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p-Bounded variation fuzzy real-valued double sequence space

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ABSTRACT. In this paper we have introduced the notion of fuzzy real-valued p-bounded variation double sequence space ${}_2bv_F^p$ $(1 \le p < \infty)$. We have studied some important properties like completeness, convergence free, solid, symmetric. Some inclusion results are also established.

Keywords: bounded variation, solid, symmetric, convergence free.

Variação de fuzzy limitada a *p* de valor real de espaços de sequência duplo

RESUMO. No presente trabalho apresentamos o conceito de variação de fuzzy limitada a p de valor real de espaços de sequência duplo ${}_2bv_F^p$ $(1 \le p < \infty)$. Foram estudadas algumas importantes propriedades como completude, convergência livre, solidez e simetria. Alguns resultados de inclusão também são estabelecidos.

Palavras-chave: variação limitada, solidez, simetria, convergência livre.

Introduction

Fuzzy set is a mathematical model of vague qualitative or quantitative data, frequently generated by the means of natural language. It is based on the generalization of the classical concepts of set and its characteristic function. The theory of fuzzy set was given by Zadeh (1965), since then many major theoretical breakthroughs have been established and successfully applied to many industrial applications. Numerous research workers are involving to develop and extend it in different directions. The introduction of fuzzy number opened many new dimensions in the field of Mathematics. With the idea of fuzzy set theory and fuzzy real numbers, existing notions in different branches of mathematics are generalized. In the field of pure mathematics the use of fuzzy set and fuzzy real numbers are very remarkable. Specifically, we may mention that these notions are used extensively in studying properties of sequence spaces. Since our work is based on fuzzy real numbers, we begin with some basic ideas on it for easy understanding of the work.

A fuzzy real number X is a fuzzy set on R, more precisely a mapping X: $R \to I$ (= [0, 1]), which associate each real number t, with its grade of membership $X(t)$.

The α-level set of a fuzzy real number X is defined by

$$[X]^{\alpha} = \begin{cases} \{t \in R : X(t) \ge \alpha\}, & \text{for } 0 < \alpha \le 1 \\ \overline{\{t \in R : X(t) > \alpha\}}, & \text{for } \alpha = 0. \end{cases}$$

A fuzzy real number X is said to be upper-semi-continuous if for each $\varepsilon > 0$, $X^{-1}([0, a+\varepsilon))$, for all $a \in I$ is open in the usual topology of R.

If there exists $t \in R$ such that $X(t) = 1$, then the fuzzy real number X is called normal.

A fuzzy real number X is said to be convex, if $X(t) \ge X(s) \wedge X(r) = \min(X(s), X(r))$, where $s < t < r$.

We denote the class of all upper-semi-continuous, normal and convex fuzzy real numbers by $R(I)$ and that of all positive fuzzy real numbers by $R^*(I)$.

For $X, Y \in R(I)$, $X \le Y$ if and only if $X^{\alpha} \le Y^{\alpha}$ for $\alpha \in [0,1]$ and "$\le$" is a partial order in $R(I)$.

The absolute value of $X \in R(I)$ is defined by

$$|X|(t) = \begin{cases} \max\{X(t), X(-t)\}, & \text{for } t \ge 0 \\ \bar{0}, & \text{otherwise.} \end{cases}$$

The set of real numbers R can be embedded into $R(I)$, for $r \in R$, $\bar{r} \in R(I)$ is defined by

$$\bar{r}(t) = \begin{cases} 1, & \text{for } t = r, \\ 0, & \text{for } t \ne r \end{cases}$$

The additive identity and multiplicative identity of $R(I)$ are denoted by $\bar{0}$ and $\bar{1}$ respectively.

For any X, Y, $Z \in R(I)$, the linear structure of $R(I)$ induces addition $X + Y$ and scalar multiplication λX, $\lambda \in R$ in terms of α-level set, defined as $[X + Y]^{\alpha} = [X]^{\alpha} + [Y]^{\alpha}$ and $[\lambda X]^{\alpha} = \lambda[X]^{\alpha}$, for each $\alpha \in [0, 1]$. A subset E of $R(I)$ is said to be bounded above if there exist a fuzzy real number μ such that $X \leq \mu$ for every $X \in E$. We called μ as the upper bound of E and it is called least upper bound if $\mu \leq \mu^{*}$ for all upper bound μ^{*} of E. A lower bound and greatest is defined similarly. The set E is said to be bounded if it is both bounded above and bounded below.

Let D be the set of all closed bounded intervals $X = [X^{L}, X^{R}]$, $Y = [Y^{L}, Y^{R}]$. Then $X \leq Y$ implies that $X^{L} \leq Y^{L}$ and $X^{R} \leq Y^{R}$. We write

$$d(X, Y) = \max\{ |X^{L}-Y^{L}|, |X^{R}-Y^{R}| \}.$$

It is straight forward that (D, d) is a complete metric space.

We consider the function $\bar{d}: R(I) \times R(I) \to R$ defined by

$$\bar{d}(X, Y) = \sup_{0 \leq \alpha \leq 1} d(X^{\alpha}, Y^{\alpha}), \text{ for } X, Y \in R(I).$$

It is well established that $\left(R(I), \bar{d}\right)$ is a complete metric space.

Preliminaries and background

In this section we discuss some fundamental concepts and properties related to the subject matter of the article.

A sequence $X = (X_k)$ of fuzzy real number is a function X from the set of positive integer into $R(I)$. The fuzzy number X_k is called the k^{th} term of the sequence.

The set E^{F} of sequences taken from $R(I)$ is said to be a sequence space of fuzzy real number if, for $(X_k), (Y_k) \in E^{F}$, $r \in R$ *i.e.* $X_k, Y_k \in R(I)$, and for all $k \in N$,

$$(X_k) + (Y_k) = (X_k + Y_k) \in E^{F} \text{ and } r(X_k) = (rX_k) \in E^{F}$$

where:

$$rX_k(t) = \begin{cases} X_k(r^{-1}t), \text{if } r \neq 0 \\ \bar{0}, \quad \text{if } r = 0. \end{cases}$$

Works on double sequence started in the early nineties. Initially it was studied by Hardy (1917), Bromwich (1965) and some others. In recent years the theory was further developed by Moričz (1991), Basarir and Solancan (1999), Savas (2007), Savas (1996), Savas and Mursaleen (2004), Tripathy and Dutta (2007, 2008, 2010), Tripathy and Sarma (2008a and b, 2009, 2011), and some others.

The notion of difference sequence in complex terms was introduced by Kizmaz (1981) and defined by

$Z(\Delta) = \{(x_k): (\Delta x_k) \in Z\}$, for $Z = \ell_{\infty}, c, c_0$ where $\Delta x_k = x_k - x_{k+1}$, for all $k \in N$.

Tripathy and Sarma (2008a) studied it for double sequence spaces. In terms of fuzzy real numbers it was studied by Basarir and Mursaleen (2003), Tripathy and Dutta (2008), Tripathy and Borgohain (2010, 2011), and others.

Hardy (1917) introduced the notions of regular convergence of double sequences and the notion of bounded variation of double sequences as follows:

Definition 2.1. A double sequence $< a_{nk} >$ is said to converge regularly if it converges in Pringsheim's sense and in addition the following limits holds:

(i) $\lim_{k \to \infty} a_{nk} = L_n$ $(n \in N)$ exist,

(ii) $\lim_{n \to \infty} a_{nk} = J_k$ $(k \in N)$ exist.

Definition 2.2. The sequence $<x_{nk}>$ is of bounded variation in (n, k), if

(i) x_{nk} is for every fixed value of n and k, of bounded variation in n or k.

(ii) the series $\sum_{n}\sum_{k} |\Delta x_{nk}|$ is convergent.

A fuzzy real-valued double sequence is a double infinite array of fuzzy real numbers. We denote a fuzzy real-valued double sequence by $<X_{nk}>$, where X_{nk} are fuzzy real numbers for each $n, k \in N$.

Definition 2.3. A fuzzy real-valued double sequence $<X_{nk}>$ is said to be convergent in Pringsheim's sense to the fuzzy real number X, if for every $\varepsilon > 0$, there exists $n_1 = n_1(\varepsilon)$, $k_1 = k_1(\varepsilon)$, such that $\bar{d}(X_{nk,} X) < \varepsilon$, for all $n \geq n_1$ and $k \geq k_1$.

Definition 2.4. A fuzzy real-valued double sequence $<X_{nk}>$ is said to be bounded if $\sup_{n,k} \bar{d}(X_{nk}, \bar{0}) < \infty$, equivalently, if there exists $\mu \in R^{*}(I)$, such that $|X_{nk}| \leq \mu$ for all $n, k \in N$.

Definition 2.5. A fuzzy real-valued double sequence $<X_{nk}>$ is said to be regularly convergent if

it is convergent in Pringsheim's sense and the followings hold:

For a given $\varepsilon>0$, there exists $n_0 = n_0(\varepsilon, k)$ and $k_0 = k_0(\varepsilon, n)$ such that

$\bar{d}(X_{nk}, L_k)< \varepsilon$, for all $n \geq n_0$, for some $L_k \in R(I)$ for each $k \in N$, and

$\bar{d}(X_{nk}, M_n)< \varepsilon$, for all $k \geq k_0$, for some $M_n \in R(I)$ for each $n \in N$.

Definition 2.6. A fuzzy real-valued double sequence space E^F is said to be normal (or solid) if $<Y_{nk}> \in E^F$, whenever $|Y_{nk}| \leq |X_{nk}|$ for all $n, k \in N$ and $<X_{nk}> \in E^F$.

Definition 2.7. A fuzzy real-valued double sequence space E^F is said to be symmetric if $<X_{\pi(n,k)}> \in E^F$, whenever $<X_{nk}> \in E^F$, where π is a permutation of N.

Definition 2.8. A fuzzy real-valued double sequence space E^F is said to be convergence free if $<X_{nk}> \in E^F$ whenever $<Y_{nk}> \in E^F$ and $Y_{nk} = \bar{0}$ implies $X_{nk} = \bar{0}$.

The notion of double difference sequences of fuzzy real numbers was introduced by Tripathy and Dutta (2008) as follows:

$Z(\Delta) = \{<X_{nk}> : <\Delta X_{nk}> \in Z\}$, for $Z = ({}_2\ell_\infty)_F$, $({}_2c)_F$, $({}_2c_0)_F$.

where $\Delta X_{nk} = X_{nk} - X_{n+1,k} - X_{n,k+1} + X_{n+1,k+1}$, for all $n, k \in N$.

The class of fuzzy real-valued bounded variation double sequences ${}_2bv_F$ was introduced by Tripathy and Dutta (2010) as follows:

$${}_2bv_F = \left\{(X_{nk}) : \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} \bar{d}(\Delta X_{nk}, \bar{0}) < \infty\right\}, \quad \text{where}$$

$\Delta X_{nk} = X_{nk} - X_{n+1,k} - X_{n,k+1} + X_{n+1,k+1}$, for all $n, k \in N$.

The class of sequence $bv_p(F)$ was introduced and studied by Talo and Basar (2008). In this article we introduce the class of p-bounded variation fuzzy real-valued double sequence ${}_2bv_F^p$ as follows:

$${}_2bv_F^p = \left\{(X_{nk}) : \sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\bar{d}(\Delta X_{nk}, \bar{0})\right)^p < \infty\right\}, (1 \leq p < \infty).$$

Where $\Delta X_{nk} = X_{nk} - X_{n+1,k} - X_{n,k+1} + X_{n+1,k+1}$, for all $n, k \in N$.

We use the following inequality throughout the article, wherever it is applicable.

Let $p = (p_k)$ be a positive sequence of real numbers with $0<p_k<\sup p_k = H$ and $D = \max(1, 2^{H-1})$. Then for all $a_k, b_k \in C$;

$|a_k + b_k|^{p_k} \leq D\left(|a_k|^{p_k} + |b_k|^{p_k}\right)$, for all $k \in N$.

Results and discussion

Theorem 3.1. The class of double sequence ${}_2bv_F^p$ $(1 \leq p < \infty)$ is a complete metric space with respect to the metric ρ defined by

$$\rho(X, Y) = \sup_n \bar{d}(X_{n1}, Y_{n1}) + \sup_k \bar{d}(X_{1k}, Y_{1k}) + \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\bar{d}(\Delta X_{nk}, \Delta Y_{nk})\right)^p\right\}^{\frac{1}{p}},$$

where $X = (X_{nk})$, $Y = (Y_{nk})$ are in ${}_2bv_F^p$. - - - - (1)

Proof: It is easily to verify that ρ, defined by (1) is a metric on ${}_2bv_F^p$. Let $<X^i>$ be a Cauchy sequence in ${}_2bv_F^p$. Then for a given $\varepsilon > 0$, $\exists\, n_0 \in I^+$ such that

$$\rho(X^i, X^j) = \sup_n \bar{d}(X_{n1}^i, X_{n1}^j) + \sup_k \bar{d}(X_{1k}^i, X_{1k}^j) + \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\bar{d}(\Delta X_{nk}^i, \Delta Y_{nk}^j)\right)^p\right\}^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0 \text{ - - - - (2)}$$

This imply that

$$\sup_n \bar{d}(X_{n1}^i, X_{n1}^j) < \varepsilon \text{ and } \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\bar{d}(\Delta X_{nk}^i, \Delta Y_{nk}^j)\right)^p\right\}^{\frac{1}{p}} < \varepsilon, \text{ for all } i, j \geq n_0$$

First we suppose that $\sup_n \bar{d}(X_{n1}^i, X_{n1}^j) < \varepsilon$, for all $i, j \geq n_0$

$\Rightarrow \bar{d}(X_{n1}^i, X_{n1}^j) < \varepsilon$, for all $i, j \geq n_0$

$\Rightarrow (X_{n1}^i)$ is a Cauchy sequence in $R(I)$, for all $n \in N$.

Since $R(I)$ is a complete metric space by the metric $\bar{d}$, so $(X_{n1}^i)_{i=1}^{\infty}$ converges for each $n \in N$. Let $\lim_{i\to\infty} X_{n1}^i = X_{n1}$, for each $n \in N$.

Similarly from (2) we have (X_{1k}^i) is a Cauchy sequence and hence convergent, so $\lim_{j\to\infty} X_{1k}^i = X_{1k}$ (say) for each $k\in N$.

Now consider that $\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\bar{d}(\Delta X_{nk}^i, \Delta Y_{nk}^j)\right)^p\right\}^{\frac{1}{p}} < \infty.$

This implies $\bar{d}(\Delta X_{nk}^i, \Delta Y_{nk}^j) < \infty$.

Thus $(\Delta X_{nk}^i)_{i=1}^{\infty}$ is a Cauchy sequence for each n, $k\in N$. Hence (ΔX_{nk}^i) converges for each n, $k\in N$.

Let us consider (ΔX_{11}^i), then (X_{11}^i), (X_{12}^i) and (X_{21}^i) are convergent and hence (X_{22}^i) converges. Next let $\lim_{i\to\infty} X_{22}^i = X_{22}$ and consider (ΔX_{12}^i), then (X_{12}^i), (X_{22}^i) and (X_{13}^i) are convergent. Thus (X_{23}^i) converges.

Proceeding in this way, we get, (X_{nk}^i) converges for each n, $k\in N$. Suppose $\lim_{i\to\infty} X_{nk}^i = X_{nk}$, for all n, $k\in N$. Now taking limit as $j \to \infty$ in (2), we have $\rho(X^i, X) < \varepsilon$ for all $i \geq n_0$. Thus for all $i \geq n_0$, we have

$$\rho(X, \bar{0}) = \rho(X, X^i) + \rho(X^i, \bar{0}) < \varepsilon + K < \infty.$$

Hence we conclude that ${}_2bv_F^p$ is a complete metric space.

Theorem 3.2. The class of double sequence ${}_2bv_F^p$ is not symmetric in general.

Proof: The proof follows from the following example:

Example 3.1. Let $p>1$. Consider the double sequence $<X_{nk}>$ defined as follows:

For all n odd,

$$X_{nn}(t) = \begin{cases} (1+nt), \text{ for } -\frac{1}{n} \leq t \leq 0; \\ (1-nt), \text{ for } 0 \leq t \leq \frac{1}{n}; \\ 0, \quad \text{otherwise.} \end{cases}$$

$X_{1k} = \bar{1}$, for all $k \geq 2$.

$X_{nk} = \bar{0}$, otherwise.

The matrix representation of the $<X_{nk}>$ is given by

$$<X_{nk}> = \begin{pmatrix} X_{11} & \bar{1} & \bar{1} & \bar{1} \text{ - - - -} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \text{ - - - -} \\ \bar{0} & \bar{0} & X_{33} & \bar{0} \text{ - - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}.$$

Then $<\Delta X_{nk}>$ is represented by

$$<\Delta X_{nk}> = \begin{pmatrix} X_{11}-\bar{1} & \bar{0} & \bar{0} & \bar{0} \text{ - - - -} \\ \bar{0} & X_{33} & -X_{33} & \bar{0} \text{ - - - -} \\ \bar{0} & -X_{33} & X_{33}+X_{44} & -X_{44} \text{ - - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}.$$

Thus we have

$$\sum_n\sum_k\left\{\bar{d}(\Delta X_{nk}, \bar{0})\right\}^p \leq \sum_n\left\{\frac{1}{n^p} + \frac{1}{(n+1)^p}\right\} < \infty$$

Hence $<X_{nk}> \in {}_2bv_F^p$.

Consider the rearrangement $<Y_{nk}>$ of $<X_{nk}>$ defined by

$Y_{nn} = X_{nn}$.

$Y_{nk} = \bar{0}$, for $n \neq k$ and n even.

For $n \neq k$ and n odd,

$$Y_{nk} = \begin{cases} \bar{1}, \text{ for } k \text{ odd}, \\ \bar{0}, \text{ otherwise.} \end{cases}$$

Thus $<Y_{nk}>$ is represented by

$$<Y_{nk}> = \begin{pmatrix} X_{11} & \bar{0} & \bar{1} & \bar{0} \text{ - - - -} \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} \text{ - - - -} \\ \bar{1} & \bar{0} & X_{33} & \bar{0} \text{ - - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}.$$

Then $<\Delta Y_{nk}>$ is represented by

$$<\Delta Y_{nk}> = \begin{pmatrix} X_{11} & -\bar{1} & \bar{1} & -\bar{1} \text{ - - - -} \\ -\bar{1} & X_{33} & -X_{33} & \bar{1} \text{ - - - -} \\ \bar{1} & -X_{33} & X_{33} & -\bar{1} \text{ - - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}$$

Clearly $\sum_n\sum_k\left\{\bar{d}(\Delta Y_{nk}, \bar{0})\right\}^p \to \infty$.

Hence $<Y_{nk}> \notin {}_2bv_F^p$.

Thus we conclude that ${}_2bv_F^p$ is not symmetric.

Theorem 3.3. The class of double sequence ${}_2bv_F^p$ is not convergence free in general.

Proof: The result follows from the following example.

Example 3.2. Consider the double sequence (X_{nk}) defined as follows:

$$X_{nn}(t) = \begin{cases} (n^2t+1), \text{for} -\frac{1}{n^2} \le t \le 0; \\ (1-n^2t), \text{for } 0 \le t \le \frac{1}{n^2}; \\ 0, \qquad \text{otherwise.} \end{cases}$$

$X_{nk} = \bar{0}$, for $n \neq k$.

Then $<\Delta X_{nk}>$ is defined as follows

For $n = k$,

$$\Delta X_{nn}(t) = \begin{cases} \left(\frac{n^2(n+1)^2}{n^2+(n+1)^2}t+1\right), \text{for} -\frac{n^2+(n+1)^2}{n(n+1)} \le t \le 0; \\ \left(1-\frac{n^2(n+1)^2}{n^2+(n+1)^2}t\right), \text{for } 0 \le t \le \frac{n^2+(n+1)^2}{n^2(n+1)^2}; \\ 0, \qquad \text{otherwise.} \end{cases}$$

for $n + 1 = k$ and for $n - 1 = k$,

$\Delta X_{nk} = X_{nn}$,

and

$\Delta X_{nk} = \bar{0}$, otherwise.

Therefore we have

$$\sum_n \sum_k \left\{\bar{d}(\Delta X_{nk}, \bar{0})\right\}^p \le \sum_n \left\{\frac{1}{n^{2p}} + \frac{1}{(n+1)^{2p}}\right\} < \infty$$

Hence $<X_{nk}> \in {}_2bv_F^p$.

Now we consider the double sequence $<Y_{nk}>$ defined by

$$Y_{nn}(t) = \begin{cases} (\frac{1}{n}t+1), \text{for} -n \le t \le 0; \\ (1-\frac{1}{n}t), \text{for } 0 \le t \le n; \\ 0, \qquad \text{otherwise.} \end{cases} \quad \text{and } Y_{nk} = \bar{0}, \text{ for } n \neq k.$$

Then $<\Delta Y_{nk}>$ is defined as follows

$$\Delta Y_{nn}(t) = \begin{cases} \left(\frac{1}{n+(n+1)}t+1\right), \text{for } -\{n+(n+1)\} \le t \le 0; \\ \left(1-\frac{1}{n+(n+1)}t\right), \text{for } 0 \le t \le \{n+(n+1)\}; \\ 0, \qquad \text{otherwise.} \end{cases}$$

$\Delta Y_{nk} = Y_{nn}$, for all $n + 1 = k$ and $n - 1 = k$,

and

$\Delta Y_{nk} = \bar{0}$, otherwise.

Thus we have

$$\sum_n \sum_k \left\{\bar{d}(\Delta Y_{nk}, \bar{0})\right\}^p \le \sum_n \left\{n^p + (n+1)^p\right\} \to \infty.$$

This implies that $<Y_{nk}> \notin {}_2bv_F^p$.

This completes the proof of the theorem.

Theorem 3.4. The class of double sequence ${}_2bv_F^p$ is not solid in general.

Proof: The result follows from the following example.

Example 3.3. Let $p>1$. Consider the double sequence $<X_{nk}>$ defined by,

$$X_{nn}(t) = \begin{cases} \left((n+1)t+1\right), \text{for} -\frac{1}{(n+1)} \le t \le 0; \\ \left(1-(n+1)t\right), \text{for } 0 \le t \le \frac{1}{(n+1)}; \\ 0, \qquad \text{otherwise.} \end{cases}$$

$X_{1k} = \bar{1}$, for all $k \ge 2$.

$X_{n1} = \overline{-1}$, for $n \ge 2$ and $X_{nk} = \bar{0}$, otherwise.

The matrix representation of $<X_{nk}>$ is given by

$$<X_{nk}> = \begin{pmatrix} X_{11} & \bar{1} & \bar{1} & \bar{1} \text{ - - - -} \\ \overline{-1} & X_{22} & \bar{0} & \bar{0} \text{ - - - -} \\ \overline{-1} & \bar{0} & X_{33} & \bar{0} \text{ - - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}$$

The matrix determined by $<\Delta X_{nk}>$ is represented as

$$<\Delta X_{nk}> = \begin{pmatrix} X_{11}+X_{22} & -X_{22} & \bar{0} & \bar{0} \text{ - - - -} \\ -X_{22} & X_{22}+X_{33} & -X_{33} & \bar{0} \text{ - - - -} \\ \bar{0} & -X_{33} & X_{33}+X_{44} & -X_{44} \text{- - - -} \\ - & - & - & - \text{ - - - -} \end{pmatrix}$$

Thus we have

$$\sum_n \sum_k \left\{\bar{d}(\Delta X_{nk}, \bar{0})\right\}^p \le 2^p \sum_n \sum_n \left\{\bar{d}(X_{nn}, \bar{0})\right\}^p + \sum_n \sum_k \left\{\bar{d}(X_{nn} + X_{n+1,k+1}, \bar{0})\right\}^p$$

$$= 2^p \sum_n \left\{\frac{1}{(n+1)^p}\right\} + \sum_n \left\{\frac{1}{(n+2)^p}\right\} < \infty$$

Thus $<X_{nk}> \in {}_2bv_F^p$.

Now consider the double sequence $<Y_{nk}>$ defined by,

$Y_{1k} = \bar{1}$, all k, even,

$Y_{n1} = \overline{-1}$, for all n even and $Y_{nk} = \bar{0}$, otherwise.

Then we have the following matrix representations.

$$<Y_{nk}> = \begin{pmatrix} \bar{0} & \bar{1} & \bar{0} & \bar{1} & - & - & - & - \\ \overline{-1} & \bar{0} & \bar{0} & \bar{0} & - & - & - & - \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & - & - & - & - \\ - & - & - & - & - & - & - & - \end{pmatrix}$$

and $$<\Delta Y_{nk}> = \begin{pmatrix} \bar{0} & \bar{1} & \overline{-1} & \bar{1} & - & - & - & - \\ \overline{-1} & \bar{0} & \bar{0} & \bar{0} & - & - & - & - \\ \bar{0} & \bar{0} & \bar{0} & \bar{0} & - & - & - & - \\ - & - & - & - & - & - & - & - \end{pmatrix}$$

Direct calculation gives

$$\sum_n \sum_k \left\{ \bar{d}(\Delta Y_{nk}, \bar{0}) \right\}^p = \infty$$

This shows that $<Y_{nk}> \notin {}_2bv_F^p$ and also it is observed that $|Y_{nk}| \leq |X_{nk}|$.

This completes the proof of the theorem.

Theorem 3.5. (*a*) ${}_2\ell_F^p \subset {}_2bv_F^p$, *for* $1 < p < \infty$ and the inclusion is strict.

(*b*) ${}_2bv_F^q \subset {}_2bv_F^p$, *for* $1 \leq q < p < \infty$ and the inclusion is strict.

Proof: (*a*) Let us consider the double sequence $<X_{nk}> \in {}_2\ell_F^p$.

$$\therefore \sum_n \sum_k \left\{ \bar{d}(X_{nk}, \bar{0}) \right\}^p < \infty \quad - - - - \quad (1)$$

We can write

$$\sum_n \sum_k \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^p \leq D^2$$

$$\left[\sum_n \sum_k \left\{ \bar{d}(X_{nk}, \bar{0}) \right\}^p + \sum_n \sum_k \left\{ \bar{d}(X_{(n+1)k}, \bar{0}) \right\}^p + \right.$$

$$\left. \sum_n \sum_k \left\{ \bar{d}(X_{n(k+1)}, \bar{0}) \right\}^p + \sum_n \sum_k \left\{ \bar{d}(X_{(n+1)(k+1)}, \bar{0}) \right\}^p \right]$$

$< \infty$. [Using (1)]

It is easy to verify that the inclusion is strict.

This completes the proof of the theorem.

(*b*) Let $<X_{nk}>$ be an element of ${}_2bv_F^p$, then

$$\sum_n \sum_k \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^p < \infty, \ 1 \leq p < \infty.$$

We choose n_0 and k_0 such that $\left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^p < 1$, for $n \geq n_0$ and $k \geq k_0$.

$\Rightarrow \bar{d}(\Delta X_{nk}, \bar{0}) < 1$, for $n \geq n_0$ and $k \geq k_0$.

$\Rightarrow \bar{d}(\Delta X_{nk}, \bar{0})^p < \bar{d}(\Delta X_{nk}, \bar{0})^q$, for $n \geq n_0$, $k \geq k_0$ and $p > q$.

For $(n, k) \in K = N \times N - \{(n, k): n \leq n_0, k \leq k_0\}$ and for $0 < q < p < \infty$, we have

$$\sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^p - \sum_{n=1}^{n_0} \sum_{k=1}^{k_0} \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^p$$

$$< \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^q - \sum_{n=1}^{n_0} \sum_{k=1}^{k_0} \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^q$$

$$\Rightarrow \sum_n \sum_k \left\{ \bar{d}(\Delta X_{nk}, \bar{0}) \right\}^q < \infty, \ 1 \leq q < \infty.$$

Thus $<X_{nk}> \in {}_2bv_F^q$.

It is easy to verify that the inclusion is strict.

Hence we conclude that ${}_2bv_F^q \subset {}_2bv_F^p$.

Conclusion

The concept of double sequence in terms of fuzzy real numbers is a very recent development. Soon after it many researchers have introduced different classes of double sequences and studied some algebraic and topological properties. The class of sequence introduced here has its importance from the point of view of its structure and norm. We have verified some important properties for the class of double sequence with some concrete examples.

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