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Domination numbers of the complete grid graphs P_k x P_n

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ABSTRACT. This paper concerns the domination numbers $\gamma(P_k \times P_n)$ for the complete P_k x P_n grid graphs for $k = 7, 8, 9$ and for all $n \geq 1$. These numbers were previously established (BONDY; MURTY, 2008; CHANG; CLARK, 1993). Here we present dominating sets by other method.

Keywords: dominating Set, domination number, transformation of a dominating set, cartesian product of two paths.

Números de dominação de gráficos *grid* completos P_k x P_n

RESUMO. Este artigo diz respeito aos números de dominação γ (P_k x P_n) para gráficos completos P_k x P_n, para $k = 7, 8, 9$ e para todo $n \geq 1$. Estes números foram previamente estabelecidos por (BONDY; MURTY, 2008; CHANG; CLARK, 1993). Aqui apresentamos conjuntos dominados por outro método.

Palavras-chave: conjunto de dominação, número de dominação, transformação de um conjunto dominante, produto cartesiano de dois caminhos.

Introduction

A dominating set in a graph is a set of vertices having the property that every vertex not in the set is adjacent to a vertex in the set. The domination number γ (G) of a graph G is the cardinality of a smallest dominating set in G.

The problem of finding the domination number of a arbitrary grid graph (= subgraph of P_k x P_n) is NP-complete (CLARK et al., 1990).

In this paper, we introduce the concept of transforming the domination from a vertex in a dominating set D of a graph $G = (V, E)$ to a vertex in $V - D$, where G is a simple connected graph. We give an algorithm using this transformation to obtain a dominating set of a graph G.

A graph $G = (V, E)$ is a mathematical structure which consists of two sets V and E, where V is finite and nonempty, and every element of E is an unordered pair $\{u, v\}$ of distinct elements of V ; we simply write uv instead of $\{u, v\}$.

The elements of V are called vertices, while the elements of E are called edges (BONDY; MURTY, 2008).

Two vertices u and v of a graph G are said to be adjacent if $uv \in E$.

The neighborhood of v is the set of all vertices of G which are adjacent to v; the neighborhood of v is denoted by $N(v)$. The closed neighborhood of v is $\bar{N}(v)$, $\bar{N}(v) = N(v) \cup \{v\}$.

The degree d (v) of a vertex v is the cardinality $|N(v)|$, $d(v) = |N(v)|$.

Material and methods

Definitions

Let D be a dominating set of a graph $G = (V, E)$.

1. We define the function C_D, which we call the weight function, as follows: C_D: $V \rightarrow N$, where N is the set of natural numbers, $C_D(v) = |\tilde{N}(v)|$, where $\tilde{N}(v) = \{w \in D: vw \in E \text{ or } w = v\}$.

i.e. the weight of v is the number of vertices in D which dominate v.

2. We say that $v \in D$ has a moving domination if there exists a vertex $w \in N(v) - D$ such that $wu \in E$ for every vertex u, $u \in \{x \in N(v): C_D(x) = 1\}$.

3. We say that a vertex $v \in D$ is a redundant vertex of D if $C_D(w) \geq 2$ for every vertex $w \in \bar{N}(v)$.

4. If $v \in D$ has a moving domination, we say that v is inefficient if transforming the domination from v to any vertex in $N(v)$ would not produce any redundant vertex.

Complete grid graph P_k x P_n:

For two vertices v_o and v_n of a graph G, a $v_o - v_n$ walk is an alternating sequence of vertices and edges $v_o, e_1, v_1, \ldots, e_n, v_n$ such that consecutive vertices and edges are incident.

A path is a walk in which no vertex is repeated. A path with n vertices is denoted by P_n, it has $n - 1$ edges; the length of P_n is $n - 1$; the cartesian product P_k x P_n of two paths is the complete grid graph with vertex set $V = \{(i, j): 1 \leq i \leq k, 1 \leq j \leq n\}$

where (u_1, u_2) (v_1, v_2) is a edge of P_k x P_n if $|u_1 - v_1| + |u_2 - v_2| = 1$ (CHANG; CLARK, 1993).

If D is a dominating set of P_k x P_n which has no redundant vertex, then a vertex $v \in D$ has a moving domination if and only if one of the following two cases occurs:

Case (1): for every vertex $w \in N(v)$, we have $C_D(w) \geq 2$.

In this case, the domination of v can be transformed to any vertex in $N(v)$ - D.

Case(2): there exists exactly one vertex $u \in N(v)$ such that $C_D(u) = 1$ in this case, the domination of v can be transformed only to u.

An algorithm for finding a dominating set of a graph P_k x P_n using a transformation of domination of vertices

1. Let P_k x $P_n = (V, E)$ be a graph of order greater than 1; $|V| = m$.

2. Let $D = V$ be a dominating set of P_k x P_n. then, for any vertex $v \in D$ we have $C_D(v) = d(v) + 1 \geq 2$.

3. Pick a vertex v_1 of D, and delete from D all vertices w, $w \in N(v_1)$ then, for $1 < n < \frac{m}{2}$, pick a vertex $v_n \in D - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$ and delete from D all vertices w, $w \in N(v_n) - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$.

4. If D contains a redundant vertex, then delete it.

Repeat this process until D has no redundant vertex.

5. Transform domination from vertices of D which have moving domination to vertices in V - D to obtain redundant vertices and go to step 4.

If no redundant vertex can be obtained by a transformation of domination of vertices of D, then stop, and the obtained dominating set D satisfies:

for every $v \in D, \exists w \in \overline{N}(v)$ such that $C_D(w) = 1$.

Example

1. Let (k, n) be the vertex in the $k-th$ row and in the $n-th$ column of the graph $G = P_7 \times P_{16}; |V| = 112$.

2. Let $D = V$, dominating set of G.

3. Pick a vertex $v_1 = (2,1) \in D$, and delete from D all vertices w, $w \in N(v_1)$, then, for $1 < n < \frac{112}{2}$, pick a vertex v_n, $v_n \in D - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$, and delete from D all vertices w, $w \in N(v_n) - \bigcup_{i=1}^{n-1} \overline{N}(v_i)$. We obtain the dominating set D (black circles) in figure 1.

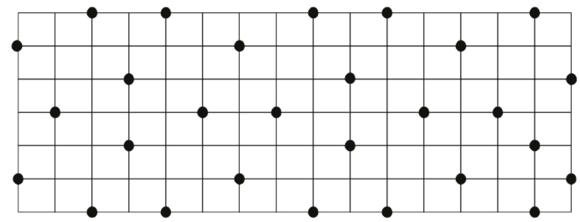


Figure 1. Dominating set (black circles).

4. Since for every vertex $v \in D$, $\exists w \in \overline{N}(v)$ such that $C_D(w) = 1$, D has no redundant vertices.

5. Transform the domination from the vertex (6, 16) to the vertex (5, 16) and delete, from D, the resulting redundant vertex (5, 15).

Therefore, the set D indicated in figure 2 (black circles) is a dominating set of $G = P_7 \times P_{16}$.

Note that D minimum dominating set (see CHANG et al., 1994) $\gamma(P_7 \times P_{16}) = 27$.

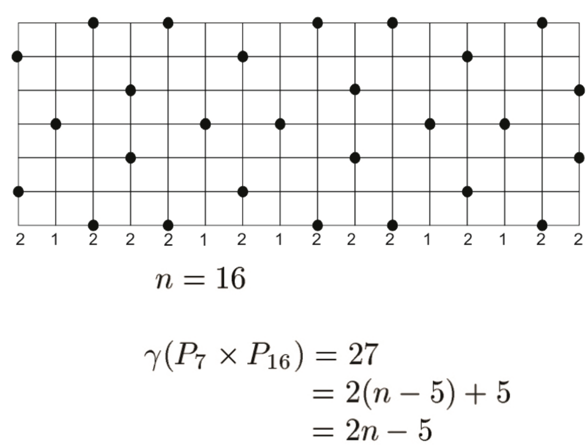
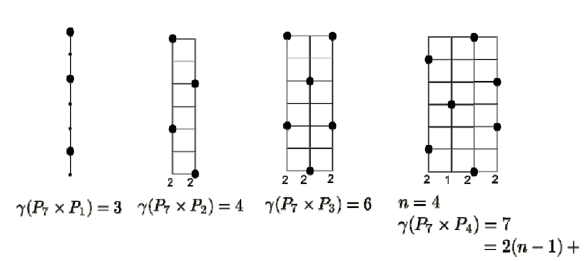
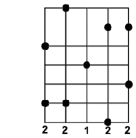


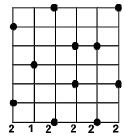
Figure 2. Domination number (black circles).

And so, we gradually get domination numbers of P_7 x P_n.

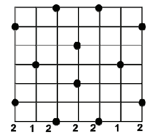




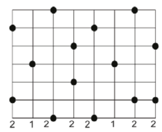
$n = 5$
$\gamma(P_7 \times P_5) = 9$
$= 2(n-1)+1$
$= 2n-1$



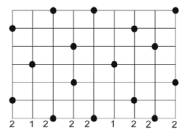
$n = 6$
$\gamma(P_7 \times P_6) = 11$
$= 2(n-1)+1$
$= 2n-1$



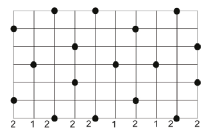
$n = 7$
$\gamma(P_7 \times P_7) = 12$
$= 2(n-2)+2$
$= 2n-2$



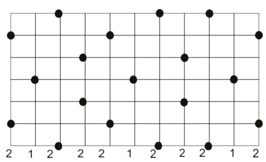
$n = 8$
$\gamma(P_7 \times P_8) = 14$
$= 2(n-2)+2$
$= 2n-2$



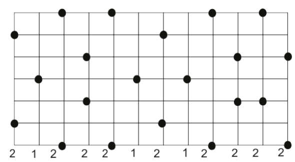
$n = 9$
$\gamma(P_7 \times P_9) = 16$
$= 2(n-2)+2$
$= 2n-2$



$n = 10$
$\gamma(P_7 \times P_{10}) = 17$
$= 2(n-3)+3$
$= 2n-3$



$n = 11$
$\gamma(P_7 \times P_{11}) = 19$
$= 2(n-3)+3$
$= 2n-3$



$n = 12$
$\gamma(P_7 \times P_{12}) = 21$
$= 2(n-3)+3$
$= 2n-3$

Table 1. Domination numbers $\gamma(P_7 \times P_n), n \geq 1$.

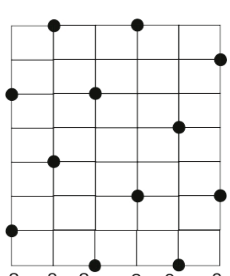
Hence:

$$\gamma(P_7 \times P_n) = \begin{cases} 3 \text{ for } n=1 \\ 4 \text{ for } n=2 \\ 6 \text{ for } n=3 \\ 2n-t \text{ for } n=3t+1, 3t+2, 3t+3; t \geq 1 \end{cases}$$

where t is a positive integer.

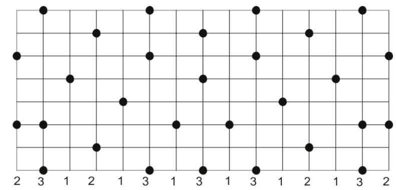
Dominating sets for P_8 x P_n Grid graph

By self-method, we give in Table 2, $\gamma(P_8 \times P_n), n \geq 1$.



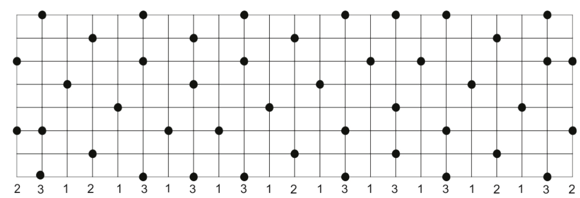
$n = 6$
$\gamma(P_8 \times P_6) = 12$
$= 2n$

And by gradually, we find that: $\gamma(P_8 \times P_n) = 2n$ for $6 \leq n \leq 14$.



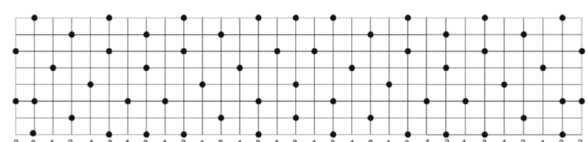
$n = 15$
$\gamma(P_8 \times P_{15}) = 29$
$= (\frac{n-5}{2})(3+1)+9$
$= 2n-1$

And by gradually, we find that $\gamma(P_8 \times P_n) = 2n-1$ for $15 \leq n \leq 22$.



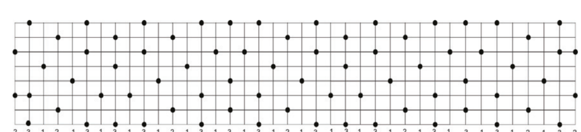
$n = 23$
$\gamma(P_8 \times P_{23}) = 44$
$= (\frac{n-7}{2})(3+1)+12$
$= 2n-2$

And by gradually, we find that $\gamma(P_8 \times P_n) = 2n-2$ for $23 \leq n \leq 30$.



$n = 31$
$\gamma(P_8 \times P_{31}) = 59$
$= (\frac{n-9}{2})(3+1)+15$
$= 2n-3$

And by gradually, we find that $\gamma(P_8 \times P_n) = 2n-3$ for $31 \leq n \leq 38$.



$n = 39$
$\gamma(P_8 \times P_{39}) = 74$
$= (\frac{n-11}{2})(3+1)+18$
$= 2n-4$

And by gradually, we find that $\gamma(P_8 \times P_n) = 2n-4$ for $39 \leq n \leq 46$.
Hence:

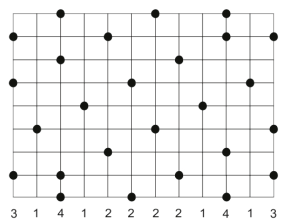
$$\gamma(P_8 \times P_n) = \begin{cases} 2n+1 \text{ for } n = 1,2,3,5 \\ 2n \text{ for } n = 4 \text{ or } 6 \leq n \leq 14 \\ 2n-(t+1) \text{ for } n = 15+8t, 16+8t, \ldots, 22+8t; t \geq 0 \end{cases}$$

where t is a integer.

Table 2. Domination numbers $\gamma(P_8 \times P_n), n \geq 1$.

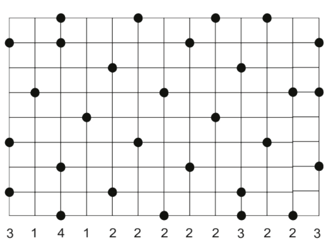
Dominating sets for P_9 x P_n Grid graph

By self-method, we give in Table 3, $\gamma(P_9 \times P_n)$, n ≥ 1



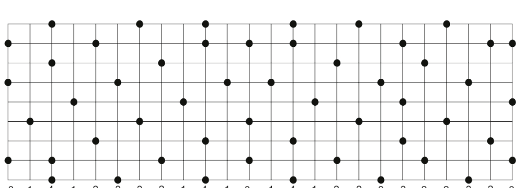
$$n = 12$$
$$\gamma(P_9 \times P_{12}) = 26 = 2(n-8) + 18 = 2n + 2$$

And by gradually, we find that $\gamma(P_9 \times P_n) = 2n + 2$ for $4 \le n \le 12$.



$$n = 13$$
$$\gamma(P_9 \times P_{13}) = 29 = 2(n-6) + 15 = 2n + 3$$

And by gradually, we find that $\gamma(P_9 \times P_n) = 2n + 3$ for $13 \le n \le 23$.



$$n = 24$$
$$\gamma(P_9 \times P_{24}) = 52 = 2(n-13) + 30 = 2n + 4$$

And by gradually, we find that $\gamma(P_9 \times P_n) = 2n + 4$ for $24 \le n \le 34$.

Hence:

$$\gamma(P_9 \times P_n) = \begin{cases} 2n+1 \text{ for } 1 \le n \le 3 \\ 2n+2 \text{ for } 4 \le n \le 12 \\ 2n + (t+3) \text{ for } n = 13+11t, 14+11t, ..., 23+11t; t \ge 0 \end{cases}$$

where t is a integer.

Table 3. Domination numbers $\gamma(P_9 \times P_n)$, n ≥ 1.

We plan to deal with P_k x P_n grid graphs, for n ≥ 1 and k ≥ 10.

Results and discussion

$$\gamma(P_7 \times P_n) = \begin{cases} 3 \text{ for } n = 1 \\ 4 \text{ for } n = 2 \\ 6 \text{ for } n = 3 \\ 2n - t \text{ for } n = 3t+1, 3t+2, 3t+3; t \ge 1 \end{cases}$$

Table1. Domination numbers $\gamma(P_7 \times P_n), n \ge 1$.

$$\gamma(P_8 \times P_n) = \begin{cases} 2n+1 \text{ for } n = 1,2,3,5 \\ 2n \text{ for } n = 4 \text{ or } 6 \le n \le 14 \\ 2n - (t+1) \text{ for } n = 15+8t, 16+8t, ..., 22+8t; t \ge 0 \end{cases}$$

Table2. Domination numbers $\gamma(P_8 \times P_n), n \ge 1$.

$$\gamma(P_9 \times P_n) = \begin{cases} 2n+1 \text{ for } 1 \le n \le 3 \\ 2n+2 \text{ for } 4 \le n \le 12 \\ 2n + (t+3) \text{ for } n = 13+11t, 14+11t, ..., 23+11t; t \ge 0 \end{cases}$$

Table3. Domination numbers $\gamma(P_9 \times P_n), n \ge 1$.

Conclusion

From these results, we note that it is possible to calculate the domination numbers quickly and without the need of a calculator in most cases, however the results published previously, you must necessarily use a calculator.

We tried this method on large grid graphs, and we obtained similar results quickly

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