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# ε-Open sets

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**ABSTRACT.** In this paper, we introduce the relatively new notion of $\zeta$ −open subset which is strictly weaker than open. We prove that the collection of all $\zeta$ −open subsets of a space forms a topology that is finer than the original one. Several characterizations and properties of this class are also given as well as connections to other well-known "generalized open" subsets.

**Keywords:** $\zeta$ −open, Countable set, Anti locally countable space.

## Conjuntos aberto $\zeta$

**RESUMO.** Este artigo apresenta um novo conceito de subconjunto aberto $\zeta$ que é muito mais fraco que o aberto. Foi comprovada que a coleta de todos os subconjuntos abertos $\zeta$ de um espaço forma uma topologia mais fina do que a original. Várias caracterizações e propriedades desta classe são também apresentadas, bem como conexões com outros bem conhecidos subconjuntos abertos generalizados.

**Palavras-chave:** aberto $\zeta$ , conjunto contável, espaço contável anti-localmente.

## Introduction

Let $(X,\Im)$ be a topological space (or simply, a space). If $A \subseteq X$ , then the closure of $A$, the interior of $A$ and the derived set of $A$ will be denoted by $Cl_\Im(A)$, $Int_\Im(A)$ and $d_\Im(A)$ , respectively. If no ambiguity appears, we use $\overline{A}, A^o$ and $A^{'}$ instead. A subset $A \subseteq X$ is called *semi-open* (simply, $SO$) (LEVINE, 1963) if there exists an open set $O \in \Im$ such that $O \subseteq A \subseteq \overline{O}$. Clearly $A$ is a semi-open set if and only if $A \subseteq Cl_\Im(Int_\Im(A))$ . A complement of a semi-open set is called *semi-closed* (simply, $SC$). The semi-interior of $A$ is the union of all semi-open subsets contained in $A$ and is denoted by $sInt(A)$. $A$ is called *preopen*( simply, $PO$) (MASHHOUR et al., 1982a and b) $A \subseteq Int_\Im(Cl_\Im(A))$. $A$ is called $\alpha-open$ (NJASTAD, 1965) if $A \subseteq Int_\Im(Cl_\Im(Int_\Im(A)))$ and $\beta-open$ (ABD EL-MONSEF et al., 1983) if $A \subseteq Cl_\Im(Int_\Im(Cl_\Im(A)))$. Finally, $A$ is called *regular-open* (simply, $RO$) (WL-DEEP et al., 1983) if $A = Int_\Im(Cl_\Im(A))$. Complements of regular-open sets are called *regular-closed* (simply, $RC$). The collection of all SO (resp., PO, RO, RC, $\alpha-$open and $\beta-$open) subsets of $X$ is denoted $SO(X,\Im)$ (resp., $PO(X,\Im)$, $RO(X,\Im)$, $RC(X,\Im)$, $\alpha(X,\Im)$ and $\beta(X,\Im)$. We remark that $\alpha(X,\Im)$ is a topological space and $\alpha(X,\Im) = SO(X,\Im) \cap PO(X,\Im)$. A space $(X,\Im)$ is called *locally countable* (*P-space, anti locally countable*, respectively) if each $x \in X$ has a countable neighborhood (countable intersections of open subsets are open, non-empty open subsets are uncountable, respectively). For the preceding notions, the reader is referred to Crossley and Hildebrand (1971), Ganster and Reilly (1990), Mashhour et al. (1982), Munkres (1975) and Tong (1986).

In this paper, we introduce the relatively new notions of $\zeta$ -open, which is weaker than the class of open subset. In section 2, we also show that the collection of all $\zeta$ -open subsets of a space $(X,\Im)$ forms a topology that is finer than $\Im$ and we investigate the connection of $\zeta$ -open notion to other classes of "generalized open" subsets as well as several characterizations of $\zeta$ -open and $\zeta$ -closed notions via the operations of interior and closure. In section 3, several interesting properties and constructions of $\zeta$ -open subsets are discussed in the case of anti locally countable spaces.

**$\zeta$-open set**

We begin this section by introducing the notion of $\zeta$-open and $\zeta$-closed subsets.

**Definition 1** *A* subset A of a space $(X,\Im)$ is called $\zeta$-open if for every $x\in A$, there exists an open subset $U\subseteq X$ containing x and such that $U\setminus sInt(A)$ is countable. The complement of an $\zeta$-open subset is called $\zeta$-closed.

Clearly every open set is $\zeta$-open, but the converse needs not be true.

**Example 1** Let $X=\{a,b\}$ and $\Im=\{\phi,X,\{a\}\}$. Set $A=\{b\}$. Then A is $\zeta$-open but not open.

Another interesting example for the infinite case is giving next.

**Example 2** Consider the real line $\Re$ with the topology $\Im=\{U\subseteq\Re:\Re\setminus U \text{ is finite or } 0\in\Re\setminus U\}$ and set $A=\Re\setminus Q\cup\{0\}$. Then A is not open while it is $\zeta$-open.

Next, we show that the collection of all $\zeta$-open subsets of a space $(X,\Im)$ forms a topology $\Im_\zeta$ that contains $\Im$.

**Theorem 1** *If* $(X,\Im)$ is a space, then $(X,\Im_\zeta)$ *is* a space such that $\Im\subseteq\Im_\zeta$.

**Proof.** We only need to show $(X,\Im_\zeta)$ is a space. Clearly since $\phi$ *and* $X$ are open, they are $\zeta$-open. If $A,B\in\Im_\zeta$ and $x\in A\cap B$, then there exist open sets U,V in X both containing x such that $U\setminus sInt(A)$ and $V\setminus sInt(B)$ are countable. Now $x\in U\cap V$ and for every

$$y\in(U\cap V)\setminus sInt(A\cap B)=(U\cap V)\setminus(sInt(A)\cap sInt(B))$$

either $y\in U\setminus sInt(A)$ or $y\in V\setminus sInt(B)$. Thus $(U\cap V)\setminus sInt(A\cap B)\subseteq U\setminus sInt(A)$ or $(U\cap V)\setminus sInt(A\cap B)\subseteq U\setminus sInt(B)$ and thus $(U\cap V)\setminus sInt(A\cap B)$ is countable. Therefore, $A\cap B\in\Im_\zeta$.

If $\{A_\alpha:\alpha\in\Delta\}$ is a collection of $\zeta$-open subsets of X, then for every $x\in\bigcup_{\alpha\in\Delta}A_\alpha$, $x\in A_\beta$ for some $\beta\in\Delta$. Hence there exists an open subset U of X containing x such that $U\setminus sInt(A)$ is countable. Now as $U\setminus sInt(\bigcup_{\alpha\in\Delta}A_\alpha)\subseteq U\setminus\bigcup_{\alpha\in\Delta}sInt(A)\subseteq U\setminus sInt(A)$, $U\setminus sInt(\bigcup_{\alpha\in\Delta}A_\alpha)$ is countable and hence $\bigcup_{\alpha\in\Delta}A_\alpha\in\Im_\zeta$.

**Corollary 1** *If* $(X,\Im)$ is a p-space, then $\Im=\Im_\zeta$.

Next we show that $\zeta$-open notion is independent of both PO and SO notions.

**Example 3** Consider $\Re$ with the standard topology. Then $Q$ is PO but not $\zeta$-open. Also $[0,1]$ is SO but not $\zeta$-open.

**Example 4** In Example 1, {b} is $\zeta$-open but neither PO nor open.

Next we characterize $\Im_\zeta$ when $X$ is a locally countable space.

**Theorem 2** If $(X,\Im)$ is a locally countable space, then $\Im_\zeta$ is the discrete topology.

**Proof.** Let $A\subseteq X$ and $x\in A$. Then there exists a countable neighborhood U of x and hence there exists an open set V containing x such that $V\subseteq U$. Clearly $V\setminus sInt(A)\subseteq U\setminus sInt(A)\subseteq U$ and thus $V\setminus sInt(A)$ is countable. Therefore A is $\zeta$-open and so $\Im_\zeta$ is the discrete topology.

**Corollary 2** If $(X,\Im)$ is a countable space, then $\Im_\zeta$ is the discrete topology.

The following result, in which a new characterization of $\zeta$-open subsets is given, will be a basic tool throughout the rest of the paper.

**Lemma 1** A subset A of a space X is $\zeta$-open if and only if for every $x\in A$, there exists an open subset U containing x and a countable subset C such that $U-C\subseteq sInt(A)$.

**Proof.** Let $A\in\Im_\zeta$ and $x\in A$, then there exists an open subset U containing x such that $U\setminus sInt(A)$) is countable. Let $C=U\setminus sInt(A)=U\cap(X\setminus sInt(A))$. Then $U-C\subseteq sInt(A)$.

Conversely, let $x\in A$. Then there exists an open subset U containing x and a countable subset C such that $U-C\subseteq sInt(A)$. Thus $U\setminus sInt(A)=C$ is countable.

The next result follows easily from the definition and the fact that the intersection of $\zeta$-closed sets is again $\zeta$-closed.

**Lemma 2** A subset A of a space X is $\zeta$-closed if and only if $Cl_\zeta(A)=A$.

We next study restriction and deletion operations.

**Theorem 3** If A is $\zeta$-open subset of X, then $\Im_\zeta|_A\subseteq(\Im|_A)_\zeta$.

**Proof.** Let $G\in \Im_{\zeta}|_A$. Then $G=H\cap A$ for some $\zeta$-open subset H. For every $x\in G$, there exist $V_H$, $V_A\in \Im$ containing x and countable sets $D_H$ and $D_A$ such that $V_H-D_H\subseteq sInt(H)$ and $V_A-D_A\subseteq sInt(A)$. Therefore, $x\in A\cap(V_H\cap V_A)\in \Im_A$, $D_H\cup D_A$ is countable and

$$\begin{aligned}A\cap(V_H\cap V_A)-(D_H\cup D_A)&\subseteq (V_H\cap V_A)\cap(X-D_H)\cap(X-D_A)\\&=(V_H-D_H)\cap(V_A-D_A)\\&\subseteq sInt(H)\cap sInt(A)\cap A\\&=sInt(H\cap A)\cap A\\&=sInt(G)\cap A\\&\subseteq sInt_A(G).\end{aligned}$$

Therefore, $G\in(\Im|_A)_{\zeta}$.

**Corollary 3** If A is open subset of X, then $\Im_{\zeta}|_A\subseteq(\Im|_A)_{\zeta}$.

In the next example, we show that if $A$ in the preceding Theorem is not $\zeta$-open, then the result needs not be true.

**Example 5** Consider $\Re$ with the standard topology and let $A=\Re\setminus Q$. Then A is not $\zeta$-open and so not open. As (0,1) is $\zeta$-open, then $D=(0,1)\cap A\in \Im_{\zeta}|_A$ while if $D\in(\Im|_A)_{\zeta}$ then for every $x\in D$, here exists $U\in\Im|_A$ and a countable $C\subseteq A$ such that $U-C\subseteq sInt(D)=\phi$. Thus $U\subseteq C$ and hence U is countable which is a contradiction.

In the next example, we show that $(\Im|_A)_{\zeta}$ needs not be a subset of $\Im_{\zeta}|_A$.

**Example 6** Consider $\Re$ with the standard topology, $A=Q$ and B=(0,2). If $B\in\Im_{\zeta}|_A$, then $B=D\cap A$ for some $D\in\Im_{\zeta}$ which is impossible as $\sqrt{2}\in B-A$. On the other hand to show $B\in(\Im|_A)_{\zeta}$, let $x\in B$. If $x\in A$, pick $q_1,q_2\in A$ such that $0<q_1<x<q_2<2$ and let $U=(q_1,q_2)\cap A$. Then $x\in U-\phi\subseteq B=sInt(B)$. If $x\notin A$, then $q_1,q_2\in\Re\setminus A$ such that $0<q_1<x<q_2<2$ and let $U=(q_1,q_2)\cap A$. Then $x\in U-\phi\subseteq B=sInt(B)$. Thus in both cases $B\in(\Im|_A)_{\zeta}$.

**Lemma 3** If X is a Lindelof space, then A-sInt(A) is countable for every closed subset $A\in\Im_{\zeta}$.

**Poof.** Let A be a closed set such that $A\in\Im_{\zeta}$. For every $x\in A$, there exists an open set $V_x$ containing x such that $V_x-sInt(A)$ is countable. Thus $\{V_x:x\in A\}$ is an open cover for A and as A is lindelof, it has a countable subcover $\{V_n:n\in N\}$. Now $A-sInt(A)=\bigcup_{n\in N}(V_n-sInt(A))$ which is countable.

**Corallary 4** If X is a second countable space, then A-sInt(A) is countable for every closed subset $A\in\Im_{\zeta}$.

**Theorem 4** Let $(X,\Im)$ be a space and $C\subset X$ is $\zeta$-closed. Then $Cl_{\Im}(C)\subseteq K\cup B$ for some closed subset K and a countable subset B.

**Proof.** Let C be $\zeta$-closed. Then X-C is $\zeta$-open and hence for every $x\in X-C$, there exists an open set U containing x and a countable set B such that $U-B\subseteq sInt(X-C)\subseteq X-Cl_{\Im}(C)$. Thus

$$\begin{aligned}&Cl_{\Im}(C)\subseteq X-(U-B)\subseteq X-(U\cap(X-B))\\&\subseteq X\cap((X-U)\cup B)\subseteq(X-U)\cup B.\end{aligned}$$

Letting K=X-U. Then K is closed such $Cl_{\Im}(C)\subseteq K\cup B$.

## Anti-locally countable spaces

In this section, several interesting properties and constructions of $\zeta$-open subsets are discussed in case of anti locally countable spaces.

**Theorem 5** A space $(X,\Im)$ is anti locally countable if and only if $(X,\Im_{\zeta})$ is anti locally countable.

**Proof.** Let $A\in\Im_{\zeta}$ and $x\in A$. Then by Lemma 1, there exists an open subset $U\subseteq X$ containing x and a countable C such that $U-C\subseteq sInt(A)$. Hence sInt(A) is uncountable and so is A. The converse follows from the fact that every open subset is $\zeta$-open.

**Corollary 5** *If* $(X,\Im)$ is anti locally countable space and A is $\zeta$-open, then $Cl_{\Im}(A)=Cl_{\Im_{\zeta}}(A)$.

**Proof.** Clearly $Cl_{\Im}(A)\supseteq Cl_{\Im_{\zeta}}(A)$. On the other hand, let $x\in Cl_{\Im}(A)$ and B be an $\zeta$-open subset containing x. Then by Lemma 1, there exists an open subset V containing x and a countable set C such that $V-C\subseteq sInt(B)$. Thus $(V-C)\cap A\subseteq sInt(B)\cap A$ and $(V\cap A)-C\subseteq sInt(B)\cap A$. As $x\in V$ and $x\in Cl_{\Im}(A)$, $V\cap A=\phi$ and then as V and A are $\zeta$-open, $V\cap A$ is $\zeta$ -open and as X is ani locally countable, $V\cap A$ is uncountable and so is

$V \cap A - C$. Thus $B \cap A$ is uncountable as it contains the uncountable set $sInt(B) \cap A$. Therefore, $B \cap A \neq \phi$ which means $x \in Cl_{\Im_\zeta}(A)$.

By a similar argument, we can easily prove the following result:

**Corollary 6** *I* $(X, \Im)$ is anti locally countable and A is $\zeta$ -closed, $Int_\Im(A) = Int_{\Im_\zeta}(A)$.

**Theorem 6** *Let* $(X, \Im)$ be an anti locally countable space. Then $\alpha(X, \Im) \subseteq \alpha(X, \Im_\zeta)$.

**Proof.** If $A \in \alpha(X, \Im)$, then $A \subseteq Int_\Im(Cl_\Im(Int_\Im(A)))$ and by Corollary 5, $A \subseteq Int_\Im(Cl_{\Im_\zeta}(Int_\Im(A)))$. Now by Corollary 6 and as $Cl_\Im(Int_\Im(A))$ is $\zeta$ -closed, $A \subseteq Int_{\Im_\zeta}(Cl_{\Im_\zeta}(Int_{\Im_\zeta}(A)))$ which means $A \in \alpha(X, \Im_\zeta)$.

The converse of the preceding result needs not be true as shown next.

**Example 7** Consider $\Re$ with the standard topology and let $A = \Re \setminus Q$. Then $A \in \alpha(X, \Im_\zeta)$, but $A \notin \alpha(X, \Im)$.

Similarly, one can show that in an anti locally countable space, $\beta(X, \Im_\zeta) \subseteq \beta(X, \Im)$.

**Theorem 7** *Let* $(X, \Im)$ be an anti locally countable space. Then $d_\Im(A) = d_{\Im_\zeta}(A)$ for every subset $A \subseteq X$.

**Proof.** If $x \in d_\Im(A)$ and V is any $\zeta$-open subset containing x, then there exists an open subset U containing x and a countable C such that $U - C \subseteq sInt(V) \subseteq V$. Thus $(U - C) \cap (A - \{x\}) \subseteq sInt(V) \cap (A - \{x\}) \subseteq V \cap (A - \{x\})$ and as $x \in d_\Im(A)$ and $V^o$ is open containing x, we have $V^o \cap (A - \{x\}) \neq \phi$ and so $V \cap (A - \{x\}) \neq \phi$. Therefore $x \in d_{\Im_\zeta}(A)$.

The converse is obvious as every open subset is $\zeta$-open.

**Theorem 8** Let $(X, \Im)$ be an anti locally countable space. Then $RO(X, \Im_\zeta) = RO(X, \Im)$.

**Proof**. If $A \in RO(X, \Im)$, then $A = Int_\Im(Cl_\Im(A))$ and by Corollary 5, $A = Int_\Im(Cl_{\Im_\zeta}(A))$. Now by Corollary 6 and as $Cl_{\Im_\zeta}(A)$ is $\zeta$-closed, $A = Int_{\Im_\zeta}(Cl_{\Im_\zeta}(A))$ which means $A \in RO(X, \Im_\zeta)$. Conversely, if $A \in RO(X, \Im_\zeta)$, then $A = Int_{\Im_\zeta}(Cl_{\Im_\zeta}(A))$. Then as A is $\zeta$-open, *by* Corollary 5, $A = Int_{\Im_\zeta}(Cl_\Im(A))$ and as $Cl_\Im(A)$ is $\zeta$ -closed being a closed set, then $A = Int_\Im(Cl_\Im(A))$ which means $A \in RO(X, \Im)$.

The converse of the preceding result needs not be true as shown next.

**Example 8** *Let* $X = \{a,b,c,d,e\}$ and $\Im = \{\phi, X, \{a\}, \{a,b\}, \{a,b,c\}, \{a,b,c,d\}\}$. Then $(X, \Im)$ is not an anti locally countable space such that $RO(X, \Im) = \{\phi, X\}$ while $RO(X, \Im_\zeta) = \Im$.

We end this section by showing that $(X, \Im_\zeta)$ is Urysohn when $(X, \Im)$ is an anti locally countable space.

**Definition 2** [8] A space $(X, \Im)$ is Urysohn if for every two distinct points x and y in X, there exists two open subsets U and V such that $x \in U$, $y \in V$ and $Cl_\Im(U) \cap Cl_\Im(V) = \phi$.

**Corollary 7** Let $(X, \Im)$ be an anti locally countable space that is Urysohn. Then $(X, \Im_\zeta)$ is Urysohn.

**Proof.** If $x \neq y$ in X, then there exists $U, V \in \Im$ such that $x \in U$, $y \in V$ and $Cl_\Im(U) \cap Cl_\Im(V) = \phi$. By Corollary *5,* $Cl_{\Im_\zeta}(U) \cap Cl_{\Im_\zeta}(V) = Cl_\Im(U) \cap Cl_\Im(V) = \phi$.

## Conclusion

The relatively new notions of $\zeta$-open, which is weaker than the class of open subset was introduced. It is showon that the collection of all $\zeta$-open subsets of a space $(X, \Im)$ forms a topology that is finer than $\Im$ and connections of $\zeta$-open notion to other classes of "generalized open" subsets as well as several characterizations of $\zeta$-open and $\zeta$-closed notions via the operations of interior and closure were investigated. Several interesting properties and constructions of $\zeta$-open subsets were discussed in the case of anti locally countable spaces.

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