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On generalized difference sequence spaces of fuzzy numbers

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ABSTRACT. The idea of difference sequence space was introduced by Kizmaz (1981) and this concept was generalized by Tripathy and Esi (2006). In this article we introduced the paranormed sequence spaces $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ of fuzzy numbers associated with the multiplier sequence $\Lambda=(\lambda_k)$ defined by a modulus function f. We study some of their properties like solidity, symmetricity, completeness etc. and prove some inclusion results.

Keywords: fuzzy numbers, paranorm, modulus function, difference sequence, multiplier sequence.

Sobre espaços de sequencias diferenciais generalizadas de números difusos

RESUMO. A ideia de espaço sequencial com diferenças foi apresentado por Kizmaz (1981), sendo este conceito generalizado por Tripathy and Esi (2006). No presente estudo, apresentamos espaços sequenciais para-normalizados $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ e $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ de números fuzzy associados com a sequência de multiplicador $\Lambda=(\lambda_k)$ definido por uma função módulo f. Analisamos algumas de suas propriedades como solidez, simetria, completude, etc e demonstramos alguns resultados de inclusão.

Palavras-chave: números fuzzy, paranorma, função módulo, sequência de diferença, sequência de multiplicador.

Introduction

The concept of fuzzy set was introduced by L A. Zadeh in the year 1965. Based on this, sequences of fuzzy numbers have been introduced by different authors and many important properties have been investigated. Applying the notion of fuzzy real numbers, different classes of fuzzy real-valued sequences have been introduced and investigated by Tripathy and Baruah (2009, 2010a and b), Tripathy and Borgohain (2008), Tripathy and Dutta (2010a), Tripathy and Sarma (2011) and many researchers on sequence spaces.

A *fuzzy real number* X is a fuzzy subset of the real line R, *i.e.*, a mapping $X: R \to I = [0,1]$ associating each real number t, with its grade of membership $X(t)$.

The α-level set of a fuzzy real number X is denoted by $[X]_\alpha$, $0 < \alpha \le 1$, where $[X]_\alpha = \{t \in R : X(t) \ge \alpha\}$. The 0-level set is the closure of the strong 0-cut *i.e.* $[X]_0 = cl(\{t \in R : X(t) > 0\})$.

A fuzzy real number X is said to be *upper-semi continuous* if for each $\varepsilon>0$, $X^{-1}([0, a+\varepsilon))$ is open in the usual topology of R.

If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called *normal*.

A fuzzy real number X is said to be *convex*, if $X(t) \ge X(s) \wedge X(r) = \min\{X(s), X(r)\}$, where $s \le t \le r$.

The class of all upper semi-continuous, normal and convex fuzzy real numbers is de-noted by $R(I)$.

The *absolute value* of $|X| \in R(I)$ is defined by (one may refer to Kaleva and Seikkala (1984)).

$$|X|\,(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{if } t \ge 0, \\ 0, & \text{otherwise.} \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

Let D be the set of all closed and bounded intervals. Define $d : R(I)\times R(I) \to R$ by

$$d(X,Y) = \sup_{0\le\alpha\le 1} \{\max\{|a_1^\alpha - b_1^\alpha|, |a_2^\alpha - b_2^\alpha|\}\},$$

where $[X]_\alpha = [a_1^\alpha, a_2^\alpha]$, $[Y]_\alpha = [b_1^\alpha, b_2^\alpha]$.

Definitions and preliminaries

Throughout the paper $p = (p_k)$ is a sequence of positive real numbers. The notion of paranormed sequences was studied at the initial stage by Simons (1965). It was further investigated by Maddox (1967), Tripathy (2004), Tripathy and Chandra (2011), Tripathy and Dutta (2010b), Tripathy and

Hazarika (2009), Tripathy and Sen (2001, 2006) and many others.

The notion of multiplier sequences $\Lambda = (\lambda_k)$ was investigated by Goes and Goes (1970) at the initial stage. It was further investigated by Kamthan (1976), Tripathy and Mahanta (2004), Tripathy and Chandra (2011) and many others.

The notion of modulus function was introduced by Nakano (1953). It was further investigated with applications to sequence spaces by Tripathy and Chandra (2011) and many others.

Definition 2.1. A *modulus function* f is a mapping $f: [0,\infty)\to [0,\infty)$ such that

(*i*) $f(x) = 0$ if and only if $x = 0$;

(ii) $f(x+ y) \le f(x) + f(y)$;

(*iii*) f is increasing;

(*iv*) f is continuous from right at 0.

Hence f is continuous everywhere in $[0, \infty)$ by (*ii*) and (*iv*).

The idea of difference sequences for real numbers was introduced by Kizmaz (1981) and it was further generalized by Tripathy and Esi (2006) as follows:

$$Z(\Delta_m) = \{(x_k)\in w : (\Delta_m x_k) \in Z\}, \text{ for } Z = c, c_0, \ell_\infty,$$

where $\Delta_m x_k = x_k - x_{k+m}$, for all $k\in N$.

Let w^F be the class of all sequences of fuzzy numbers. Throughout the paper w^F, c^F, c_0^F and ℓ_∞^F denote the classes of *all*, *convergent*, *null* and *bounded* sequences of fuzzy real numbers respectively.

Definition 2.2. A sequence (X_n) of fuzzy real numbers is said to converge to the fuzzy real number X_0 if for every $\varepsilon > 0$ there exists a positive integer $n_0\in N$ such that $\bar{d}\,(X_k,X_0) < \varepsilon$ for all $n \ge n_0$.

Definition 2.3. A sequence (X_n) of fuzzy real numbers is said to be *bounded* if $\sup_n \bar{d}\left(X_n,\bar{0}\right)<\infty$

Definition 2.4. A fuzzy real valued sequence space E^F is said to be *normal* (or *solid*) if $(Y_n)\in E^F$, whenever $|Y_n| \le |X_n|$ for all $n\in N$ and $(X_n)\in E^F$.

Definition 2.5. A fuzzy real valued sequence space E^F is said to be monotone if E^F contains the canonical pre-images of all its step spaces.

Remark 2.1. If a class of sequences of fuzzy numbers is solid, then it is monotone.

Definition 2.6. A fuzzy real valued sequence space E^F is said to be *symmetric* if $(X_{\pi(n)}) \in E^F$, whenever $(X_n)\in E^F$ for all $n\in N$, where π is a permutation of N.

Definition 2.7. A fuzzy real valued sequence space E^F is said to be *convergence free* if $(X_n)\in E^F$, whenever $(Y_n)\in E^F$ and $Y_n = \bar{0}$ implies $X_n = \bar{0}$.

For (a_k) and (b_k) two sequences of complex terms and $p = (p_k)\in \ell_\infty$, we have the following known inequality:

$$|a_k + b_k|^{p_k} \le D(|a_k|^{p_k} + |b_k|^{p_k}),$$

where $H = \sup_k p_k$ and $D= \max\{1, 2^{H-1}\}$.

Recently the paranormed sequence spaces $c(f,\Lambda,\Delta_m,p)$, $c_0(f,\Lambda,\Delta_m,p)$ and $\ell_\infty(f,\Lambda,\Delta_m,p)$ are introduced by Tripathy and Chandra (2011). We now give the fuzzy analogue of these classes of sequences as follows.

Definition 2.8. Let f be a modulus function, then for a given multiplier sequence $\Lambda=(\lambda_k)$, we introduce the following fuzzy real valued sequence spaces.

$$c^F(f,\Lambda,\Delta_m,p) = \{(X_k)\in w^F : (f(\bar{d}(\Delta_m(\lambda_k X_k), L)))^{p_k} \to 0, \text{ as } k\to\infty, \text{ for some } L\in R(I)\},$$

$$c_0^F(f,\Lambda,\Delta_m,p) = \{(X_k)\in w^F : (f(\bar{d}(\Delta_m(\lambda_k X_k), \bar{0})))^{p_k} \to 0, \text{ as } k\to\infty\},$$

$$\ell_\infty^F(f,\Lambda,\Delta_m,p) = \{(X_k)\in w^F : \sup_k (f(\bar{d}(\Delta_m(\lambda_k X_k))))^{p_k} < \infty\}.$$

When $f(x) = x$, for all $x\in [0, \infty)$, the above classes of sequences are denoted by $c^F(\Lambda,\Delta_m,p)$, $c_0^F(\Lambda,\Delta_m,p)$ and $\ell_\infty^F(\Lambda,\Delta_m,p)$ respectively. When $\lambda_k = 1$ for all $k\in N$, these classes of sequences are denoted by $c^F(f, \Delta_m, p)$, $c_0^F(f, \Delta_m, p)$, $\ell_\infty^F(f, \Delta_m,p)$ respectively.

When $f(x) = x$, for all $x\in [0, \infty)$, $\lambda_k = 1$ and $p_k = 1$ for all $k\in N$, we represent these classes of sequences by $c^F(\Delta_m)$, $c_0^F(\Delta_m)$, $\ell_\infty^F(\Delta_m)$ respectively. Further taking $m=1$, we get the spaces $c^F(\Delta)$, $c_0^F(\Delta)$, $\ell_\infty^F(\Delta)$ respectively.

Similarly taking different combinations of restrictions, we will get different paranormed sequences of fuzzy numbers from these classes of sequences.

Results

Theorem 3.1. The classes of sequences $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ are closed under addition and multiplication.

Proof: We prove the theorem for the class of sequences $c_0^F(f,\Lambda,\Delta_m,p)$. The other classes can be proved similarly.

Suppose (X_k), $(Y_k) \in c_0^F(f,\Lambda,\Delta_m,p)$. Then we have

$$(f(\bar{d}(\Delta_m(\lambda_k X_k),\bar{0})))^{p_k} \to 0, \text{ as } k\to\infty \quad (1)$$

and

$$(f(\bar{d}(\Delta_m(\lambda_k Y_k),\bar{0})))^{p_k} \to 0, \text{ as } k\to\infty \quad (2)$$

Now for $a, b\in R$, we have

$$(f(\bar{d}(\Delta_m(\lambda_k(aX_k+bY_k)),\bar{0})))^{p_k}$$
$$\leq D[a(f(\bar{d}(\Delta_m(\lambda_k X_k),\bar{0})))^{p_k} + b(f(\bar{d}(\Delta_m(\lambda_k Y_k),\bar{0})))^{p_k}]$$

$\to 0$, as $k\to\infty$ by (1) and (2).

This shows that $(aX_k + bY_k)\in c_0^F(f,\Lambda,\Delta_m,p)$ and hence the class of sequences $c_0^F(f,\Lambda,\Delta_m,p)$ is closed under the addition and multiplication.

Theorem 3.2. Let $p=(p_k)\in \ell_\infty$. Then the classes of sequences $c^F(f, \Lambda, \Delta_m, p)$, $c_0^F(f, \Lambda, \Delta_m, p)$ and $\ell_\infty^F(f, \Lambda, \Delta_m, p)$ are are paranormed spaces, paranormed by g, define by

$$g(X)= \sup_k (f(\bar{d}(\Delta_m(\lambda_k X_k),\bar{0})))^{\frac{p_k}{M}},$$

where

$M=\max(1, \sup_k p_k)$ and $X=(X_k)$.

Proof: Clearly, $g(X) \geq 0$, $g(-X) = g(X)$ and $g(X+Y) \leq g(X)+g(Y)$.

Next we show the continuity of the product. Let α be fixed and $g(X)\to 0$. Then it is obvious that $g(\alpha X)\to 0$. Let $\alpha\to 0$ and X be fixed. Since f is continuous, we have

$$f(|\alpha|\bar{d}(\Delta_m(\lambda_k X_k),\bar{0}))\to 0, \text{ as } \alpha\to 0, \text{ for all } k\in N.$$

Thus we have, $\sup_k (f(\bar{d}(\Delta_m(\lambda_k X_k),\bar{0})))^{\frac{p_k}{M}}\to 0$, as $\alpha\to 0$.

Hence $g(\alpha X)\to 0$ as $\alpha\to 0$.

Therefore g is a paranorm.

Proposition 3.3. *$c_0^F(f,\Lambda,\Delta_m,p) \subset c^F(f,\Lambda,\Delta_m,p) \subset \ell_\infty^F(f,\Lambda,\Delta_m,p)$ and the inclusions are proper.*

Proof: Easy, so omitted.

Theorem 3.4. *The classes of sequences $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f,\Lambda,\Delta_m,p)$* are neither solid nor monotone in general, but the class of sequences, $c_0^F(f,\Lambda,p)$ and $\ell_\infty^F(f,\Lambda,p)$ are solid and as such are monotone.

Proof: Let (X_k) be a given sequence and (α_k) be a sequence of scalars such that $|\alpha_k|\leq 1$, for all $k\in N$. Then we have, $(f(\bar{d}(\alpha_k\lambda_k X_k),\bar{0})))^{p_k} \leq (f(\bar{d}(\lambda_k X_k),\bar{0})))^{p_k}$ for all $k\in N$. The solidness of $c_0^F(f,\Lambda,p)$ *and* $\ell_\infty^F(f,\Lambda,p)$ follows from the above inequality.

The monotonicity of these two classes of sequences follows by Remark 2.1.

The first part of the proof follows from the following examples.

Example 3.1: Let $f(x) = x$, for all $x\in [0,\infty)$; $m=2$, $\lambda_k=2$, for all $k\in N$; $p_k = 1$ for all k odd and $p_k = 2$ for all k even. Consider the sequence (X_k) defined by $X_k = A$ for all $k\in N$, where

$$A(t) = \begin{cases} t+1, & \text{for } -1\leq t\leq 0; \\ 1-t, & \text{for } 0\leq t\leq 1; \\ 0, & \text{otherwise.} \end{cases}$$

Then clearly $(X_k)\in c^F(f,\Lambda,\Delta_2,p)$. For E, a class of sequences, consider its J-step space E_J defined as follows.

When $(X_k)\in E_J$, then its canonical pre-image $(Y_k)\in E_J$ is given by

$$Y_k = \begin{cases} X_k, & \text{for } k \text{ even,} \\ \bar{0}, & \text{for } k \text{ odd.} \end{cases}$$

Then $(Y_k)\notin (c^F(\Delta_2,p))_J$. Thus the class of sequences $c^F(f,\Lambda,p)$ is not monotone. Hence is not solid. Hence the class of sequences $c^F(f,\Lambda,p)$ is not monotone in general by Remark 2.1.

Example 3.2: Let $f(x) = x$, for all $x\in [0,\infty)$; $m=3$, $\lambda_k = 2+k^{-1}$, $p_k = 2$ for all k odd and $p_k=3$ for all k even. Consider the sequence (X_k) defined by $X_k = \bar{k}$ for all $k\in N$. Then clearly $(X_k)\in \ell_\infty^F(f,\Lambda,\Delta_3,p)$.

Now consider the sequence (Y_k) defined by $(Y_k) = (X_1,\bar{0},X_3,\bar{0},X_5,\bar{0})$. Then $(Y_k)\notin \ell_\infty^F(f,\Lambda,\Delta_3,p)$. Hence $\ell_\infty^F(f,\Lambda,\Delta_3,p)$ is not solid. Thus the class of sequences $\ell_\infty^F(f,\Lambda,p)$ is not monotone in general by Remark 2.1.

Similarly one can construct examples to show that the class of sequences $c_0^F(f,\Lambda,\Delta_m,p)$ is neither solid nor monotone in general.

Theorem 3.5. *The classes of sequences* $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ *and* $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ *are not convergence free in general.*

Proof: The result follows from the following example.

Example 3.3. Let $f(x) = x$, for all $x\in [0, \infty)$, $m = 2$, $\lambda_k = 2$, for all $k\in N$, $p_k = 2$ for all k odd and $p_k = 3$ for all k even. Consider the sequence (X_k) defined as in Example 3.1. Clearly $X_k\in c^F(f,\Lambda,\Delta_2,p)$. Consider the sequence (Y_k) defined by $Y_k = A$ for all k odd and $Y_k = A_k$ for all k even, where

$$A_k(t) = \begin{cases} \frac{t}{k}+1, \text{ for } -k \le t \le 0; \\ 1-\frac{t}{k}, \text{ for } 0 \le t \le k; \\ 0, \text{ otherwise.} \end{cases}$$

for all k even.

Then $(Y_k)\notin c^F(f,\Lambda,\Delta_2,p)$. Hence the class of sequences $c^F(f, \Lambda,\Delta_m, p)$ is not convergence free.

Similar examples can be constructed to show that the classes of sequences $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f, \Lambda,\Delta_m, p)$ are not convergence free.

Theorem 3.6. *The classes of sequences* $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ *and* $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ *are not symmetric in general.*

Proof. The result follows from the following example.

Example 3.4. Let $f(x) =x$, for all $x\in [0, \infty)$; $m = 2$, $\lambda_k =3$, $p_k = 2$ for all k odd and $p_k = 3$ for all k even. Consider the sequence $(X_k) = (A, B, A, B,\ldots)$, where the fuzzy number A is defined as in Example 3.1 and the fuzzy number B is defined by

$$B(t) = \begin{cases} \frac{t}{2}+1, \text{ for } -2 \le t \le 0, \\ 1-\frac{t}{2}, \text{ for } 0 \le t \le 2, \\ 0, \text{ otherwise.} \end{cases}$$

Then $(X_k)\in c^F(f,\Lambda,\Delta_2,p)$. Consider its rearrangement (Y_k) of (X_k) defined by $(Y_k) = (A, B, B, A, A, B, B, A, A, \ldots)$. Then $(Y_k)\notin c^F(f,\Lambda,\Delta_2,p)$. Hence the class of sequences $c^F(f,\Lambda,\Delta_m,p)$ is not symmetric.

Similar examples can be constructed to show that the classes of sequences $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ are not symmetric.

Conclusion

In this article we have introduced and studied different properties of the classes of sequences $c^F(f,\Lambda,\Delta_m,p)$, $c_0^F(f,\Lambda,\Delta_m,p)$ and $\ell_\infty^F(f,\Lambda,\Delta_m,p)$ of fuzzy numbers and have investigated their different properties. The idea applied can be used for introducing many other classes of sequences and study their similar properties.

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