A Covering Property with respect to Generalized Preopen Sets

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ABSTRACT: In this paper, we introduce and study the notion of $\mu$-precompact spaces on the observation that each $\mu$-preopen set of a generalized topological space is contained in a $\mu$-open set. The $\mu$-precompactness is weaker than $\mu$-compactness but stronger than weakly $\mu$-compactness of generalized topological spaces.

Key Words: $\mu$-preopen, $\mu$-compact, weakly $\mu$-compact, $\mu$-precompact.

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1. Introduction

Let $(X, \mathcal{P})$ be a topological space. We find that certain subsets like semi-open sets (Levine [10], also called $\beta$-sets by Njåstad [13]), pre-open sets (Mashhour et al. [11]), semi-pre-open sets (Andrijević [1], also called $\beta$-open sets by El-Monsef et al. [9]), $\alpha$-sets (Njåstad [13]) of a topological space $X$ possess properties more or less similar to those of open sets of $X$. Also topological properties generated by sets like semi-open, pre-open etc. had impacts in developing the study of classical objects, see e.g. [7,8,18]. On this observation, Császár [6] introduced and studied $\gamma$-open sets in $X$. Again following the properties of $\gamma$-open sets of a topological space, Császár [4] introduced and studied the concept of generalized topology.

Let $X$ be a nonempty set and $\mu$ be a subcollection of the power set $\text{exp}(X)$ of $X$. $\mu$ is called a generalized topology on $X$ if $\emptyset \in \mu$ and the union of arbitrary number of elements of $\mu$ is again a member of $\mu$. A nonempty set $X$ endowed with a generalized topology $\mu$ is called a generalized topological space and it is denoted by $(X, \mu)$. We write GT (resp. GTS) to denote the generalized topology $\mu$ (resp. generalized topological space $(X, \mu)$). An element of $\mu$ is called a $\mu$-open set of $(X, \mu)$. The complement of a $\mu$-open set is called a $\mu$-closed set of $(X, \mu)$. A generalized topological space $(X, \mu)$ is called strong [3] (also called $\mu$-space by Noiri [14]) if $X \in \mu$. For brevity, we retain the term $\mu$-space due to Noiri [14] to mean the strongly generalized topological space $(X, \mu)$ as well.

Henceforth, we write $X$ to denote a GTS or $\mu$-space to be understood from the context. For a subset $A$ of a GTS $X$, the generalized closure [2] of $A$ is denoted by $c_\mu(A)$ which is the intersection of all $\mu$-closed sets containing $A$ and the generalized interior [2] of $A$ is denoted by $i_\mu(A)$ which is the union of all $\mu$-open sets contained in $A$. 

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in $A$. It can be proved that a subset $A$ of $X$ is $\mu$-open (resp. $\mu$-closed) if and only if $A = i_\mu(A)$ (resp. $A = c_\mu(A)$). Also for any subset $A$ of $X$, we have $c_\mu(A) = X - i_\mu(X - A)$.

Throughout the paper, $N$ denotes the set of natural numbers and $R$, the set of real numbers.

2. $\mu$-precompact spaces

We begin by recalling some known definitions and results to use in the sequel.

**Definition 2.1** (CsáSZár [2]). A subset $A$ of $X$ is called $\mu$-preopen if $A \subset i_\mu(c_\mu(A))$ and $\mu$-semiopen if $A \subset c_\mu(i_\mu(A))$.

**Definition 2.2** (Sarsak [17]). A subset $A$ of a GTS $X$ is called $\mu$-regularly closed if $A = c_\mu(i_\mu(A))$. The complement of a $\mu$-regularly closed set is called a $\mu$-regularly open set. So a subset $A$ of a GTS is $\mu$-regularly open if $A = i_\mu(c_\mu(A))$.

Note that if $G$ is a $\mu$-open set in $X$, then $i_\mu(c_\mu(G))$ is $\mu$-regularly open in $X$.

We see that a subset $A$ of $X$ is $\mu$-preopen if and only if there exists a $\mu$-open set $G$ such that $A \subset G \subset c_\mu(A)$. Also a subset $A$ of $X$ is $\mu$-semiopen if and only there exists a $\mu$-open set $G$ such that $G \subset A \subset c_\mu(G)$.

We write ‘$\mu$-open collection’ and ‘$\mu$-preopen collection’ to mean a collection consisting $\mu$-open sets and $\mu$-preopen sets respectively of a $\mu$-space. A cover of a $\mu$-space $X$ is a collection $\mathcal{A}$ of subsets of $X$ such that $\bigcup_{A \in \mathcal{A}} A = X$. $\mathcal{A}$ is called a $\mu$-open cover (resp. $\mu$-preopen cover) of $X$ if $\mathcal{A}$ is a $\mu$-open collection (resp. $\mu$-preopen collection) of $X$ and covers $X$. The terms ‘regularly $\mu$-open collection’, ‘regularly $\mu$-preopen collection’, ‘$\mu$-semiopen collection’ ‘$\mu$-semiopen cover’ are apparent.

**Definition 2.3** (Sarsak [16]). A $\mu$-space is called $\mu$-compact if each $\mu$-open cover of $X$ has a finite subcover.

**Definition 2.4** (Sarsak [17]). A $\mu$-space is called weakly $\mu$-compact (briefly, w$\mu$-compact) if each $\mu$-open cover $\mathcal{G}$ of $X$ has a finite subcollection $\mathcal{G}_{n}$ such that $\bigcup_{G \in \mathcal{G}_{n}} c_\mu(G) = X$.

**Definition 2.5** (Sarsak [15]). A $\mu$-space is called $\mu$-$S$-closed if each $\mu$-semiopen cover $\mathcal{G}$ of $X$ has a finite subcollection $\mathcal{G}_{n}$ such that $\bigcup_{G \in \mathcal{G}_{n}} c_\mu(G) = X$.

We now introduce the following.

**Definition 2.6.** Let $\mathcal{F}$ be a $\mu$-preopen collection of $X$. For each $A \in \mathcal{F}$, there exists a $\mu$-open set $U$ such that $A \subset U \subset c_\mu(A)$. We define $\mathcal{U} = \{U \mid A \in \mathcal{F}, A \subset U \subset c_\mu(A)\}$. Then $\mathcal{U}$ is said to be a ‘$\mu$-open super collection’ of $\mathcal{F}$.

It follows that there always exists a $\mu$-open super collection of a $\mu$-preopen collection of a $\mu$-space $X$. We also see that $\mathcal{U}$ is a cover of $X$ if $\mathcal{F}$ is a cover of $X$. In this case, $\mathcal{U}$ is said to be a $\mu$-open super cover of the $\mu$-preopen cover $\mathcal{F}$.

**Definition 2.7.** A $\mu$-space $X$ is said to be $\mu$-precompact if each $\mu$-preopen cover of $X$ has a finite $\mu$-open super cover.
If $\mathcal{V}$ is a finite $\mu$-open super cover of a $\mu$-preopen cover $\mathcal{I}$ of a $\mu$-precompact space $X$, then for each $U \in \mathcal{V}$, there exists a $\mu$-preopen set $A \in \mathcal{I}$ such that $A \subseteq U \subseteq c_\mu(A)$. Thus we have a finite subcollection \{\(A \mid U \in \mathcal{V}, A \subseteq U \subseteq Cl(A)\}\) of $\mathcal{I}$ corresponding to $\mathcal{V}$.

It is easy to see that a $\mu$-compact space is a $\mu$-precompact space and a $\mu$-precompact space is a weakly $\mu$-compact space but reverse implication relations are not true.

**Example 2.8.** On $R$, we define $\mu = \{0, R\} \cup \{(-\infty, n) \mid n \in \mathbb{N}\} \cup \{[1, \infty]\}$. The $\mu$-space $(R, \mu)$ is $\mu$-precompact but not a $\mu$-compact space.

**Lemma 2.9.** If $A$ is $\mu$-preopen in $X$, then $i_\mu(c_\mu(A))$ is $\mu$-regularly open in $X$.

**Proof:** Since $A$ is a $\mu$-preopen set in $X$, there exists a $\mu$-open set $G$ such that $A \subseteq G \subseteq c_\mu(A)$ which implies that $c_\mu(A) = c_\mu(G)$. Thus we have $i_\mu(c_\mu(A)) = i_\mu(c_\mu(G))$. Since $i_\mu(c_\mu(G))$ is $\mu$-regularly open, $i_\mu(c_\mu(A))$ is $\mu$-regularly open in $X$.

**Example 2.10** (cf. Example 1 [12]). We define $\mu = \{0, (-\infty, b), (-\infty, b]\}$ where $b \in R$. So $(X, \mu)$ is a GTS. We put $A = (-\infty, a)$, $a \in R$ and $a > b$. We see that $i_\mu(c_\mu(A)) = (-\infty, b]$, $i_\mu(c_\mu((-\infty, b])) = (-\infty, b]$. It means that $i_\mu(c_\mu(A))$ is $\mu$-regularly open in $(X, \mu)$. As $A \not\subseteq i_\mu(c_\mu(A))$, $A$ is not $\mu$-preopen in $X$.

So we conclude that the converse of Lemma 2.9 need not be true in general.

**Theorem 2.11.** A $\mu$-space $X$ is $\mu$-precompact if and only if each $\mu$-preopen cover $\mathcal{I}$ of $X$ has a finite $\mu$-regularly open super cover $\{i_\mu(c_\mu(A)) \mid A \in \mathcal{I}\}$ where $\mathcal{I}$ is a finite subcollection of $\mathcal{I}$.

**Proof:** By $\mu$-precompactness of $X$, we obtain a finite $\mu$-open super cover $\mathcal{V}$ of $\mathcal{I}$. For each $G \in \mathcal{I}$, there exists $A \in \mathcal{I}$ such that $A \subseteq G \subseteq c_\mu(A)$ which implies that $A \subseteq G \subseteq i_\mu(c_\mu(A)) \subseteq c_\mu(A)$. We put $\mathcal{I} = \{A \in \mathcal{I} \mid G \in \mathcal{I}, A \subseteq G \subseteq c_\mu(A)\}$. It means that $\mathcal{I}$ is a finite subcollection of $\mathcal{I}$, $\mathcal{I}$ being a cover of $X$, $\{i_\mu(c_\mu(A)) \mid A \in \mathcal{I}\}$ is also a cover of $X$. By Lemma 2.9, $i_\mu(c_\mu(B))$ is regularly open for each $B \in \mathcal{I}$. So $\mathcal{I}$ is a finite subcollection of $\mathcal{I}$ such that $\{i_\mu(c_\mu(B)) \mid B \in \mathcal{I}\}$ is a $\mu$-regularly open super cover of the $\mu$-preopen cover $\mathcal{I}$ of $X$.

Conversely, since $i_\mu(c_\mu(A))$ is $\mu$-open and $A \subseteq i_\mu(c_\mu(A)) \subseteq c_\mu(A)$ for each $A \in \mathcal{I}$, $\{i_\mu(c_\mu(A)) \mid A \in \mathcal{I}\}$ is a finite $\mu$-open super cover of $\mathcal{I}$. So $X$ is $\mu$-precompact.

**Theorem 2.12.** In a $\mu$-space $X$, the following statements are equivalent.

1. $X$ is $\mu$-precompact.

2. Each $\mu$-preopen cover $\mathcal{I}$ of $X$ has a finite subcollection $\mathcal{B}$ such that $\{i_\mu(c_\mu(B)) \mid B \in \mathcal{B}\}$ covers $X$.

3. If $\mathcal{E}$ is a collection of $\mu$-preclosed sets of $X$ such that $\bigcap_{E \in \mathcal{E}} E = \emptyset$, then there exists a finite subcollection $\mathcal{F}$ of $\mathcal{E}$ such that $\bigcap_{F \in \mathcal{F}} i_\mu(c_\mu(F)) = \emptyset$. 

Proof: (a) ⇒ (b): Follows from Theorem 2.11.

(b) ⇒ (c): Let $\mathcal{E} = \{E_\alpha \mid \alpha \in \Delta\}$ be a collection of $\mu$-preclosed sets such that $\bigcap_{\alpha \in \Delta} E_\alpha = \emptyset$. It means that $\{X - E_\alpha \mid \alpha \in \Delta\}$ is a $\mu$-preopen cover of $X$. By (b), we find a finite subcollection $\{X - E_{\alpha_k} \mid \alpha_k \in \Delta, k \in \{1,2,\ldots,n\}\}$ of $\{X - E_\alpha \mid \alpha \in \Delta\}$ such that $\{i_\mu(c_\mu(X - E_{\alpha_k})) \mid k \in \{1,2,\ldots,n\}\}$ covers $X$. It means that $X - \bigcup_{k=1}^n i_\mu(c_\mu(X - E_{\alpha_k})) = \emptyset$ and hence $\bigcap_{k=1}^n i_\mu(E_{\alpha_k}) = \emptyset$.

(c) ⇒ (a): Let $X$ be a $\mu$-space satisfying (c). Suppose $\mathcal{W} = \{W_\alpha \mid \alpha \in A\}$ is a $\mu$-preopen cover of $X$. So we find that $\mathcal{E} = \{X - W_\alpha \mid \alpha \in A\}$ is a collection of $\mu$-preclosed sets such that $\bigcap_{\alpha \in A} X - W_\alpha = \emptyset$. By (c), we obtain a finite subcollection $\{X - W_{\alpha_k} \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ such that $\bigcap_{k=1}^n i_\mu(X - W_{\alpha_k}) = \emptyset$ which in turn implies that $\bigcup_{k=1}^n i_\mu(W_{\alpha_k}) = X$. So $\{W_{\alpha_k} \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ is a finite subcollection $\mathcal{W}$ such that $\{i_\mu(c_\mu(W_{\alpha_k})) \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ covers $X$. Then by Theorem 2.11, $X$ is $\mu$-precompact. □

Definition 2.13. A collection $\mathcal{A}$ of subsets of $X$ is called a $\mu$-proximate cover of $X$ if $c_\mu(\bigcup_{\alpha \in \mathcal{A}} A) = X$.

Theorem 2.14. Each $\mu$-preopen cover of a $\mu$-precompact space $X$ has a finite $\mu$-proximate $\mu$-preopen cover.

Proof: Let $\mathcal{A} = \{A_\alpha \mid \alpha \in \Delta\}$ be a $\mu$-preopen cover of a $\mu$-precompact space $X$. By $\mu$-precompactness of $X$, we obtain a finite $\mu$-open super cover $\{G_1, G_2,\ldots,G_n\}$ of $\mathcal{A}$. For each $k \in \{1,2,\ldots,n\}$, there exist an $\alpha_k \in \Delta$ such that $A_{\alpha_k} \subset G_k \subset c_\mu(G_{\alpha_k})$. Since $\{G_1, G_2,\ldots,G_n\}$ is a cover of $X$, we have $X = \bigcup_{k=1}^n c_\mu(A_{\alpha_k}) = c_\mu(\bigcup_{k=1}^n A_{\alpha_k})$. So $\{A_{\alpha_1}, A_{\alpha_2},\ldots,A_{\alpha_n}\}$ is a finite $\mu$-proximate $\mu$-preopen cover of $X$. □

Definition 2.15 (Császár [3]). A $\mu$-space $X$ is called $\mu$-extremally disconnected if $c_\mu(G)$ is $\mu$-open for each $\mu$-open set $G$ of $X$.

Theorem 2.16. A w\mu-compact and $\mu$-extremally disconnected space is a $\mu$-precompact space.

Proof: Let $\mathcal{E} = \{E_\alpha \mid \alpha \in A\}$ be a $\mu$-preopen cover of a w\mu-compact $\mu$-extremally disconnected $\mu$-space $X$. For each $\alpha \in A$, there exists a $\mu$-open set $G_\alpha$ such that $E_\alpha \subset G_\alpha \subset c_\mu(E_\alpha) = c_\mu(G_\alpha)$. We see that $\mathcal{G} = \{G_\alpha \mid \alpha \in A\}$ is a $\mu$-preopen cover of $X$. Since $X$ is w\mu-compact, we obtain a finite subcollection $\{G_{\alpha_k} \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ such that $\{c_\mu(G_{\alpha_k}) \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ covers $X$. By $\mu$-extremal disconnectedness of $X$, we see that $\{c_\mu(G_{\alpha_k}) \mid \alpha_k \in A, k \in \{1,2,\ldots,n\}\}$ is a finite $\mu$-open super cover of $\mathcal{E}$. □

Definition 2.17. A $\mu$-semiopen set $A$ in $X$ is said to be covered if $G \subset A \subset c_\mu(G)$ for some $\mu$-open set $G$, then there exists a $\mu$-open set $H$ such that $G \subset A \subset H \subset c_\mu(G)$.

Lemma 2.18. A covered $\mu$-semiopen set in $X$ is $\mu$-preopen in $X$. 
Proof: Let $A$ be a covered $\mu$-semiopen set and $G \subset A \subset c_\mu(G)$ for some $\mu$-open set. Then $c_\mu(A) = c_\mu(G)$. Also we have another $\mu$-open set $H$ such that $G \subset A \subset H \subset c_\mu(G)$ which implies that $A \subset i_\mu(c_\mu(G)) = i_\mu(c_\mu(A))$. Hence $A$ is $\mu$-preopen. 

In Example 2.8, $[1, \infty)$ is $\mu$-open and hence it is both $\mu$-semiopen and $\mu$-preopen. But there exist no $\mu$-open set $G$ such that $[1, \infty) \subset G$. So $[1, \infty)$ is not covered $\mu$-semiopen. So we conclude that the converse of Lemma 2.18 may not be true.

Theorem 2.19. If each $\mu$-semiopen set of a $\mu$-precompact space $X$ is covered, then $X$ is $\mu$-S-closed also.

Proof: Let $\mathcal{F}$ be a $\mu$-semiopen cover of $X$. By Lemma 2.18, $\mathcal{F}$ is a $\mu$-preopen cover of $X$. By Theorem 2.11, $\mathcal{F}$ has a finite subcollection $\mathcal{F}$ such that $\{i_\mu(c_\mu(A)) \mid A \in \mathcal{F}\}$ covers $X$. For each $A \in \mathcal{F}$, we have $A \subset i_\mu(c_\mu(A)) \subset c_\mu(A)$. So $\mathcal{F}$ is a finite subcollection of $\mathcal{F}$ such that $\{(c_\mu(A) \mid A \in \mathcal{F}\}$ covers $X$ and so $X$ is $\mu$-S-closed.

A subset $A$ of a $\mu$-space is said to $\mu$-precompact with respect to $X$ if each $\mu$-preopen cover with respect to $X$ of $A$ has a finite $\mu$-open super cover. In view of Theorem 2.11, it can be showed that a subset $A$ of $X$ is $\mu$-precompact with respect to $X$ if each $\mu$-preopen cover $\mathcal{F}$ with respect to $X$ of $A$ has a finite subcollection $\mathcal{F}$ such that $\{i_\mu(c_\mu(G)) \mid G \in \mathcal{F}\}$ covers $A$.

Theorem 2.20. If each proper $\mu$-regularly closed set of a $\mu$-space $X$ is $\mu$-precompact with respect to $X$, then $X$ is $\mu$-precompact.

Proof: Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \Delta\}$ be a $\mu$-preopen cover of $X$. Since $\mathcal{F}$ is a cover of $X$, there exits an $A \in \mathcal{F}$ such that $A \neq \emptyset$. By Lemma 2.9, $i_\mu(c_\mu(A))$ is $\mu$-regularly open in $X$ and so $X - i_\mu(c_\mu(A))$ is $\mu$-regularly closed in $X$. By the assumption, we get a finite subcollection $\{A_\alpha_k \mid \alpha_k \in \Delta, k \in \{1, 2, \ldots, n\}\}$ such that $X - i_\mu(c_\mu(A)) \subset \bigcup_{k=1}^n i_\mu(c_\mu(A_\alpha_k))$ and thus $X \subset \bigcup_{k=1}^n i_\mu(c_\mu(A_\alpha_k)) \cup i_\mu(c_\mu(A))$. Therefore by Theorem 2.11, $X$ is $\mu$-precompact.

Recall that a nonempty collection $\mathcal{C}$ of nonempty subsets of a set $S$ is called a filter base [19, p. 78] if $C_1, C_2 \in \mathcal{C}$, then $C_3 \subset C_1 \cap C_2$ for some $C_3 \in \mathcal{C}$. A filter base is called maximal [19, p. 80] if its not properly contained into another filter base. A filter base is always contains in a maximal filter base [19, p. 80].

Definition 2.21. A filter base $\mathcal{F}$ on a $\mu$-space $X$ is called $p_\mu$-converges to a point $x \in X$ if for each $\mu$-preopen set $A$ of $X$ with $x \in A$, there exists $F \in \mathcal{F}$ such that $F \subset i_\mu(c_\mu(A))$.

Definition 2.22. A filter base $\mathcal{F}$ on a $\mu$-space $X$ is called $p_\mu$-accumulates to a point $x \in X$ if for each $\mu$-preopen set $A$ of $X$ with $x \in A$, $F \cap i_\mu(c_\mu(A)) \neq \emptyset$ for each $F \in \mathcal{F}$.
Lemma 2.23. If a filter base $\mathcal{F}$ in $X$ $p_\mu$-converges to a point $x \in X$, then the filter base is $p_\mu$-accumulates to $x$.

Proof: By $p_\mu$-convergence of $\mathcal{F}$ to $x \in X$, there exists $F \in \mathcal{F}$ such that $F \subseteq i_\mu(c_\mu(A))$ for each $\mu$-preopen set $A$ with $x \in A$. Let $E \in \mathcal{F}$. Then there exists $D \in \mathcal{F}$ such that $D \subseteq E \cap F \subseteq F \subseteq i_\mu(c_\mu(A))$. So $D \cap i_\mu(c_\mu(A)) \neq \emptyset$. As $D \subseteq E$, we have $E \cap i_\mu(c_\mu(A)) \neq \emptyset$. So $\mathcal{F}$ $p_\mu$-accumulates to $x \in X$. \hfill $\Box$

Lemma 2.24. Let $\mathcal{F}$ be a maximal filter base in $X$. Then $\mathcal{F}$ $p_\mu$-converges to $x \in X$ iff and only if $\mathcal{F}$ is $p_\mu$-accumulates to $x \in X$.

Proof: Since $\mathcal{F}$ is a filter base, $\mathcal{F}$ is $p_\mu$-accumulates to $x \in X$ by Lemma 2.23 if $\mathcal{F}$ is $p_\mu$-converges to $x \in X$.

Conversely, let $\mathcal{F}$ be a maximal filter base $\mathcal{F}$ $p_\mu$-accumulate to $x \in X$. If $\mathcal{F}$ does not $p_\mu$-converges to $x$, then for each $F \in \mathcal{F}$, there exists a $\mu$-preopen set $A$ containing $x$ such that $F \not\subseteq i_\mu(c_\mu(A))$ i.e. $F \cap c_\mu(i_\mu(X - A)) \neq \emptyset$. We put $\mathcal{E} = \mathcal{F} \cup \{F \cap c_\mu(i_\mu(X - A)) \mid F \in \mathcal{F}\}$. Then $\mathcal{E}$ is a filter base properly containing $\mathcal{F}$, a contradiction to the fact that $\mathcal{F}$ is a maximal filter base. \hfill $\Box$

Theorem 2.25. The following statements are equivalent:

1. $X$ is $\mu$-precompact.

2. Each filter base $p_\mu$-accumulates to some $x_0 \in X$.

3. Each maximal filter base $p_\mu$-converges in $X$.

Proof: $(a) \Rightarrow (b)$: Suppose that there exists a filter base $\mathcal{F} = \{F_\alpha \mid \alpha \in A\}$ in $X$ and $\mathcal{F}$ does not $p_\mu$-accumulates in $X$. It means that for each $x \in X$, there exists a $\mu$-preopen set $A$ containing $x$ and an $F_\alpha(x) \in \mathcal{F}$ such that $F_\alpha(x) \cap i_\mu(c_\mu(A)) = \emptyset$. So $\mathcal{F} = \{A_x \mid x \in X\}$ is a $\mu$-preopen cover of $X$. By Theorem 2.11, $\mathcal{F}$ has a finite subcollection $A_{x_1}, A_{x_2}, \ldots, A_{x_n}$ such that $\{i_\mu(c_\mu(A_{x_k})) \mid k \in \{1, 2, \ldots, n\}\}$ covers $X$. As $\mathcal{F}$ is a filter base, there exists an $F_0 \in \mathcal{F}$ such that $F_0 \subseteq \bigcap_{k=1}^n F_{\alpha(x_k)}$. It means that $F_0 \cap i_\mu(c_\mu(A_{x_k})) = \emptyset$ for each $k \in \{1, 2, \ldots, n\}$. Now $F_0 = F_0 \cap X = F_0 \cap (\bigcup_{k=1}^n i_\mu(A_{x_k}))) = \bigcup_{k=1}^n (F_0 \cap i_\mu(A_{x_k})) = \emptyset$, a contradiction to the fact that $F_0 \neq \emptyset$.

$(b) \Rightarrow (c)$: Let $\mathcal{F}$ be a maximal filter base in $X$. By $(ii)$, $\mathcal{F}$ $p_\mu$-accumulates to some $x_0 \in X$. $\mathcal{F}$ being a maximal filter base in $X$, $\mathcal{F}$ $p_\mu$-converges to $x_0 \in X$ by Lemma 2.24.

$(c) \Rightarrow (a)$: Let $\mathcal{F} = \{A_\alpha \mid \alpha \in \Delta\}$ be a $\mu$-preopen cover of $X$. If possible, let $X$ be not $\mu$-precompact. Then for each finite subcollection $\Delta_0$ of $\Delta$, we have $\bigcup_{\alpha \in \Delta_0} i_\mu(c_\mu(A_\alpha)) \neq X$ which implies that $\bigcap_{\alpha \in \Delta_0} c_\mu(i_\mu(X - A_\alpha)) \neq \emptyset$. We put $F_{\Delta_0} = \bigcap_{\alpha \in \Delta_0} c_\mu(i_\mu(X - A_\alpha))$. Let $\Lambda$ be the collection of all finite subcollection of $\Delta$. We write $\mathcal{F} = \{F_\lambda \mid \lambda \in \Lambda\}$ (each $F_\lambda$ bears the meaning as of $F_{\Delta_0}$). We see that $\mathcal{F}$ is a filter base on $X$ and hence there exists a maximal filter base $\mathcal{M}$ containing $\mathcal{F}$. By $(c)$, $\mathcal{M}$ $p_\mu$-converges to some point $x_0 \in X$ and so $\mathcal{M}$ $p_\mu$-accumulates to
some point \( x_0 \in X \) by Lemma 2.24. As \( \mathcal{S} \) is a cover of \( X \), there exists \( A_0 \in \mathcal{S} \) such that \( x_0 \in A_0 \). Then by construction, \( c_\mu(i_\mu(X - A_0)) \in \mathcal{M} \). Since \( \mathcal{M} \) \( p_\mu \)-accumulates to \( x_0 \) and \( x_0 \in A_0 \), we see that \( M \cap i_\mu(c_\mu(A_0)) \neq \emptyset \) for each \( M \in \mathcal{M} \), in particular, \( c_\mu(i_\mu(X - A_0)) \cap i_\mu(c_\mu(A_0)) \neq \emptyset \), a contradiction to the fact that \( c_\mu(i_\mu(X - A_0)) \cap i_\mu(c_\mu(A_0)) = \emptyset \). □

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