omega_{i,j}^\mu), \omega_{i,j}^\mu(0) = T_k(\omega_i), |\omega_{i,j}^\mu| \leq k, \tag{5.28}
\]

\[
\omega_{i,j}^\mu \to T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \text{ in } W_0^{1,\infty}L_\psi(Q) \tag{5.29}
\]

for the modular convergence as \( j \to \infty \),

\[
T_k(u)_\mu + \exp(-\mu t)T_k(w_i) \to T_k(u) \text{ in } W_0^{1,\infty}L_\psi(Q) \tag{5.30}
\]

for the modular convergence as \( \mu \to \infty \).

Let now the function \( \rho_m \) defined on \( \mathbb{R} \) with \( m \geq k \) by:

\[
\rho_m(s) = \begin{cases} 
1 & \text{if } |s| \leq m, \\
m + 1 - |s| & \text{if } m \leq |s| \leq m + 1, \\
0 & \text{if } |s| \geq m + 1.
\end{cases} \tag{5.31}
\]

we set

\[
R_m(s) = \int_0^s \rho_m(r)dr, \quad \theta_{i,j}^{\mu,m} = T_k(u_n) - \omega_{i,j}^\mu.
\]
Using the admissible test function $Z_{i,j,n}^{\mu,m} = \phi(\theta_{i,j}^{\mu,n})\rho_m(u_n)$ as test function in $(\mathcal{P}_n)$ leads to

\[
\begin{aligned}
&\frac{\partial u_n}{\partial t} Z_{i,j,n}^{\mu,m} + \int_Q a(x,t,u_n,\nabla u_n) \cdot \left( \nabla T_k(u_n) - \nabla \omega_{i,j}^{\mu} \right) \phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} a(x,t,u_n,\nabla u_n) \cdot \nabla u_n \phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n) \, dx \, dt \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x,t,u_n) \cdot \nabla u_n \phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n) \, dx \, dt \\
&+ \int_Q \Phi_n(x,t,u_n) \cdot (\nabla T_k(u_n) - \nabla \omega_{i,j}^{\mu}) \phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt \\
&+ \int_Q g_n(x,t,u_n,\nabla u_n) Z_{i,j,n}^{\mu,m} \, dx \, dt \\
&= \int_Q f_n Z_{i,j,n}^{\mu,m} \, dx \, dt.
\end{aligned}
\]

(5.32)

Since $g_n(x,t,u_n,\nabla u_n)\phi(\theta_{i,j}^{\mu,n})\rho_m(u_n) \geq 0$ on $\{|u_n| > k\}$, yields

\[
\begin{aligned}
&\frac{\partial u_n}{\partial t} Z_{i,j,n}^{\mu,m} + \int_Q a(x,t,u_n,\nabla u_n) \cdot \left( \nabla T_k(u_n) - \nabla \omega_{i,j}^{\mu} \right) \phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} a(x,t,u_n,\nabla u_n) \cdot \nabla u_n \phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n) \, dx \, dt \\
&+ \int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x,t,u_n) \cdot \nabla u_n \phi(\theta_{i,j}^{\mu,n})\rho_m'(u_n) \, dx \, dt \\
&+ \int_Q \Phi_n(x,t,u_n) \cdot (\nabla T_k(u_n) - \nabla \omega_{i,j}^{\mu}) \phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt \\
&+ \int_{\{|u_n| \leq k\}} g_n(x,t,u_n,\nabla u_n) \phi(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt \\
&\leq \int_Q f_n Z_{i,j,n}^{\mu,m} \, dx \, dt.
\end{aligned}
\]

(5.33)

Denoting by $\epsilon(n,j,\mu,i)$ any quantity such that

\[
\lim_{i \to \infty} \lim_{\mu \to \infty} \lim_{j \to \infty} \lim_{n \to \infty} \epsilon(n,j,\mu,i) = 0.
\]

Now, we prove below the following results for any fixed $k \geq 0$.

\[
\int_Q f_n Z_{i,j,n}^{\mu,m} \, dx \, dt = \epsilon(n,j,\mu).
\]

(5.34)

\[
\int_Q \Phi_n(x,t,u_n) \cdot (\nabla T_k(u_n) - \nabla \omega_{i,j}^{\mu}) \phi'(\theta_{i,j}^{\mu,n})\rho_m(u_n) \, dx \, dt = \epsilon(n,j,\mu).
\]

(5.35)
\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x, t, u_n) \cdot \nabla u_n \phi(\theta_{i,j}^m n) \rho_m(u_n) \, dx \, dt = \epsilon(n, j, \mu). \tag{5.36}
\]

\[
\frac{\partial u_n}{\partial t}, Z_{i,j,n}^m \geq \epsilon(n, j, \mu, i). \tag{5.37}
\]

\[
\int_{\{m \leq |u_n| \leq m+1\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \phi(\theta_{i,j}^m n) \rho_m(u_n) \, dx \, dt \leq \epsilon(n, j, \mu, m). \tag{5.38}
\]

\[
\int_Q \left[ a\left(x, t, T_k(u_n)\right), \nabla T_k(u_n) \right] - a\left(x, t, T_k(u_n)\right), \nabla T_k(u_n) \chi_a) \right] \times \left[ \nabla T_k(u_n) - \nabla T_k(u_n) \chi_a \right] \, dx \, dt \leq \epsilon(n, j, \mu, i). \tag{5.39}
\]

**Proof of (5.34)**:

By the almost everywhere convergence of \( u_n \), we have \( \phi(T_k(u_n) - \omega_{i,j}^\mu) \rho_m(u_n) \to \phi(T_k(u) - \omega_{i,j}^\mu) \rho_m(u) \) weakly-$^\ast$ in \( L^\infty(Q) \) as \( n \to \infty \), and then,

\[
\int_Q f_n \phi(T_k(u_n) - \omega_{i,j}^\mu) \rho_m(u_n) \, dx \, dt \to \int_Q f \phi(T_k(u) - \omega_{i,j}^\mu) \rho_m(u) \, dx \, dt,
\]

so that, \( \phi(T_k(u) - \omega_{i,j}^\mu) \rho_m(u) \to \phi(T_k(u) - T_k(u)\mu - \exp(-\mu t)T_k(u))\rho_m(u) \) weakly star in \( L^\infty(Q) \) as \( j \to \infty \), and finally,

\[
\phi(T_k(u) - T_k(u)\mu - \exp(-\mu t)T_k(u))\rho_m(u) \to 0 \text{ weakly star as } \mu \to \infty.
\]

Then, we deduce that,

\[
f_n \phi(T_k(u_n) - \omega_{i,j}^\mu) \rho_m(u_n)) = \epsilon(n, j, \mu). \tag{5.40}
\]

**Proof of (5.35) and (5.36)**:

Similarly, Lebesgue’s convergence theorem shows that,

\[
\Phi_n(x, t, u_n) \rho_m(u_n) \to \Phi(x, t, u) \rho_m(u) \text{ strongly in } (E_{\varphi}(Q)^N) \text{ as } n \to \infty,
\]

and

\[
\Phi_n(x, t, u_n) \chi_{\{m \leq |u_n| \leq m+1\}} \phi'(T_k(u_n) - \omega_{i,j}^\mu) \to \Phi(x, t, u) \chi_{\{m \leq |u_n| \leq m+1\}} \phi'(T_k(u) - \omega_{i,j}^\mu) \text{ strongly in } (E_{\varphi}(Q)^N).
\]

Then by virtue of

\[
\nabla T_k(u_n) \to \nabla T_k(u) \text{ weak star in } (L^\infty_{\varphi}(Q)^N),
\]

and \( \nabla u_n \chi_{\{m \leq |u_n| \leq m+1\}} = \nabla T_{m+1}(u_n) \chi_{\{m \leq |u_n| \leq m+1\}} \) a. e. in \( Q \), one has,

\[
\int_Q \Phi_n(x, t, u_n) \cdot (\nabla T_k(u_n) - \nabla \omega_{i,j}^\mu) \phi'(T_k(u_n) - \omega_{i,j}^\mu) \rho_m(u_n) \, dx \, dt 
\]

\[
\to \int_Q \Phi(x, t, u) \nabla (\nabla T_k(u) - \nabla \omega_{i,j}^\mu) \phi'(T_k(u) - \omega_{i,j}^\mu) \rho_m(u) \, dx \, dt
\]
as \( n \to \infty \), and

\[
\int_{\{m \leq |u_n| \leq m+1\}} \Phi_n(x, t, u_n) \phi(T_k(u_n)) - \omega_{i,j}^{\nu} \nabla u_n \rho_m' u_n \, dx \, dt \\
\to \int_{\{m \leq |u_n| \leq m+1\}} \Phi(x, t, u) \phi(T_k(u)) - \omega_{i,j}^{\nu} \nabla u \rho_m' u \, dx \, dt
\]

as \( n \to +\infty \).
Thus, by using the modular convergence of \( \omega_{i,j}^{\mu} \) as \( j \to +\infty \) and letting \( \mu \) tend to infinity, we get (5.35) and (5.36).

**Proof of (5.37)**: Since \( u_n \in W^{1,2}_0 L_p(Q) \), there exists a smooth function \( u_{n\sigma} \) such that:

\[
u_{n\sigma} \to u_n \text{ for the modular convergence in } W^{1,2}_0 L_p(Q) \cap L^2(Q),
\]

\[rac{\partial u_{n\sigma}}{\partial t} \to \frac{\partial u_n}{\partial t} \text{ for the modular convergence in } W^{-1,2} L_p(Q) + L^2(Q).
\]

Then,

\[
\langle \frac{\partial u_n}{\partial t}, Z_{\mu,m}^{\mu,m} \rangle = \lim_{\sigma \to 0^+} \int_Q \left( u_{n\sigma} \right)' \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \nabla u_{n\sigma} \rho_m' u_{n\sigma} \, dx \, dt
\]

\[
= \lim_{\sigma \to 0^+} \int_Q R_m(u_{n\sigma})' \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \nabla u_{n\sigma} \rho_m' u_{n\sigma} \, dx \, dt
\]

\[
= \lim_{\sigma \to 0^+} \int_Q \left( R_m(u_{n\sigma}) - T_k(u_{n\sigma}) \right)' \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \nabla u_{n\sigma} \rho_m' u_{n\sigma} \, dx \, dt
\]

\[
+ \int_Q \left( T_k(u_{n\sigma})' \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \nabla u_{n\sigma} \right) \rho_m' u_{n\sigma} \, dx \, dt
\]

\[
= \lim_{\sigma \to 0^+} \int_{\Omega} \left( [R_m(u_{n\sigma}) - T_k(u_{n\sigma}) \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu}] \right) dx
\]

\[
- \int_Q \left( R_m(u_{n\sigma}) - T_k(u_{n\sigma}) \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \right)' \, dx \, dt
\]

\[
+ \int_Q \left( T_k(u_{n\sigma})' \phi(T_k(u_{n\sigma})) - \omega_{i,j}^{\nu} \right) \, dx \, dt
\]

\[
= \lim_{\sigma \to 0^+} \left[ I_1(\sigma) + I_2(\sigma) + I_3(\sigma) \right].
\]

Observe that for \(|s| \leq k\), we have \( R_m(s) = T_k(s) = s \) and for \(|s| > k\) we have \(|R_m(s)| \geq |T_k(s)|\) and, since both \( R_m(s) \) and \( T_k(s) \) have the same sign of \( s \), we
deduce that sign \((s)\)(\(R_m(s) - T_k(s)\)) \(\geq 0\). Consequently

\[
I_1(\sigma) = \left[ \int_{\{|u_{n\sigma}| > k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi(T_k(u_{n\sigma})) - \omega_{i,j}^\mu \right] dx dt
\]

\[
\geq - \int_{\{|u_{n\sigma}| > k\}} (R_m(u_{n\sigma}(0)) - T_k(u_{n\sigma}(0)))\phi(T_k(u_{n\sigma}(0)) - \omega_{i,j}^\mu(0)) dx
\]

and so, by letting \(\sigma \to 0^+\) in the last integral, we get

\[
\limsup_{\sigma \to 0^+} I_1(\sigma) \geq - \int_{\{|u_0| > k\}} (R_m(u_0) - T_k(u_0))\phi(T_k(u_0) - T_k(w)) dx.
\]

Letting \(n \to \infty\), the right hand side of the above inequality clearly tends to

\[
- \int_{\{|u_0| > k\}} (R_m(u_0) - T_k(u_0))\phi(T_k(u_0) - T_k(w)) dx
\]

which obviously goes to 0 as \(i \to \infty\).

Which yields that

\[
\limsup_{\sigma \to 0^+} I_1(\sigma) \geq \epsilon(n, i).
\]

About \(I_2(\sigma)\), since \((R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}))' = 0\), one has

\[
I_2(\sigma) = \int_{\{|u_{n\sigma}| > k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma})) - \omega_{i,j}^\mu(\omega_{i,j}^\mu)' dx dt
\]

\[
= \mu \int_{\{|u_{n\sigma}| > k\}} (R_m(u_{n\sigma}) - T_k(u_{n\sigma}))\phi'(T_k(u_{n\sigma}) - \omega_{i,j}^\mu)(T_k(v_j) - T_k(u_{n\sigma})) dx dt
\]

by using the fact \(\phi' \geq 0\) and that \((R_m(u_{n\sigma}) - T_k(u_{n\sigma}))(T_k(u_{n\sigma}) - \omega_{i,j}^\mu)x_{\{|u_{n\sigma}| > k\}} \geq 0\) and so by letting \(\sigma \to 0^+\) in the last integral, we get

\[
\limsup_{\sigma \to 0^+} I_2(\sigma) \geq \mu \int_{\{|u_n| > k\}} (R_m(u_n) - T_k(u_n))\phi'(T_k(u_n) - \omega_{i,j}^\mu)(T_k(v_j) - T_k(u_n)) dx dt,
\]

and since, as it can be easily seen, the last integral is of the form \(\epsilon(n, j)\), we deduce that

\[
\limsup_{\sigma \to 0^+} I_2(\sigma) \geq \epsilon(n, j).
\]

For what concerns \(I_3(\sigma)\), one

\[
I_3(\sigma) = \int_Q (R_m(u_{n\sigma}) - \omega_{i,j}^\mu)' \phi(T_k(u_{n\sigma}) - \omega_{i,j}^\mu) dx dt
\]

\[
+ \int_Q (\omega_{i,j}^\mu)' \phi(T_k(u_{n\sigma}) - \omega_{i,j}^\mu) dx dt
\]
and then, by setting $\xi(s) = \int_0^s \phi(\eta) d\eta$ and integrating by parts

$$I_3(\sigma) = \left[ \int_\Omega \xi(T_k(u_n)) - \omega_{i,j}^u \right]_0^T + \mu \int_Q (T_k(v_j) - \omega_{i,j}^u)\phi(T_k(u_n)) - \omega_{i,j}^u) dx \, dt,$$

Since $\xi \geq 0$ and $(T_k(v_j) - \omega_{i,j}^u)\phi(T_k(u_n)) - \omega_{i,j}^u) \geq 0$, yields

$$I_3(\sigma) \geq - \int_\Omega \xi(T_k(u_n(0)) - T_k(w_i)) dx + \mu \int_Q (T_k(v_j) - T_k(u_n))\phi(T_k(u_n)) - \omega_{i,j}^u) dx \, dt,$$

so that,

$$\limsup_{\sigma \to 0^+} I_3(\sigma) \geq - \int_\Omega \xi(T_k(u_n(0)) - T_k(w_i)) dx + \mu \int_Q (T_k(v_j) - T_k(u_n))\phi(T_k(u_n)) - \omega_{i,j}^u) dx \, dt.$$

Hence, by letting $n \to \infty$ in the last side, we obtain

$$\limsup_{\sigma \to 0^+} I_3(\sigma) \geq - \int_\Omega \xi(T_k(u_0) - T_k(w_i)) dx + \mu \int_Q (T_k(v_j) - T_k(u))\phi(T_k(u)) - \omega_{i,j}^u) dx \, dt + \epsilon(n).$$

since the first integral of the last side is of the from $\epsilon(i)$ while the second one is of the form $\epsilon(j)$, we deduce that

$$\limsup_{\sigma \to 0^+} I_3(\sigma) \geq \epsilon(n, j, i).$$

where we have used the fact that (recall that $|\omega_{i,j}^u| \leq k$)

$$\int_Q G_k(u)\phi'(T_k(u) - \omega_{i,j}^u)(T_k(u) - \omega_{i,j}^u) dx \, dt = \int_{\{u > k\}} (u - k)\phi'(k - \omega_{i,j}^u)(k - \omega_{i,j}^u) dx \, dt$$

$$+ \int_{\{u < -k\}} (u + k)\phi'(-k - \omega_{i,j}^u)(-k - \omega_{i,j}^u) dx \, dt \geq 0.$$

Combining these estimates, we conclude that

$$(u_{n,i}^i, \phi(T_k(u_n)) - \omega_{i,j}^u)\rho_m(u_n)) \geq \epsilon(n, j, i).$$

Proof of (5.38) : Concerning the third term of the right hand side of (5.33) we obtain that

$$| \int_{\{m \leq |u_n| \leq m + 1\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \phi(\theta_{n,i,j}^{m} \rho_m(u_n)) dx \, dt |$$

$$\leq \phi(2k) \int_{\{m \leq |u_n| \leq m + 1\}} a(x, t, u_n, \nabla u_n) \cdot \nabla u_n dx \, dt.$$
Then by (5.16) we deduce that,
\[
\left| \int_Q a(x, t, u_n, \nabla u_n) \cdot \nabla u_n \phi(\theta_{n,j}^i) \rho_m(u_n) \, dx \, dt \right| \leq \varepsilon(n, \mu, m). \tag{5.42}
\]

**Proof of (5.39):** Now, concerning the sixth term of the right hand side of (5.33), We can write
\[
\left| \int_{\{|u_n| \leq \delta\}} g_n(x, t, u_n, \nabla u_n) \phi(T_k(u_n) - \omega_{\mu,j}^i) \rho_m(u_n) \, dx \, dt \right| \leq b(k) \int_Q c_2(x, t) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt
\]
\[
+ \frac{b(k)}{\alpha} \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt. \tag{5.43}
\]

Since $c_2(x, t)$ belongs to $L^1(Q)$ it is easy to see that
\[
b(k) \int_Q c_2(x, t) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt = \varepsilon(n, j, \mu).
\]

On the other hand, the second term of the right hand side of (5.43) reads as
\[
\frac{b(k)}{\alpha} \int_Q a(T_k(u_n), \nabla T_k(u_n)) \nabla T_k(u_n) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt
\]
\[
= \frac{b(k)}{\alpha} \int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \right)
\]
\[
\times \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt
\]
\[
+ \frac{b(k)}{\alpha} \int_Q a(x, t, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt
\]
\[
+ \frac{b(k)}{\alpha} \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \nabla T_k(v_j) \chi^s_j \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt
\]
and, as above, by letting successively first $n$, then $j, \mu$ and finally $s$ go to infinity, we can easily see that each one of last two integrals of the right-hand side of the last equality is of the form $\varepsilon(n, j, \mu)$.

This implies that
\[
\left| \int_{\{|u_n| \leq \delta\}} g_n(x, t, u_n, \nabla u_n) \phi(T_k(u_n) - \omega_{\mu,j}^i) \rho_m(u_n) \, dx \, dt \right|
\]
\[
\leq \frac{b(k)}{\alpha} \int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi^s_j) \right)
\]
\[
\times \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi^s_j \right) \phi(T_k(u_n) - \omega_{\mu,j}^i) \, dx \, dt + \varepsilon(n, j, \mu).
\]
Combining (5.33), (5.38), (5.37) (5.40), (5.43) and (5.44), we get

$$
\int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right)
\times \left( \phi'(T_k(u_n)) - \omega_{\mu,j}^s - \frac{b(k)}{\alpha} |\phi(T_k(u_n)) - \omega_{\mu,j}^s| \right) dx \, dt
\leq \varepsilon(n, j, \mu, i, s, m).
$$

and so, thanks to (5.26),

$$
\int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right)
\times \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) dx \, dt
\leq 2 \varepsilon(n, j, \mu, i, s, m).
$$

On the other hand, we have

$$
\int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u) \chi^s) \right)
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx \, dt
$$

$$
- \int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right)
\times \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) dx \, dt
$$

$$
= \int_Q a(x, t, T_k(u_n), \nabla T_k(u_n)) \left( \nabla T_k(v_j) \chi_j^s - \nabla T_k(u) \chi^s \right) dx \, dt
$$

$$
- \int_Q a(x, t, T_k(u_n), \nabla T_k(u) \chi^s) \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx \, dt
$$

$$
+ \int_Q a(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) dx \, dt
$$

and, as it can be easily seen, each integral of the right-hand side is of the form $\varepsilon(n, j, s)$, implying that

$$
\int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u) \chi^s) \right)
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) dx \, dt
$$

$$
= \int_Q \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(v_j) \chi_j^s) \right)
\times \left( \nabla T_k(u_n) - \nabla T_k(v_j) \chi_j^s \right) dx \, dt + \varepsilon(n, j, s).
$$
For \( r \leq s \), we have
\[
0 \leq \int_{Q^r} \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u), \nabla T_k(u)) \right) \\
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \, dx \, dt
\leq \int_{Q^r} \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)) \right) \\
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \, dx \, dt
\leq \int_{Q} \left( a(x, t, T_k(u), \nabla T_k(u)) - a(x, t, T_k(u), \nabla T_k(u) \chi^s) \right) \\
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \chi^s \right) \, dx \, dt
\leq \int_{Q} \left( a(x, t, T_k(u), \nabla T_k(u)) - a(x, t, T_k(u), \nabla T_k(v_j) \chi_j^s) \right) \\
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \chi_j^s \right) \, dx \, dt + \varepsilon(n, j, s).
\]
\[
\leq \varepsilon(n, j, \mu, i, s, m).
\]

Hence, by passing to the limit sup over \( n \) and the limit successively on \( j \to \infty, \mu \to \infty, i \to \infty, s \to \infty, \) and \( m \to \infty, \) we get
\[
\limsup_{n \to \infty} \int_{Q^r} \left[ \left( a(x, t, T_k(u_n), \nabla T_k(u_n)) - a(x, t, T_k(u_n), \nabla T_k(u)) \right) \\
\times \left( \nabla T_k(u_n) - \nabla T_k(u) \right) \right] \, dx \, dt = 0.
\]

Using a similar tools as in \([16]\), we get
\[
T_k(u_n) \to T_k(u) \text{ for the modular convergence in } W^{1,s}_0 L(Q).
\]

Which implies that exists a subsequence still denote by \( u_n \) such that
\[
\nabla u_n \to \nabla u \text{ a.e. in } Q.
\]

We deduce then that,for all \( k > 0, \) one has
\[
a(x, t, T_k(u_n), \nabla T_k(u_n)) \to a(x, t, T_k(u), \nabla T_k(u)) \\
\text{weak star in } (L^1(\Psi(Q)))^N \text{ for } \sigma(\Pi L^\psi \Pi E_\phi).
\]
Step 5: Equi-integrability of $g_n(x, u_n, \nabla u_n)$.

We shall now prove that $g_n(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u)$ strongly in $L^1(Q)$ by using Vitli’s theorem.

Since $g_n(x, t, u_n, \nabla u_n) \to g(x, t, u, \nabla u)$ a.e. in $Q$, thanks to (5.22) and (5.44) and Vitli’s theorem, it suffices to prove that $g_n(x, t, u_n, \nabla u_n)$ are uniformly equi-integrable in $Q$.

Let $E \subset Q$ be a measurable subset of $Q$. Then for any $m > 0$, one has

$$
\int_E |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt = \int_{E \cap \{u_n \leq m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt
+ \int_{E \cap \{u_n > m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt.
$$

On the one hand,

$$
\int_{E \cap \{u_n > m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \frac{1}{m} \int_Q g_n(x, t, u_n, \nabla u_n) \nabla u_n \, dx \, dt \leq \frac{C}{m}
$$

where $C$ is the constant in (3.4). Therefore, there exists $m = m(\varepsilon)$ large enough such that

$$
\int_{E \cap \{u_n \leq m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \frac{\varepsilon}{2} \quad \forall n.
$$

On the other hand

$$
\int_{E \cap \{u_n \leq m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \int_E |g_n(x, t, T_m(u_n), \nabla T_m(u_n))| \, dx \, dt
\leq b(m) \int_E \left( c_2(x, t) + \varphi(x, \nabla|T_m(u_n)) \right) \, dx \, dt
\leq b(m) \int_E \left( c_2(x, t) + \frac{1}{\alpha} d(x, t) \right) \, dx \, dt
+ \frac{b(m)}{\alpha} \int_E a(x, t, T_m(u_n), \nabla T_m(u_n)) \cdot \nabla T_m(u_n) \, dx \, dt
$$

where we have used (3.4). Therefore, it is easy to see that there exists $\nu > 0$ such that

$$
|E| < \nu \implies \int_{E \cap \{u_n \leq m\}} |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \frac{\varepsilon}{2} \quad \forall n.
$$

Consequently,

$$
|E| < \nu \implies \int_E |g_n(x, t, u_n, \nabla u_n)| \, dx \, dt \leq \varepsilon \quad \forall n.
$$
Which shows that \( g_n(x, t, u_n, \nabla u_n) \) are uniformly equi-integrable in \( Q \) as required.

Moreover, we get
\[
g_n(x, t, u_n, \nabla u_n) \rightarrow g(x, t, u, \nabla u) \quad \text{strongly in } L^1(Q).
\] (5.47)

**Step 6: Passage to the limit.**

Let \( v \in W^{1,x}_0 L^p(Q) \) such that \( \frac{\partial w}{\partial t} \in W^{-1,x} L^p(Q) + L^1(Q) \). There exists a prolongation \( \overline{v} \) of \( v \) such that (see the proof of Lemma 4.7 and Theorem 4.6. in [1])
\[
\begin{align*}
\overline{v} & = v \quad \text{on } Q, \\
\overline{v} & \in W^{1,x}_0 L^p(\Omega \times \mathbb{R}) \cap L^1(\Omega \times \mathbb{R}) \cap L^\infty(\Omega \times \mathbb{R}), \\
\text{and } \frac{\partial \overline{v}}{\partial t} & \in W^{-1,x} L^p(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}).
\end{align*}
\]

By Lemma 4.7, there exists a sequence \( (w_j)_j \) in \( D(\Omega \times \mathbb{R}) \) such that \( w_j \rightarrow \overline{v} \) in \( W^{1,x}_0 L^p(\Omega \times \mathbb{R}) \) and \( \frac{\partial w_j}{\partial t} \rightarrow \frac{\partial \overline{v}}{\partial t} \) in \( W^{-1,x} L^p(\Omega \times \mathbb{R}) + L^1(\Omega \times \mathbb{R}) \) for the modular convergence and
\[ \|w_j\|_{\infty, Q} \leq (N + 2)\|v\|_{\infty, Q}. \]

Go back to approximate equations \( (P_n) \) and use \( T_k(u_n - w_j) \chi_{[0, \tau]} \) for every \( \tau \in [0, T] \), as a test function one has

\[
\begin{align*}
\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j) \, dx \, dt & = \int_{Q_\tau} a(x, t, u_n, \nabla u_n) \cdot \nabla T_k(u_n - w_j) \, dx \, dt \\
& \quad + \int_{Q_\tau} \Phi_n(x, t, u_n) \cdot \nabla T_k(u_n - w_j) \, dx \, dt \\
& \quad + \int_{Q_\tau} g_n(x, t, u_n, \nabla u_n) T_k(u_n - w_j) \, dx \, dt \\
& \quad \leq \int_{Q_\tau} f_n T_k(u_n - w_j) \, dx \, dt.
\end{align*}
\] (5.48)

For the first term of (5.48), we get
\[
\begin{align*}
\int_{Q_\tau} \frac{\partial u_n}{\partial t} T_k(u_n - w_j) \, dx \, dt & = \left[ \int_{\Omega} T_k(u_n - w_j) \, dx \right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u_n - w_j) \, dx \, dt \\
& = \left[ \int_{\Omega} T_k(u - w_j) \, dx \right]_0^\tau + \int_{Q_\tau} \frac{\partial w_j}{\partial t} T_k(u - w_j) \, dx \, dt \\
& \quad + \varepsilon(n) \\
& = \int_{Q_\tau} \frac{\partial u}{\partial t} T_k(u - w_j) \, dx \, dt.
\end{align*}
\]
For the second term of (5.48), we have if $|u_n| > \lambda$ then $|u_n - w_j| \geq |u_n| - \|w_j\|_{\infty} > k$, therefore $\{|u_n - w_j| \leq k\} \subseteq \{|u_n| \leq k + (N + 2)\|v\|_{\infty}\}$, which implies that, we get

$$\liminf_{n \to +\infty} \int_Q a(x, t, u_n, \nabla u_n) \nabla T_k(u_n - w_j) \, dx \, dt$$

$$\geq \int_Q a(x, t, T_{k+(N+2)\|v\|_{\infty}}(u), \nabla T_{k+(N+2)\|v\|_{\infty}}(u))$$

$$\left( \nabla T_{k+(N+2)\|v\|_{\infty}}(u) - \nabla w_j \right) \chi_{\{|u-w_j| \leq k\}} \, dx \, dt,$$

$$= \int_Q a(x, t, u, \nabla u)(\nabla u - \nabla w_j) \chi_{\{|u-w_j| \leq k\}} \, dx \, dt$$

$$= \int_Q a(x, t, u, \nabla u) \nabla T_k(u - w_j) \, dx \, dt. \quad (5.49)$$

Since $\nabla T_k(u_n - w_j) \to \nabla T_k(u - w_j)$ in $L_\varphi(Q)$ as $n \to +\infty$, we have (as $n \to +\infty$)

$$\int_Q \Phi_n(x, t, u_n) \cdot \nabla T_k(u_n - w_j) \, dx \, dt$$

$$\to \int_Q \Phi(x, t, u) \cdot \nabla T_k(u - w_j) \, dx \, dt.$$

Consequently, by using the strong convergence of $(g_n(x, t, u_n, \nabla u_n))_n$ and $((f_n))_n$, one has

$$\int_Q \frac{\partial u}{\partial t} T_k(u - w_j) \, dx \, dt$$

$$+ \int_Q a(x, t, u, \nabla u) \cdot \nabla T_k(u - w_j) \, dx \, dt$$

$$+ \int_Q \Phi(x, t, u) \cdot \nabla T_k(u - w_j) \, dx \, dt$$

$$+ \int_Q g(x, t, u, \nabla u) T_k(u - w_j) \, dx \, dt \leq \int_Q fT_k(u - w_j) \, dx \, dt. \quad (5.50)$$

Thus, by using the modular convergence of $j$, we achieve this step.

As a conclusion of Step 1 to Step 6, the proof of Theorem 5.1 is complete.

References


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