Acta Scientiarum
http://www.uem.br/acta
ISSN printed: 1806-2563
ISSN on-line: 1807-8664
Doi: 10.4025/actascitechnol.v35i1.15523

Some Generalized Lacunary statistically difference double semi-normed sequence spaces defined by Orlicz function

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ABSTRACT. In this article, we have introduced the idea of statistically convergent generalized difference lacunary double sequence spaces $[\overline{w}^2(M,\Delta^n,p,q)]_\theta$, $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ and defined over a semi norm space (X, q). Also we have study some basic properties and obtained some inclusion relations between them.

Keywords: statistical convergent, P-convergent, difference sequence, lacunary sequence, Orlicz function.

Espaços sequenciais semi-normais duplos com diferença estatística lacunar generalizada definidos pela função de Orlicz

RESUMO. Este trabalho apresenta a ideia de espaços sequenciais duplos com diferença lacunar generalizada e convergente $[\overline{w}^2(M,\Delta^n,p,q)]_\theta$ e $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ definidos em um espaço semi normalizado. Além disso, foram também estudadas algumas propriedades básicas e obtidas algumas relações de inclusão entre elas.

Palavras-chave: estatística convergente, P-convergente, sequência de diferença, sequência lacunar, função de orlicz.

Introduction

The concept involving statistical convergence plays a vital role not only in pure mathematics but also in other branches of mathematics especially in information theory, computer science and biological science.

Let ℓ_∞, c and c_0 be the Banach spaces of bounded, convergent and null sequences $x=(x_k)$ with the usual norm $\|x\|=\sup_k |x_k|$. In order to extend the notion of convergence of sequences, statistical convergence was introduced by Fast (1951) and Schoenberg (1959) independently. Later on it was further investigated by Fridy (1985), Mursaleen and Mohiuddine (2009a and b), Mohiuddine and Danish Lohani (2009), Mohiuddine et al. (2010), Šalát (1980), Tripathy (2003), Tripathy and Sen (2001) and many others. The idea depends on the notion of density (natural or asymptotic) of subsets of N. A subset E of N is said to have natural density $\delta(E)$ if

$$\delta(E)=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\chi_E(k)$$

exists, where χ_E denotes the characteristic function of E.

A sequence $(x_k)\in w$ is said to be *statistically convergent* to a number ℓ if for every $\varepsilon>0$ the set

$$\{k\in\mathbb{N}:|x_k-\ell|\geq\varepsilon\}$$

has natural density zero.

Kizmaz (1981) introduced the notion of difference sequence spaces as follows:

$$X(\Delta)=\{x=(x_k)\in w:(\Delta x_k)\in X\}$$

for $X=\ell_\infty, c$ and c_o. Later on, the notion was generalized by Et and Çolak (1995) as follows:

$$X(\Delta^m)=\{x=(x_k)\in w:(\Delta^m x_k)\in X\}$$

for $X=\ell_\infty, c$ and c_o, where

$$\Delta^m x=(\Delta^m x_k)=(\Delta^{m-1}x_k-\Delta^{m-1}x_{k+1}), \Delta^0 x=x$$

and also this generalized difference notion has the following binomial representation:

$$\Delta^m x_k = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x_{k+i} \text{ for all } k \in \mathbb{N}.$$

Subsequently, difference sequence spaces were studied by Esi (2009a and b), Esi and Tripathy (2008), Tripathy et al. (2005) and many others.

An Orlicz function M is a function $M : [0,\infty) \to [0,\infty)$, which is continuous, convex, nondecreasing function such that $M(0) = 0$, $M(x) > 0$, for $x > 0$ and $M(x) \to \infty$ as $x \to \infty$. If convexity of Orlicz function is replaced by $M(x + y) \leq M(x) + M(y)$ then this function is called the modulus function and characterized by Ruckle (1973). An Orlicz function M is said to satisfy Δ_2 – *condition* for all values u, if there exists $K > 0$ such that $M(2u) \leq KM(u), u \geq 0$.

Remark 1.1. An Orlicz function satisfies the inequality $M(\lambda x) \leq \lambda M(x)$ for all λ with $0 < \lambda < 1$.

Lindenstrauss and Tzafriri (1971) used the idea of Orlicz function to construct the sequence space

$$l_M = \left\{ (x_k) : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r} \right) < \infty, \textit{for some } r > 0 \right\}$$

which is a Banach space normed by

$$\| (x_k) \| = \inf \left\{ r > 0 : \sum_{k=1}^{\infty} M\left(\frac{|x_k|}{r} \right) \leq 1 \right\}.$$

The space l_M is closely related to the space l_p, which is an Orlicz sequence space with $M(x) = |x|^p$, for $1 \leq p < \infty$.

In a later stage different Orlicz sequence spaces were introduced and studied by Tripathy and Mahanta (2004), Esi (1999, 2009a and b, 2010), Esi and Et (2000), Parashar and Choudhary (1994) and many others.

The following well-known inequality will be used throughout the article. Let $p = (p_k)$ be any sequence of positive real numbers with $0 < p_k \leq \sup_k p_k = G, D = \max\{1, 2^{G-1}\}$ then

$$|a_k + b_k|^{p_k} \leq D(|a_k|^{p_k} + |b_k|^{p_k})$$

for all $k \in \mathbb{N}$ and a_k, $b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max\{1, |a|^G\}$ for all $a \in \mathbb{C}$.

Let w^2 denote the set of all double sequences of complex numbers. By the convergence of a double sequence we mean the convergence in the Pringsheim sense that is, a double sequence $x = (x_{k,l})$ has Pringsheim limit L (denoted by $P - \lim x = L$) provided that given $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that $|x_{k,l} - L| < \varepsilon$ for $k,l > N$ (PRINGSHEIM, 1900). And we called it as "P - *convergent*". We shall denote the space of all P - *convergent* sequences by c^2. The double sequence $x = (x_{k,l})$ is bounded if and only if there exists a positive number M such that $|x_{k,l}| < M$ for all k and l. We shall denote the space of all bounded double sequences by l_∞^2. The zero single sequence will be denoted by $\theta = (\theta, \theta, \theta, \ldots)$ and the zero double sequence will be denoted by $\theta^2 = (\theta)$.

The notion of asymtotic density for subsets of $\mathbb{N} \times \mathbb{N}$ was introduced by Tripathy (2003). A subset E of $\mathbb{N} \times \mathbb{N}$ is said to have density $\rho(E)$ if

$$\rho(E) = \lim_{p,q \to \infty} \frac{1}{pq} \sum_{n \leq p} \sum_{k \leq q} \chi_E(n,k)$$

exists.

The notion of statistically convergent double sequences was introduced by Mursaleen and Edely (2003) and Tripathy (2003) independently.

A double sequence $(x_{k,l})$ is said to be *statistically convergent* to ℓ in Pringsheim's sense if for every $\varepsilon > 0$,

$$\rho(\{(k,l) \in \mathbb{N} \times \mathbb{N} : |x_{k,l} - L| \geq \varepsilon\}) = 0.$$

The double sequence $\theta_{r,s} = \{(k_r, l_s)\}$ is called double lacunary sequence if there exist two increasing sequence of integers such that

$$k_o = 0,\ h_r = k_r - k_{r-1} \to \infty,\ as\ r \to \infty$$

and

$$l_0 = 0, \overline{h}_s = l_s - l_{s-1} \to \infty\ as\ s \to \infty.$$

Notations: $k_{r,s} = k_r l_s, h_{r,s} = h_r \overline{h}_s$ and $\theta_{r,s}$ is determined by

$$I_{r,s} = \{(k,l) : k_{r-1} < k \leq k_r \textit{ and } l_{s-1} < l \leq l_s\}$$

$$q_r = \frac{k_r}{k_{r-1}}, \overline{q}_s = \frac{l_s}{l_{s-1}} \textit{ and } q_{r,s} = q_r \overline{q}_s.$$

The set of all double lacunary sequences is denoted by $N_{\theta_{r,s}}$ and defined by Savas and Patterson (2006) as follows:

$$N_{\theta_{r,s}} = \left\{ x = (x_{k,l}) : P - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l) \in I_{r,s}} | x_{k,l} - L | = 0, \textit{for some } L \right\}$$

New types of double sequence spaces

In this presentation our goal is to extend a few results known in the literature from ordinary (single) difference sequences to difference double sequences. Some studies on double sequence spaces can be found in Gökhan and Çolak (2004, 2005 2006).

Let M be an Orlicz function and $p = (p_{k,l})$ be a factorable double sequence of strictly positive real numbers and $\theta_{r,s}$ be a lacunary sequence. Let X be a seminormed space over the complex field □ with the seminorm q. We now define the following new statistically convergent generalized difference lacunary double sequence spaces:

$$[\overline{w}^2(M,\Delta^n,p,q)]_\theta - \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{q(\Delta^n x_{k,l} - L)}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \textit{ and } L \right\},$$

$$[\overline{w}_o^2(M,\Delta^n,p,q)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0. \right\},$$

and

$$[w_\infty^2(M,\Delta^n,p,q)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : \sup_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho} \right) \right]^{p_{k,l}} < \infty, \text{ for some } \rho > 0. \right\},$$

where:

$$\Delta^u x = (\Delta^u x_{k,l}) = (\Delta^{u-1} x_{k,l} - \Delta^{u-1} x_{k,l+1} - \Delta^{u-1} x_{k+1,l} + \Delta^{u-1} x_{k+1,l+1})$$

$$(\Delta^1 x_{k,l}) = (\Delta x_{k,l})$$

$$= (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}),\ \Delta^0 x_{k,l} = x_{k,l}$$

and also this generalized difference double notion has the following binomial representation:

$$\Delta^n x_{k,l} = \sum_{i=0}^{n} \sum_{j=0}^{n} (-1)^{i+j} \binom{n}{i} \binom{m}{j} x_{k+i,l+j}$$

Some double sequence spaces are obtained by specializing $\theta_{r,s}$ M, p, q and n. Here are some examples:

(i) If $\theta_{r,s} = \{(k_r, l_s)\} = \{(2^r, 2^s)\}$, $M(x)=x$, $n = 0$, $p_{k,l} = 1$ for all $k,l \in$ □ and $q(x) = |x|$ then we obtain the double sequence spaces $[\overline{w}^2], [\overline{w}_0^2]$ and $[w_\infty^2]$.

(ii) If $M(x)=x$, $n = 0$, $p_{k,l} = 1$ for all $k,l \in$ □ and $q(x) = |x|$ then we obtain the double sequence spaces $[\overline{w}^2]_\theta, [\overline{w}_0^2]_\theta$ and $[w_\infty^2]_\theta$.

(iii) If $M(x)=x$, $n = 0$ for all $k,l \in$ □ and $q(x) = |x|$ then we obtain the double sequence spaces $[\overline{w}^2(p)]_\theta, [\overline{w}_0^2(p)]_\theta$ and $[w_\infty^2(p)]_\theta$ defined as follows:

$$[\overline{w}^2(p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left(|x_{k,l} - L| \right)^{p_{k,l}} = 0, \text{for some } L \right\},$$

$$[\overline{w}_o^2(p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left(|x_{k,l}| \right)^{p_{k,l}} = 0 \right\},$$

and

$$[w_\infty^2(p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : \sup_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left(|x_{k,l}| \right)^{p_{k,l}} < \infty \right\}.$$

(iv) If $n = 0$ and $q(x) = |x|$ then we obtain the double sequence spaces $[\overline{w}^2(M,p)]_\theta, [\overline{w}_0^2(M,p)]_\theta$ and $[w_\infty^2(M,p)]_\theta$ defined as follows:

$$[\overline{w}^2(M,p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \textit{ and } L \right\},$$

$$[\overline{w}_0^2(M,p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$[w_\infty^2(M,p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : \sup_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \textit{for some } \rho > 0 \right\}.$$

(v) If $n=1$ and $q(x) = |x|$ then we obtain the double sequence spaces $[\overline{w}^2(M,\Delta,p)]_\theta, [\overline{w}_0^2(M,\Delta,p)]_\theta$ and $[w_\infty^2(M,\Delta,p)]_\theta$ defined as follows:

$$[\overline{w}^2(M,\Delta,p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|\Delta x_{k,l} - L|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \textit{ and } L \right\},$$

$$[\overline{w}_0^2(M,\Delta,p)]_\theta = \left\{ x = (x_{k,l}) \in w^2 : st_2 - \lim_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|\Delta x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} = 0, \text{ for some } \rho > 0 \right\},$$

and

$$[w_\infty^2(M,\Delta,p)]_\theta = \left\{ x = (x_{k,l}) : \sup_{r,s} (h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \square \left[M\left(\frac{|\Delta x_{k,l}|}{\rho} \right) \right]^{p_{k,l}} < \infty, \textit{for some } \rho > 0 \right\},$$

where

$$(\Delta x_{k,l}) = (x_{k,l} - x_{k,l+1} - x_{k+1,l} + x_{k+1,l+1}).$$

Main Results

Theorem 3.1. *Let* $p=(p_{k,l})$ *be bounded. The classes of* $[\overline{w}^2(M,\Delta^n,p,q)]_\theta$, $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ *and* $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ *are linear spaces over the complex field* $\mathbb{C}$.

Proof. We give the proof for the space $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ and for the others spaces the proof can be obtained in a similar way. Let $x=(x_{k,l}), y=(y_{k,l})\in[w_\infty^2(M,\Delta^n,p,q)]_\theta$. Then we have

$$\sup_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}}<\infty, \text{for some } \rho_1>0 \tag{3.1}$$

and

$$\sup_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}<\infty, \text{for some } \rho_2>0. \tag{3.2}$$

Let $\alpha,\beta\in\mathbb{C}$ be scalars and $\rho=\max\{2|\alpha|\rho_1,2|\beta|\rho_2\}$. Since M is non-decreasing convex function, we have

$$\left[M\left(\frac{q(\Delta^n(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}}$$

$$\le\left[M\left(\frac{q(\Delta^n(\alpha x_{k,l}))}{\rho}\right)\right]^{p_{k,l}}+\left[M\left(\frac{q(\Delta^n(\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}}$$

$$\le D\left\{\left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right)\right]^{p_{k,l}}+\left[M\left(\frac{q(\Delta^n y_{k,l})}{\rho}\right)\right]^{p_{k,l}}\right\}$$

where $D=\max(1,2^{H-1}), H=\sup_{k,l}p_{k,l}<\infty$.

Now, from (3.1) and (3.2), we have

$$\sup_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n(\alpha x_{k,l}+\beta y_{k,l}))}{\rho}\right)\right]^{p_{k,l}}<\infty.$$

Therefore $\alpha x+\beta y\in[w_\infty^2(M,\Delta^n,p,q)]_\theta$. Hence $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ is a linear space.

Theorem 3.2. The double sequence spaces $[\overline{w}^2(M,\Delta^n,p,q)]_\theta$ $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ and $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ are seminormed spaces, seminormed by

$$f((x_{k,l}))=\sum_{k=1}^{n}q(x_{k,1})+\sum_{l=1}^{n}q(x_{1,l})+\inf\left\{\rho>0:\sup_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n y_{k,l})}{\rho}\right)\right]^{p_{k,l}}\le 1\right\}.$$

Proof. Since q is a seminorm, so we have $f((x_{k,l}))\ge 0$ for all $x=(x_{k,l})$ $f(\theta^2)=0$ and $f((\lambda x_{k,l}))=|\lambda|f((x_{k,l}))$ for all scalars λ.

Now, let $x=(x_{k,l})$, $y=(y_{k,l})\in[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ Then there exist $\rho_1>0$, $\rho_2>0$ such that

$$\sup_{k,l}M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)\le 1 \text{ and } \sup_{k,l}M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right)\le 1.$$

Let $\rho=\rho_1+\rho_2$ Then we have,

$$\sup_{k,l}M\left(\frac{q(\Delta^n(x_{k,l}+y_{k,l}))}{\rho}\right)$$

$$\le\left(\frac{\rho_1}{\rho_1+\rho_2}\right)\sup_{k,l}M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)+\left(\frac{\rho_2}{\rho_1+\rho_2}\right)\sup_{k,l}M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right)\le 1.$$

Since $\rho_1,\rho_2>0$ so we have,

$$f((x_{k,l})+(y_{k,l}))=\sum_{k=1}^{n}q(x_{k,1}+y_{k,1})+\sum_{l=1}^{n}q(x_{1,l}+y_{1,l})$$

$$+\inf\left\{\rho>0:\sup_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n y_{k,l})}{\rho}\right)\right]^{p_{k,l}}\le 1\right\}$$

$$\le\sum_{k=1}^{n}q(x_{k,1})+\sum_{l=1}^{n}q(x_{1,l})+\inf\left\{\rho_1>0:\sup_{k,l}(h_{r,s})^{-1}M\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)\le 1\right\}$$

$$+\sum_{k=1}^{n}q(y_{k,1})+\sum_{l=1}^{n}q(y_{1,l})+\inf\left\{\rho_2>0:\sup_{k,l}(h_{r,s})^{-1}M\left(\frac{q(\Delta^n y_{k,l})}{\rho_2}\right)\le 1\right\}$$

$$=f((x_{k,l}))+f((y_{k,l})).$$

Therefore *f* is a seminorm.

Theorem 3.3. Let $(X,\ q)$ be a complete seminormed space. Then the spaces $[\overline{w}^2(M,\Delta^n,p,q)]_\theta$ $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ and $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ are complete seminormed spaces seminormed by *f*.

Proof. We prove the theorem for the space $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$. The other cases can be establish following similar technique. Let $x^i=(x_{k,l}^i)$ be a Cauchy sequence in

$[\bar{w}_0^2(M,\Delta^n,p,q)]_\theta$. Let $\varepsilon > 0$ be given and for $b > 0$, choose x_0 fixed such that $M\left(\frac{bx_0}{2}\right) \geq 1$ and there exists $m_0 \in \mathbb{N}$ such that

$$f(x^i - x^j) = f\left((x_{k,l}^i) - (x_{k,l}^j)\right) < \frac{\varepsilon}{bx_o}, \text{for all } i, j \geq m_o.$$

By definition of seminorm, we have

$$\sum_{k=1}^{n} q\left(x_{k,1}^i - x_{k,1}^j\right) + \sum_{l=1}^{n} q\left(x_{1,l}^i - x_{1,l}^j\right) + \inf\left\{\rho > 0: \sup_{k,l} M\left(\frac{q\left(\Delta^n x_{k,l}^i - \Delta^n x_{k,l}^j\right)}{\rho}\right) \leq 1\right\} < \frac{\varepsilon}{bx_o}.$$
(3.3)

This implies that

$$\sum_{k=1}^{n} q\left(x_{k,1}^i - x_{k,1}^j\right) + \sum_{l=1}^{n} q\left(x_{1,l}^i - x_{1,l}^j\right) < \varepsilon$$

This shows that $(x_{k,1}^i)$ and $(x_{1,l}^j)(k,l \leq n)$ are Cauchy sequences in (X, q) Since (X, q) is complete, so there exists $x_{k,1}, x_{1,l} \in X$ such that

$$\lim_{i\to\infty} x_{k,1}^i = x_{k,1} \text{ and } \lim_{j\to\infty} x_{1,l}^j = x_{1,l} (k,l \leq n)$$

Now from (3.3), we have

$$M\left(\frac{q\left(\Delta^n\left(x_{k,l}^i - x_{k,l}^j\right)\right)}{f\left(\left(x_{k,l}^i\right) - \left(x_{k,l}^j\right)\right)}\right) \leq 1 \leq M\left(\frac{bx_o}{2}\right), \text{for all } i, j \geq m_o .$$
(3.4)

This implies

$$q\left(\Delta^n\left(x_{k,l}^i - x_{k,l}^j\right)\right) \leq \frac{bx_o}{2} \cdot \frac{\varepsilon}{bx_o} = \frac{\varepsilon}{2}, \text{for all } i, j \geq m_o$$

So $\left(\Delta^n(x_{k,l}^i)\right)$ is a Cauchy sequence in (X, q). Since (X, q) is complete, there exists $x_{k,l} \in X$ such that $\lim_i \Delta^n(x_{k,l}^i) = x_{k,l}$ for all $k,l \in \mathbb{N}$ Since M is continuous, so for $i \geq m_0$, on taking limit as $j \to \infty$ we have from (3.4),

$$M\left(\frac{q\left(\Delta^n(x_{k,l}^i) - \lim_{j\to\infty} \Delta^n x_{k,l}^j\right)}{\rho}\right) \leq 1 \to M\left(\frac{q\left(\Delta^n(x_{k,l}^i) - x_{k,l}\right)}{\rho}\right) \leq 1.$$

On taking the infimum of such ρ's, we have

$$f\left(\left(x_{k,l}^i - x_{k,l}\right)\right) < \varepsilon, \text{for all } i \geq m_o .$$

Thus $\left(x_{k,l}^i - x_{k,l}\right) \in [\bar{w}_0^2(M,\Delta^n,p,q)]_\theta$. By linearity of the space $[\bar{w}_0^2(M,\Delta^n,p,q)]_\theta$ we have for all $i \geq m_0$,

$$(x_{k,l}) = (x_{k,l}^i) + \left(x_{k,l}^i - x_{k,l}\right) \in [\bar{w}_0^2(M,\Delta^n,p,q)]_\theta$$

Thus $[\bar{w}_0^2(M,\Delta^n,p,q)]_\theta$ is a complete semi-normed space.

Theorem 3.4. **(a)** *If* $0 < \inf_{k,l} p_{k,l} \leq p_{k,l} < 1$ *then* $[Z^2(M,\Delta^u,p,q)]_\theta \subset [Z^2(M,\Delta^u,q)]_\theta$

(b) *If* $1 < p_{k,l} \leq \sup_{k,l} p_{k,l} < \infty$ *then* $[Z^2(M,\Delta^u,q)]_\theta \subset [Z^2(M,\Delta^u,p,q)]_\theta$ *where* $Z^2 = \bar{w}^2, \bar{w}_0^2$ *and* w_∞^2.

Proof. The first part of the result follows from the inequality

$$M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right) \leq \left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right)\right]^{p_{k,l}}$$

and the second part of the result follows from the inequality

$$\left[M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right)\right]^{p_{k,l}} \leq M\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right).$$

Theorem 3.5. *Let* M_1 *and* M_2 *be Orlicz functions satisfying* Δ_2*-condition. If* $\beta = \lim_{t\to\infty} \frac{(M_2(t))}{t} \geq 1$ *then* $[Z^2(M_1,\Delta^n,p,q)]_\theta = [Z^2(M_1 \circ M_2,\Delta^n,p,q)]_\theta$ *where* $Z^2 = \bar{w}^2, \bar{w}_0^2$ *and* w_∞^2.

Proof. We prove it for $Z^2 = \bar{w}_0^2$ and the other cases will follows on applying similar techniques. Let $x = (x_{k,l}) \in [\bar{w}_0^2(M_1,\Delta^n,p,q)]_\theta$ then

$$st_2 - \lim_{r,s}(h_{r,s})^{-1} \sum_{(k,l)\in I_{r,s}} \left[M_1\left(\frac{q\left(\Delta^{n-1} x_{k,l}\right)}{\rho}\right)\right]^{p_{k,l}} = 0$$

Let $0<\varepsilon<1$ and δ with $0<\delta<1$ such that $M_2(t)<\varepsilon$ for $0\le t<\delta$. Let

$$y_{k,l}=M_1\left(\frac{q(\Delta^n x_{k,1})}{\rho}\right)$$

and consider

$$[M_2(y_{k,l})]^{p_{k,l}}=[M_2(y_{k,l})]^{p_{k,l}}+[M_2(y_{k,l})]^{p_{k,l}} \quad (3.6)$$

where the first term is over $y_{k,l}\le\delta$ and the second is over $y_{k,l}>\delta$ From the first term in (3.6), using the Remark 1.1

$$[M_2(y_{k,l})]^{p_{k,l}}<[M_2(2)]^H+[(y_{k,l})]^{p_{k,l}} \quad (3.7)$$

On the other hand, we use the fact that

$$y_{k,l}<\frac{y_{k,l}}{\delta}<1+\frac{y_{k,l}}{\delta}$$

Since M_2 is non-decreasing and convex, it follows that

$$M_2(y_{k,l})<M_2\left(1+\frac{y_{k,l}}{\delta}\right)<\frac{1}{2}M_2(2)+\frac{1}{2}M_2\left(\frac{2y_{k,l}}{\delta}\right)$$

Since M_2 satisfies Δ_2-condition, we have

$$M_2(y_{k,l})<\frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2)+\frac{1}{2}K\frac{y_{k,l}}{\delta}M_2(2)=K\frac{y_{k,l}}{\delta}M_2(2)$$

Thus from the second term in (3.6) we have

$$[M_2(y_{k,l})]^{p_{k,l}}\le\max\left(1,(KM_2(2)\delta^{-1})^H\right)[(y_{k,l})]^{p_{k,l}} \quad (3.8)$$

By (3.7) and (3.8), taking limit in the Pringsheim sense, we have $x=(x_{k,l})\in[\overline{w}_0^2(M_1\circ M_2,\Delta^n,p,q)]_\theta$. Observe that in this part of the proof we did not need $\beta\ge1$ Now, let $\beta\ge1$ and $x=(x_{k,l})\in[\overline{w}_0^2(M_1,\Delta^n,p,q)]_\theta$. Since $\beta\ge1$ we have $M_2(t)\ge\beta t$ for all $t\ge0$. It follows that $x=(x_{k,l})\in[\overline{w}_0^2(M_1\circ M_2,\Delta^n,p,q)]_\theta$ implies $x=(x_{k,l})\in[\overline{w}_0^2(M_1,\Delta^n,p,q)]_\theta$. This implies $[\overline{w}_0^2(M_1,\Delta^n,p,q)]_\theta=[\overline{w}_0^2(M_1\circ M_2,\Delta^n,p,q)]_\theta$

Theorem 3.6. *Let M, M_1 and M_2 be Orlicz functions, q, q_1 and q_2 be seminorms. Then*

(i) $[Z^2(M_1,\Delta^n,p,q)]_\theta\cap[Z^2(M_2,\Delta^n,p,q)]_\theta\subset[Z^2(M_1+M_2,\Delta^n,p,q)]_\theta$.

(ii) $[Z^2(M,\Delta^n,p,q_1)]_\theta\cap[Z^2(M,\Delta^n,p,q_2)]_\theta\subset[Z^2(M,\Delta^n,p,q_1+q_2)]_\theta$.

(iii) *If q_1 is stronger than q_2, then* $[Z^2(M,\Delta^n,p,q_1)]_\theta\subset[Z^2(M,\Delta^n,p,q_2)]_\theta$, *where $Z^2=\overline{w}^2,\overline{w}_0^2$ and w_∞^2.*

Proof. (i) We establish it for only $Z^2=\overline{w}_0^2$ The rest of the cases are similar. Let $x=(x_{k,l})\in[\overline{w}_0^2(M_1,\Delta^n,p,q)]_\theta\cap[\overline{w}_0^2(M_2,\Delta^n,p,q)]_\theta$ Then

$$st_2-\lim_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M_1\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}}=0 \text{ for some } \rho_1>0,$$

$$st_2\quad\lim_{r,s}(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M_2\left(\frac{q(\Delta^n x_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}=0 \text{ for some } \rho_2>0.$$

Let $\rho=\max\{\rho_1,\rho_2\}$ The result follows from the following inequality

$$\left[(M_1+M_2)\left(\frac{q(\Delta^n x_{k,l})}{\rho}\right)\right]^{p_{k,l}}$$

$$\le D\left\{\left[M_1\left(\frac{q(\Delta^n x_{k,l})}{\rho_1}\right)\right]^{p_{k,l}}+\left[M_2\left(\frac{q(\Delta^n x_{k,l})}{\rho_2}\right)\right]^{p_{k,l}}\right\}$$

The proofs of (ii) and (iii) follow obviously.

The proof of the following result is routine work.

Proposition 3.7. For any Orlicz function M, if $q_1\equiv(\textit{equivalent to})\,q_2$, then $[Z^2(M,\Delta^n,p,q_1)]_\theta=[Z^2(M,\Delta^n,p,q_2)]_\theta$ where $Z^2=\overline{w}^2,\overline{w}_0^2$ and w_∞^2.

Let E^2 be a double sequence space. Then E^2 is called

(a) solid (or normal) if $(\alpha_{k,l}x_{k,l})\in E^2$ whenever $(x_{k,l})\in E^2$ for all double sequences $(\alpha_{k,l})$ of scalars with $|\alpha_{k,l}|\le1$ for all $k,l\in$

(b) monotone provided E^2 contains the canonical preimages of all its step spaces.

It is a well known result that if E^2 is normal then it is monotone.

Theorem 3.8. The spaces $[\overline{w}_0^2(M,\Delta^n,p,q)]_\theta$ and $[w_\infty^2(M,\Delta^n,p,q)]_\theta$ are normal as well as monotone.

Proof. Let $(\alpha_{k,l})$ be a double sequences of scalars such that $|\alpha_{k,l}|\leq 1$ for all $k,l\in$ Since M is monotone, we get for some $\rho>0$

$$(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n(\alpha_{k,l}x_{k,l}))}{\rho}\right)\right]^{p_{k,l}}$$

$$(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\sup|\alpha_{k,l}|\frac{q(\Delta^n(x_{k,l}))}{\rho}\right)\right]^{p_{k,l}}$$

$$(h_{r,s})^{-1}\sum_{(k,l)\in I_{r,s}}\left[M\left(\frac{q(\Delta^n(x_{k,l}))}{\rho}\right)\right]^{p_{k,l}}$$

which leads us to the desired results.

Conclusion

In this article we defined some new sequence spaces by double lacunary summability method by combining the concept of Orlicz function and statistical convergence. Further, we proved some topological and algebraic properties of the resulting spaces.

Acknowledgements

The authors expresses there heartfelt gratitude to the anonymous reviewer for such excellent comments and suggestions which have enormously enhanced the quality and presentation of this paper.

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Received on December 15, 2011.
Accepted on April 11, 2012.