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α^{γ}-open sets, α^{γ}-functions and some new separation axioms

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ABSTRACT. In this paper, a new kind of set called an α^{γ}-open set is introduced and investigated using the γ-operator due to Ogata. Such sets are used for studying new types of mappings, viz. α^{γ}-continuous, $\alpha^{(\gamma,\beta)}$-irresolute, etc. Finally, new separation axioms: α^{γ}-T_i (i = 0, ½, 1, 2), α^{γ}-D_i (i = 0, 1, 2), and a new notion of the graph of a function called an α^{γ}-closed graph.

Keywords: α^{γ}-open, α^{γ}-g closed, α^{γ}-continuous, $\alpha^{(\gamma,\beta)}$-irresolute, α^{γ}D-set, α^{γ}-closed graph.

α^{γ}-aberto conjunto, α^{γ}-funções e alguns novos axiomas de separação

RESUMO. Neste artigo, um novo tipo de conjunto chamado α^{γ}-aberto conjunto é introduzido e investigado usando o γ-operador devido a Ogata. Esses jogos são usados para estudar novos tipos de mapeamentos, viz. α^{γ}-contínuo, $\alpha^{(\gamma,\beta)}$-irresoluto, etc. Finalmente, novos axiomas de separação: α^{γ}-T_i (i = 0, ½, 1, 2), α^{γ}-D_i (i = 0, 1, 2), e uma nova noção do gráfico de uma função chamado um α^{γ}-gráfico fechado.

Palavras-chave: α^{γ}-aberto, α^{γ}-g fechado, α^{γ}-contínuo, $\alpha^{(\gamma,\beta)}$-irresoluto, α^{γ}D-conjunto, α^{γ}-gráfico fechado.

Introduction

In 1965 Njastad (1965) introduced α-open sets, Kasahara (1979) defined an operation α on a topological space to introduce α-closed graphs. Following the same technique, Ogata (1991) defined an operation on a topological space and introduced γ-open sets.

In this paper, we introduce the notion of α^{γ}-open sets, α^{γ}-continuity and $\alpha^{(\gamma,\beta)}$-irresoluteness in topological spaces. By utilizing these notions we introduce some weak separation axioms. Also we show that some basic properties α^{γ}-T_i (i = 0, ½, 1, 2), α^{γ}-D_i (i = 0, 1, 2) spaces and we offer a new notion of the graph of a function called an α^{γ}-closed graph and investigate some of their fundamental properties.

Throughout the paper spaces X and Y mean topological spaces. For a subset A of a space X, cl(A) and int(A) represent the closure of A and the interior of A, respectively.

Preliminaries

A subset A of X is called α-open if $A \subseteq int(cl(int(A)))$. The complement of α-open set is called α-closed set. The family of all α-open sets of X is denoted by $\alpha O(X)$. For a subset A of X, the union of all α-open sets of X contained in A is called the α-interior (in short $\alpha int(A)$) of A, and the intersection of all α-closed sets of X containing A is called the α-closure (in short $\alpha cl(A)$) of A. An operation γ (KASAHARA, 1979) on a topology τ is a mapping from τ in to power set P(X) of X such that $V \subseteq (V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of at V. A subset A of X with an operation γ on τ is called γ-open (OGATA, 1991) if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Then, τ_{γ} denotes the set of all γ-open set in X. Clearly $\tau_{\gamma} \subseteq \tau$. Complements of γ-open sets are called γ-closed. The γ-closure (OGATA, 1991) of a subset A of X with an operation γ on τ is denoted by τ_{γ}-cl(A) and is defined to be the intersection of all γ-closed sets containing A, and the γ-interior (OGATA, 1991) of A is denoted by τ_{γ}-int(A) and defined to be the union of all γ-open sets of X contained in A. A topological (X, τ) with an operation γ on τ is said to be γ-regular (OGATA, 1991) if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V. It is also to be noted that $\tau = \tau_{\gamma}$ if and only if X is a γ-regular space (OGATA, 1991).

α^{γ}-open sets

Definition 3.1. Let (X, τ) be a topological space, γ an operation on τ and $A \subseteq X$. Then A is called an α^{γ}-open set if $A \subseteq int(\tau_{\gamma}\text{-}cl(int(A)))$.

$\alpha^{\gamma}O(X)$ denotes the collection of all α^{γ}-open sets of (X, τ), and $\alpha^{\gamma}O(X, x)$ is the collection of all α^{γ}-open sets containing the point x of X.

A subset A of X is called α^{γ}-closed if and only if its complement is α^{γ}-open. Moreover, $\alpha^{\gamma}C(X)$ denotes the collection of all α^{γ}-closed sets of (X, τ).

It can be shown that a subset A of X is α^γ-closed if and only if $cl(\tau_\gamma\text{-}int(cl(A))) \subseteq A$.

Remark 3.2.

(1) Every α-open set is α^γ-open, while in a γ-regular space these concepts are equivalent.

(2) Every γ-open set is α^γ-open, but the converse may not be true.

Example 3.3. Let X = {a, b, c} and τ = {φ, {a}, {b}, {a, b}, X}. Define an operation γ on τ by γ(A) = {a} if A = {a} and γ(A) = A ∪ {c} if A ≠ {a}. Clearly, τ_γ = {φ, {a}, X}.

(1) Then {a, c} is α^γ-open but not α-open.

(2) Also {a, c} is α^γ-open but not γ-open.

Theorem 3.4. An arbitrary union of α^γ-open sets is α^γ-open.

Proof. Let $\{A_k: k \in I\}$ be a family of α^γ-open sets. Then for each k,

$A_k \subseteq int(\tau_\gamma\text{-}cl(int(A_k)))$ and so

$\cup_k A_k \subseteq \cup_k int(\tau_\gamma\text{-}cl(int(A_k)))$

$\subseteq int(\cup_k \tau_\gamma\text{-}cl(int(A_k)))$

$\subseteq int(\tau_\gamma\text{-}cl((\cup_k int(A_k)))$

$\subseteq int(\tau_\gamma\text{-}cl(int(\cup_k A_k)))$.

Thus, $\cup_k A_k$ is α^γ-open.

Remark 3.5.

(1) An arbitrary intersection of α^γ-closed sets is α^γ-closed.

(2) The intersection of even two α^γ-open sets may not be α^γ-open.

Example 3.6. Let X = {a, b, c} and τ = {φ, {a}, {b}, {a, b}, X}. Define an operation γ on τ by γ(A) = A if A = {a, b} and γ(A) = X otherwise. Clearly, τ_γ = {φ, {a, b}, X} and $\alpha^\gamma O(X)$ = {φ, {a}, {b}, {a, b}, {a, c}, {b, c}, X}, take A = {a, c} and B = {b, c}. Then A ∩ B = {c}, which is not an α^γ-open set.

Definition 3.7. Let A be a subset of a topological space (X, τ) and γ an operation on τ.

(1) The union of all α^γ-open sets contained in A is called the α^γ-interior of A and denoted by $\alpha^\gamma int(A)$.

(2) The intersection of all α^γ-closed sets containing A is called the α^γ-closure of A and denoted by $\alpha^\gamma cl(A)$.

(3) The set denoted by $\alpha^\gamma D(A)$ and defined by {x: for every α^γ-open set U containing x, U ∩ (A\{x}) ≠ φ} is called the α^γ-derived set of A.

(4) The α^γ-frontier of A, denoted by $\alpha^\gamma Fr(A)$ is defined as $\alpha^\gamma cl(A) \setminus \alpha^\gamma cl(X \setminus A)$.

We now state the following theorem without proof.

Theorem 3.8. Let (X, τ) be a topological space and γ an operation on τ. For any subsets A, B of X we have the following:

(1) A is α^γ-open if and only if $A = \alpha^\gamma int(A)$.

(2) A is α^γ-closed if and only if $A = \alpha^\gamma cl(A)$.

(3) If A ⊆ B then $\alpha^\gamma int(A) \subseteq \alpha^\gamma int(B)$ and $\alpha^\gamma cl(A) \subseteq \alpha^\gamma cl(B)$.

(4) $\alpha^\gamma int(A) \cup \alpha^\gamma int(B) \subseteq \alpha^\gamma int(A \cup B)$.

(5) $\alpha^\gamma int(A \cup B) \subseteq \alpha^\gamma int(A) \cap \alpha^\gamma int(B)$.

(6) $\alpha^\gamma cl(A) \cup \alpha^\gamma cl(B) \subseteq \alpha^\gamma cl(A \cup B)$.

(7) $\alpha^\gamma cl(A \setminus B) \subseteq \alpha^\gamma cl(A) \cap \alpha^\gamma cl(B)$.

(8) $\alpha^\gamma int(X \setminus A) = X \setminus \alpha^\gamma cl(A)$.

(9) $\alpha^\gamma cl(X \setminus A) = X \setminus \alpha^\gamma int(A)$.

(10) $\alpha^\gamma int(A) = A \setminus \alpha^\gamma D(X \setminus A)$.

(11) $\alpha^\gamma cl(A) = A \cup \alpha^\gamma D(A)$.

(12) $\tau_\gamma\text{-}int(A) \subseteq \alpha^\gamma int(A)$.

(13) $\alpha^\gamma cl(A) \subseteq \tau_\gamma\text{-}cl(A)$.

Theorem 3.9. Let A be a subset of a topological space (X, τ) and γ be an operation on τ. Then x ∈ $\alpha^\gamma cl(A)$ if and only if for every α^γ-open set V of

X containing x, A ∩ V≠ φ.

Proof. Let x ∈$\alpha^\gamma cl(A)$ and suppose that V∩ A = φ for some α^γ-open set V which contains x. Then (X \ V) is α^γ-closed and A ⊆ (X \ V), thus $\alpha^\gamma cl(A)$ ⊆ (X \ V). But this implies that x ∈(X \ V), a contradition. Therefore V ∩ A ≠ φ.

Conversely, Let A ⊆ X and x ∈X such that for each α^γ-open set U which contains x, U ∩ A ≠ φ. If x ∉ $\alpha^\gamma cl(A)$, there is an α^γ-closed set F such that A ⊆ F and x ∉ F. Then (X\F) is an α^γ-open set with x ∈ (X\F), and thus (X \ F) ∩ A ≠ φ, which is a contradiction.

Definition 3.10. A subset A of a topological space (X, τ) with an operation γ on τ is called τ-α^γ-open (resp. α^γ-γ-open) if $int(A) = \alpha^\gamma int(A)$ (resp. $\tau_\gamma\text{-}int(A) = \alpha^\gamma int(A)$).

Definition 3.11. A subset A of a topological space (X, τ) with an operation γ on τ is called an α^γ-generalized closed set (α^γ-g closed, for short) if

$\alpha^\gamma cl(A)$ ⊆ U whenever A ⊆ U and U is an α^γ-open set in X.

The complement of an α^γ-g closed set is called an α^γ-g open set. Clearly, A is α^γ-g open if and only if F ⊆ $\alpha^\gamma int(A)$ whenever F ⊆ A and F is α^γ-closed in X.

Theorem 3.12. Every α^γ-closed set is α^γ-g closed.

Proof. A set A ⊆ X is α^γ-closed if and only if $\alpha^\gamma cl(A)$ = A. Thus $\alpha^\gamma cl(A)$ ⊆ U for every U ∈ $\alpha^\gamma O(X)$ containing A.

Theorem 3.13. A subset A of topological space (X, τ) with an operation γ on τ, is α^γ-g closed if and only if $\alpha^\gamma cl(\{x\})$ ∩ A ≠ φ, holds for every x ∈ $\alpha^\gamma cl(A)$.

Proof. Let A be an α^γ-g closed set in X and suppose if possible there exists an x ∈$\alpha^\gamma cl(A)$ such that $\alpha^\gamma cl(\{x\})$ ∩ A ≠ φ. Therefore A ⊆ X \ $\alpha^\gamma cl(\{x\})$,

and so $\alpha^\gamma cl(A)$ ⊆ X\$\alpha^\gamma cl(\{x\})$. Hence x ∈$\alpha^\gamma cl(A)$, which is a contradiction.

Conversely, suppose that the condition of the therem holds and let U be any α^γ-open set such that A ⊆ U and let x ∈ $\alpha^\gamma cl(A)$. By assumption, there exists a z ∈$\alpha^\gamma cl(\{x\})$ and z ∈ A ⊆ U. Thus by the Theorem 3.9,

$U\cap\{x\}\neq\varphi$. Hence $x \in U$, which implies $\alpha^\gamma cl(A) \subseteq U$.

Theorem 3.14. Let A be an α^γ-g closed set in a topological space (X, τ) with operation γ on τ. Then $\alpha^\gamma cl(A)\backslash A$ does not contain any nonempty α^γ-closed set.

Proof. If possible, let F be an α^γ-closed set such that $F \subseteq \alpha^\gamma cl(A)\backslash A$ and $F \neq \varphi$. Then $F \subseteq X\backslash A$ which implies $A \subseteq X\backslash F$. Since A is α^γ-g closed and $X\backslash F$ is α^γ-open, therefore $\alpha^\gamma cl(A) \subseteq X\backslash F$, that is $F \subseteq X\backslash\alpha^\gamma cl(A)$.

Hence $F \subseteq \alpha^\gamma cl(A) \cap (X\backslash\alpha^\gamma cl(A)) = \varphi$. This shows that, $F = \varphi$ which is a contradiction.

Theorem 3.15. In a topological space (X, τ) with an operation γ on τ, either $\{x\}$ is α^γ-closed or $X\backslash\{x\}$ is α^γ-g closed.

Proof. Suppose that $\{x\}$ is not α^γ-closed, then $X\backslash\{x\}$ is not α^γ-open. Then X is the only α^γ-open set such that $X\backslash\{x\}\subseteq X$. Hence $X\backslash\{x\}$ is α^γ-g closed set.

α^γ-Functions

Definition 4.1. Let (X, τ) and (Y, σ) be two topological spaces and γ an operation on τ. Then a function $f: (X, \tau) \to (Y, \sigma)$ is said to be α^γ-continuous at a point $x \in X$ if for each open set V of Y containing f(x), there exists an α^γ-open set U of X containing x such that $f(U) \subseteq V$.

If f is α^γ-continuous at each point x of X, then f is called α^γ-continuous on X.

Theorem 4.2. Let (X, τ) be a topological space with an operation γ on τ. For a function $f: (X, \tau) \to (Y, \sigma)$, the following statements are equivalent:

(1) f is α^γ-continuous.

(2) $f^{-1}(V)$ is α^γ-open set in X, for each open set V in Y.

(3) $f^{-1}(V)$ is α^γ-closed set in X, for each closed set V in Y.

(4) $f(\alpha^\gamma cl(U)) \subseteq cl(f(U))$, for each subset U of X.

(5) $\alpha^\gamma cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$, for each subset V of Y.

(6) $f^{-1}(int(V)) \subseteq \alpha^\gamma int(f^{-1}(V))$, for each subset V of Y.

(7) $int(f(U)) \subseteq f(\alpha^\gamma int(U))$, for each subset U of X.

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3). Obvious.

(3) $\Rightarrow$ (4). Let U be any subset of X. Then $f(U) \subseteq cl(f(U))$ and $cl(f(U))$ is closed set in Y. Hence $U \subseteq f^{-1}(cl(f(U)))$. By (3), we have $f^{-1}(cl(f(U)))$ is α^γ-closed set in X. Therefore, $\alpha^\gamma cl(U) \subseteq f^{-1}(cl(f(U)))$.

Hence $f(\alpha^\gamma cl(U)) \subseteq cl(f(U))$.

(4) $\Rightarrow$ (5). Let V be any subset of Y. Then $f^{-1}(V)$ is a subset of X. By (4), we have $f(\alpha^\gamma cl(f^{-1}(V))) \subseteq cl(f(f^{-1}(V))) = cl(V)$. Hence $\alpha^\gamma cl(f^{-1}(V)) \subseteq f^{-1}(cl(V))$.

(5) $\Leftrightarrow$ (6). Let V be any subset of Y. Then apply (5) to $Y\backslash V$ we obtain

$\alpha^\gamma cl(f^{-1}(Y\backslash V)) \subseteq f^{-1}(cl(Y\backslash V)) \Leftrightarrow \alpha^\gamma cl(X\backslash f^{-1}(V)) \subseteq f^{-1}(Y\backslash int(V)) \Leftrightarrow X\backslash\alpha^\gamma int(f^{-1}(V)) \subseteq X\backslash f^{-1}(int(V)) \Leftrightarrow f^{-1}(int(V)) \subseteq \alpha^\gamma int(f^{-1}(V))$. Therefore, $f^{-1}(int(V)) \subseteq \alpha^\gamma int(f^{-1}(V))$.

(6) $\Rightarrow$ (7). Let U be any subset of X. Then f(U) is a subset of Y. By (6), we have $f^{-1}(int(f(U))) \subseteq \alpha^\gamma int(f^{-1}(f(U)))=\alpha^\gamma int(U)$. Therefore, $int(f(U)) \subseteq f(\alpha^\gamma int(U))$.

(7) $\Rightarrow$ (1). Let $x \in X$ and let V be any open set of Y containing f(x). Then $x \in f^{-1}(V)$ and $f^{-1}(V)$ is a subset of X. By (7), we have $int(f(f^{-1}(V))) \subseteq f(\alpha^\gamma int(f^{-1}(V)))$. Then $int(V) \subseteq f(\alpha^\gamma int(f^{-1}(V)))$. Since V is an open set. Then $V \subseteq f(\alpha^\gamma int(f^{-1}(V)))$ implies that $f^{-1}(V) \subseteq \alpha^\gamma int(f^{-1}(V))$. Therefore, $f^{-1}(V)$ is α^γ-open set in X which contains x and clearly $f(f^{-1}(V)) \subseteq V$. Hence f is α^γ-continuous.

Theorem 4.3. For a function $f: (X, \tau) \to (Y, \sigma)$ with an operation γ on τ, the following statements are equivalent:

(1) $f^{-1}(V)$ is α^γ-open set in X, for each open set V in Y.

(2) $\alpha^\gamma Fr(f^{-1}(V)) \subseteq f^{-1}(Fr(V))$, for each subset V in Y.

Proof. (1) $\Rightarrow$ (2). Let V be any subset of Y. Then, we have $f^{-1}(Fr(V))=f^{-1}(cl(V)\backslash int(V))=f^{-1}(cl(V))\backslash f^{-1}(int(V)) =\alpha^\gamma cl(f^{-1}(V))\backslash f^{-1}(int(V))\supseteq\alpha^\gamma cl(f^{-1}(V))\backslash\alpha^\gamma int f^{-1}(int(V)) \supseteq \alpha^\gamma cl(f^{-1}(V))\backslash\alpha^\gamma int(f^{-1}(V))= \alpha^\gamma Fr\, f^{-1}(V)$, and hence $f^{-1}(Fr(V)) \supseteq \alpha^\gamma Fr(f^{-1}(V))$.

(2) $\Rightarrow$ (1). Let V be open in Y and $F = Y\backslash V$. Then by (2), we obtain $\alpha^\gamma Fr(f^{-1}(F)) \subseteq f^{-1}(Fr(F)) \subseteq f^{-1}(cl(F)) = f^{-1}(F)$ and hence $\alpha^\gamma cl(f^{-1}(F)) = \alpha^\gamma int(f^{-1}(F)) \cup \alpha^\gamma Fr(f^{-1}(F)) \subseteq f^{-1}(F)$. Thus $f^{-1}(F)$ is α^γ-closed and hence $f^{-1}(V)$ is α^γ-open in X.

Theorem 4.4. let (X, τ) be a topological space with an operation γ on τ and let $f: (X, \tau) \to (Y, \sigma)$ be a function. Then

$X\backslash\alpha^\gamma C(f)=\cup\{\alpha^\gamma Fr(f^{-1}(V)): V \in \sigma, f(x) \in V, x \in X\}$,

where $\alpha^\gamma C(f)$ denotes the set of points at which f is α^γ-continuous.

Proof. Let $x \in X\backslash\alpha^\gamma C(f)$. Then there exists $V \in \sigma$ containing f(x) such that $f(U) \nsubseteq V$, for every α^γ-open set U containing x. Hence $U \cap [X \backslash f^{-1}(V)] \neq \varphi$ for every α^γ-open set U containing x. Therefore, by

Theorem 3.9, $x \in \alpha^\gamma cl(X\backslash f^{-1}(V))$. Then $x \in f^{-1}(V)\cap \alpha^\gamma cl(X\backslash f^{-1}(V)) \subseteq \alpha^\gamma Fr(f^{-1}(V)$. So, $X\backslash\alpha^\gamma C(f) \subseteq \cup\{\alpha^\gamma Fr(f^{-1}(V)):V \in \sigma, f(x) \in V, x \in X\}$

Conversely, let $x \notin X\backslash\alpha^\gamma C(f)$. Then for each $V \in \sigma$ containing f(x), $f^{-1}(V)$ is an α^γ-open set containing x. Thus $x \in \alpha^\gamma int(f^{-1}(V)$ and hence $x \notin \alpha^\gamma Fr(f^{-1}(V))$, for every $V \in \sigma$ containing f(x). Therefore,

$X\backslash\alpha^\gamma C(f) \supseteq \cup\{\alpha^\gamma Fr(f^{-1}(V)):V \in \sigma, f(x) \in V, x \in X\}$.

Remark 4.5. Every γ-continuous function is α^γ-continuous, but the converse is not true.

Example 4.6. Let $X = \{a, b, c\}$, $\tau = \{\varphi, \{a\}, X\}$ and $\sigma = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. Define an operation γ on τ by $\gamma(A) = A$ if $A=\{a\}$ and $\gamma(A) = A \cup \{b\}$ if $A \neq$

{a}. Define a function f: (X, τ) → (X, σ) as follows: f(x) = a if x = a, f(x) = a if x = b and f(x) = c if x = c.

Then f is α^{γ}-continuous but not γ-continuous at b, because {a, b} is an open set in (X, σ) containing f(b) = a, there exist no α^{γ}-open set U in (X, τ) containing b such that f(U) ⊆ {a, b}.

Remark 4.7. Let γ and β be operations on the topological spaces (X, τ) and (Y, σ), respectively. If the functions f: (X, τ) → (Y, σ) and g: (Y, σ) → (Z, υ) are α^{γ}-continuous and continuous, respectively, then gof is α^{γ}-continuous.

Definition 4.8. Let (X, τ) be a topological space with an operation γ on τ. A function f: (X, τ) → (Y, σ) is called τ-α^{γ}-continuous (resp. α^{γ}-γ-continuous) if for each open set V in Y, $f^{-1}(V)$ is τ-α^{γ}-open (resp. α^{γ}-γ-open) in X.

Theorem 4.9. Let f: (X, τ) → (Y, σ) be a mapping and γ an operation on τ. Then the following are equivalent:

(1) f is γ-continuous.

(2) f is α^{γ}-continuous and α^{γ}-γ-continuous.

Proof. (1) ⇒ (2). Let f be γ-continuous. Then f is α^{γ}-continuous. Now, let G be any open set in Y, then $f^{-1}(G)$ is γ-open in X. Then τ_{γ}-int($f^{-1}(G)$) = $f^{-1}(G)$ = α^{γ}int($f^{-1}(G)$). Thus, $f^{-1}(G)$ is α^{γ}-γ-open in X. Therefore f is α^{γ}-γ-continuous.

(2) ⇒ (1). Let f be α^{γ}-continuous and α^{γ}-γ-continuous. Then for any open set G in Y, $f^{-1}(G)$ is both α^{γ}-open and α^{γ}-γ-open in X. So

$f^{-1}(G)$ = α^{γ}int($f^{-1}(G)$) = τ_{γ}-int($f^{-1}(G)$).

Thus $f^{-1}(G)$ is γ-open and hence f is γ-continuous.

Theorem 4.10. Let f: (X, τ) → (Y, σ) be τ-α^{γ}-continuous, where γ is an operation on τ. Then f is continuous if and only if f is α^{γ}-continuous.

Proof. Let V ∈ σ. Since f is continuous as well as τ-α^{γ}-continuous, $f^{-1}(V)$ is open as well as τ-α^{γ}-open in X and hence $f^{-1}(V)$ = int($f^{-1}(V)$) = α^{γ}int($f^{-1}(V)$) ∈ α^{γ}O(X). Therefore, f is α^{γ}-continuous.

Conversely, let V ∈ σ. Then $f^{-1}(V)$ is α^{γ}-open and τ-α^{γ}-open. So $f^{-1}(V)$ = α^{γ}int($f^{-1}(V)$) = int($f^{-1}(V)$). Hence $f^{-1}(V)$ is open in X. Therefore f is continuous.

Definition 4.11. A function f: (X, τ) → (Y, σ), where γ and β are operations on τ andσ, respectively, is called α^{β}-g-closed if for every α^{γ}-closed set F in X, f(F) is α^{β}-g closed in Y.

Definition 4.12. Let (X, τ) and (Y, σ) be two topological spaces and γ, β operations on τ, σ, respectively. A mapping f: (X, τ) → (Y, σ) is called $\alpha^{(\gamma,\beta)}$-irresolute at x if and only if for each α^{β}-open set V in Y containing f(x), there exists an α^{γ}-open set U in X containing x such that f(U) ⊆ V.

If f is $\alpha^{(\gamma,\beta)}$-irresolute at each point x ∈ X, then f is called $\alpha^{(\gamma,\beta)}$-irresolute on X.

Theorem 4.13. let (X, τ), (Y, σ) be topological spaces and γ, β operations on τ, σ, respectively. If f: (X, τ) → (Y, σ) is $\alpha^{(\gamma,\beta)}$-irresolute and α^{β}-g-closed, and A is α^{γ}-g closed in X, then f(A) is α^{β}-g closed.

Proof. Suppose A is an α^{γ}-g closed set in X and that U is an α^{β}-open set in Y such that f(A) ⊆ U. Then A ⊆ $f^{-1}(U)$. Since f is $\alpha^{(\gamma,\beta)}$-irresolute, $f^{-1}(U)$ is α^{γ}-open set in X. Again A is an α^{γ}-g closed set, therefore α^{γ}cl(A) ⊆ $f^{-1}(U)$ and hence f(α^{γ}cl(A)) ⊆ U. Since f is an α^{β}-g-closed map, f(α^{γ}cl(A)) is an α^{β}-g closed set in Y. Therefore, α^{β}cl(f(α^{γ}cl(A)) ⊆ U, which implies α^{β}cl(f(A)) ⊆ U.

We now state the following theorem without proof.

Theorem 4.14. Let f: (X, τ) → (Y, σ) be a mapping and γ, β operations on τ, σ, respectively. Then the following are equivalent:

(1) f is $\alpha^{(\gamma,\beta)}$-irresolute.

(2) The inverse image of each α^{β}-open set in Y is an α^{γ}-open set in X.

(3) The inverse image of each α^{β}-closed set in Y is an α^{γ}-closed set in X.

(4) α^{γ}cl($f^{-1}(V)$) ⊆ f^{-1}(α^{β}cl(V)), for all V ⊆ Y.

(5) f(α^{γ}cl(U)) ⊆ α^{β}cl(f(U)), for all U ⊆ X.

(6) α^{γ}Fr($f^{-1}(V)$) ⊆ f^{-1}(α^{β}Fr(V)), for all V ⊆ Y.

(7) f(α^{γ}D(U)) ⊆ α^{β}cl(f(U)), for all U ⊆ X.

(8) f^{-1}(α^{β}int(V)) ⊆ α^{γ}int($f^{-1}(V)$), for all V ⊆ Y.

α^{γ}-Separation Axioms

Definition 5.1. A topological space (X, τ) with an operation γ on τ is said to be

(1) α^{γ}-T_0 if for each pair of distinct points x, y in X, there exists an α^{γ}-open set U such that either x ∈ U and y ∉ U or x ∉U and y ∈ U.

(2) α^{γ}-T_1 if for each pair of distinct points x, y in X, there exist two α^{γ}-open sets U and V such that x ∈ U but y ∉ U and y ∈ V but x ∉ V.

(3) α^{γ}-T_2 if for each distinct points x, y in X, there exist two disjoint α^{γ}-open sets U and V containing x and y respectively.

(4) α^{γ}-$T_{½}$ if every α^{γ}-g closed set is α^{γ}-closed.

Theorem 5.2. A topological space (X, τ) with an operation γ on τ is α^{γ}-T_0 if and only if for each pair of distinct points x, y of X, α^{γ}cl({x}) ≠ α^{γ}cl({y}).

Theorem 5.3. The following statements are equivalent for a topological space (X, τ) with an operation γ on τ:

(1) (X, τ) is α^{γ}-$T_{½}$.

(2) Each singleton {x} of X is either α^γ-closed or α^γ-open.

Theorem 5.4. A topological space (X, τ) with an operation γ on τ is α^γ-T_1 if and only if the singletons are α^γ-closed sets.

Theorem 5.5. The following statements are equivalent for a topological space (X, τ) with an operation γ on τ:

(1) X is α^γ-T_2.

(2) Let x ∈ X. For each y ≠ x, there exists an α^γ-open set U containing x such that y ∉ α^γcl(U).

(3) For each x ∈X, ∩{α^γcl(U): U ∈α^γO(X) and x ∈ U}= {x}.

Corollary 5.6. If (X, τ) is a topological space and γ be an operation on τ, then the following statements are hold:

(1) Every α^γ-T_1 space is α^γ-$T_{½}$.

(2) Every α^γ-$T_{½}$ space is α^γ-T_0.

Proof. (1) By deinition and Theorem 5.4 we prove it.

(2) Let x and y be any two distinct points of X. By Theorem 5.3, the singleton set {x} is α^γ-closed or α^γ-open.

(a) If {x} is α^γ-closed, then X\{x} is α^γ-open. So y ∈ X\{x} and x ∉ X\{x}. Therefore, we have X is α^γ-T_0.

(b) If {x} is α^γ-open. Then x ∈{x} and y ∉ {x}. Therefore, we have X is α^γ-T_0.

Definition 5.7. A subset A of a topological space X is called an α^γDifference set (in short α^γD-set) if there are U, V ∈α^γO(X) such that U ≠ X and A = U\V.

It is true that every α^γ-open set U different from X is an α^γD-set if A = U and V = φ. So, we can observe the following.

Remark 5.8. Every proper α^γ-open set is a α^γD-set.

Now we define another set of separation axioms called α^γ-D_i, i = 0, 1, 2 by using the α^γD-sets.

Definition 5.9. A topological space (X, τ) with an operation γ on τ is said to be

(1) α^γ-D_0 if for any pair of distinct points x and y of X there exists an α^γD-set of X containing x but not y or an α^γD-set of X containing y but not x.

(2) α^γ-D_1 if for any pair of distinct points x and y of X there exists an α^γD-set of X containing x but not y and an α^γD-set of X containing y but not x.

(3) α^γ-D_2 if for any pair of distinct points x and y of X there exist disjoint α^γD-set G and E of X containing x and y, respectively.

Remark 5.10. For a topological space (X, τ) with an operation γ on τ, the following properties hold:

(1) If (X, τ) is α^γ-T_i, then it is α^γ-T_{i-1}, for i = 1, 2.

(2) If (X, τ) is α^γ-T_i, then it is α^γ-D_i, for i = 0, 1, 2.

(3) If (X, τ) is α^γ-D_i, then it is α^γ-D_{i-1}, for i = 1, 2.

Theorem 5.11. A space X is α^γ-D_1 if and only if it is α^γ-D_2.

Proof. Necessity. Let x; y ∈X, x ≠ y. Then there exist α^γD-sets G_1, G_2 in X such that x ∈G_1, y ∉ G_1 and y ∈G_2, x ∉ G_2. Let G_1 = U_1\$U_2$ and G_2 = U_3\$U_4$, where U_1, U_2, U_3 and U_4 are α^γ-open sets in X. From x ∉ G_2, it follows that either x ∉ U_3 or x ∈ U_3 and x ∈U_4. We discuss the two cases separately.

(i) x ∉ U_3. By y ∉ G_1 we have two subcases:

(a) y ∉ U_1. From x ∈ U_1\$U_2$, it follows that x ∈ U_1\(U_2 ∪ U_3), and by y ∈U_3\$U_4$ we have y ∈U_3\(U_1 ∪ U_4). Therefore (U_1\(U_2 ∪ U_3))∩(U_3\(U_1∪U_4)) = φ.

(b) y ∈ U_1 and y ∈ U_2. We have x ∈U_1\$U_2$, and y ∈$U_2$. Therefore ($U_1$\$U_2$) ∩ U_2 = φ.

(ii) x ∈U_3 and x ∈U_4. We have y ∈U_3\$U_4$ and x ∈ U_4. Hence (U_3\$U_4$) ∩ U_4 = φ. Therefore X is α^γ-D_2.

Sufficiency. Follows from Remark 5.10 (3).

Theorem 5.12. A space is α^γ-D_0 if and only if it is α^γ-T_0.

Proof. Suppose that X is α^γ-D_0. Then for each distinct pair x, y ∈X, at least one of x, y, say x, belongs to an α^γD-set G but y ∉ G. Let G = U_1\U_2 where U_1 ≠ X and U_1, U_2 ∈α^γO(X). Then x ∈ U_1, and for y ∉ G we have two cases: (a) y ∉ U_1, (b) y ∈ U_1 and y ∈U_2.

In case (a), x ∈ U_1 but y ∉ U_1.

In case (b), y ∈U_2 but x ∉ U_2.

Thus in both the cases, we obtain that X is α^γ-T_0.

Conversely, if X is α^γ-T_0, by Remark 5.10 (2), X is α^γ-D_0.

Corollary 5.13. If (X, τ) is α^γ-D_1, then it is α^γ-T_0.

Proof. Follows from Remark 5.10 (3) and Theorem 5.12.

Definition 5.14. A point x ∈X which has only X as the α^γ-neighborhood is called an α^γ-neat point.

Theorem 5.15. For an α^γ-T_0 topological space (X, τ) the following are equivalent:

(1) (X, τ) is α^γ-D_1.

(2) (X, τ) has no α^γ-neat point.

Proof. (1) ⇒ (2). Since (X, τ) is α^γ-D_1, then each point x of X is contained in an α^γD-set A = U\V and thus in U. By definition U ≠ X. This implies that x is not an α^γ-neat point.

(2) ⇒ (1). If X is α^γ-T_0, then for each distinct pair of points x, y ∈ X, at least one of them, x (say) has an α^γ-neighborhood U containing x and not y. Thus U which is different from X is a α^γD-set. If X has no α^γ-neat point, then y is not an α^γ-neat point. This means that there exists an α^γ-neighbourhood V of y such that V ≠ X. Thus y ∈ V\U but not x and V\U is an α^γ D-set. Hence X is α^γ-D_1.

Corollary 5.16. An α^γ-T_0 space X is not α^γ-D_1 if and only if there is a unique α^γ-neat point in X.

Proof. We only prove the uniqueness of the α^γ-neat point. If x and y are two α^γ-neat points in X, then since X is α^γ-T_0, at least one of x and y, say x, has an α^γ-neighborhood U containing x but not y.

Hence U $\neq$ X. Therefore x is not an α^γ-neat point which is a contradiction.

Definition 5.17. A topological space (X, τ) with an operation γ on τ, is said to be α^γ-symmetric if for x and y in X, $x \in \alpha^\gamma cl(\{y\})$ implies $y \in \alpha^\gamma cl(\{x\})$.

Theorem 5.18. If (X, τ) is a topological space with an operation γ on τ, then the following are equivalent:

(1) (X, τ) is α^γ-symmetric space.

(2) Every singleton is α^γ-g closed, for each $x \in X$.

Proof. (1) ⇒ (2). Assume that $\{x\} \subseteq U \in \alpha^\gamma O(X)$, but $\alpha^\gamma cl(\{x\}) \not\subseteq U$. Then $\alpha^\gamma cl(\{x\}) \cap X\backslash U \neq \varphi$. Now, we take $y \in \alpha^\gamma cl(\{x\}) \cap X\backslash U$, then by hypothesis $x \in \alpha^\gamma cl(\{y\}) \subseteq X \backslash U$ and $x \notin U$, which is a contradiction. Therefore {x} is α^γ-g closed, for each $x \in X$.

(2) ⇒ (1). Assume that $x \in \alpha^\gamma cl(\{y\})$, but $y \notin \alpha^\gamma cl(\{x\})$. Then $\{y\} \subseteq X\backslash\alpha^\gamma cl(\{x\})$ and hence $\alpha^\gamma cl(\{y\}) \in X\backslash\alpha^\gamma cl(\{x\})$. Therefore $x \in X\backslash\alpha^\gamma cl(\{x\})$, which is a contradiction and hence $y \in \alpha^\gamma cl(\{x\})$.

Corollary 5.19. If a topological space (X, τ) with an operation γ on τ is an α^γ-T_1 space, then it is α^γ-symmetric.

Proof. In an α^γ-T_1 space, every singleton is α^γ-closed (Theorem 5.4) and therefore is α^γ-g closed (Theorem 3.12). Then by Theorem 5.18, (X, τ) is α^γ-symmetric.

Corollary 5.20. For a topological space (X, τ) with an operation γ on τ, the following statements are equivalent:

(1) (X, τ) is α^γ-symmetric and α^γ-T_0.

(2) (X, τ) is α^γ-T_1.

Proof. By Remark 5.10 and Corollary 5.19, it suffices to prove only (1) ⇒ (2).

Let $x \neq y$ and by α^γ-T_0, we may assume that $x \in U \subseteq X\backslash\{y\}$for some $U \in \alpha^\gamma O(X)$. Then $x \notin \alpha^\gamma cl(\{y\})$ and hence $y \notin \alpha^\gamma cl(\{x\})$. There exists an α^γ-open set V such that $y \in V \subseteq X\backslash\{x\}$ and thus (X, τ) is an α^γ-T_1 space.

Corollary 5.21. For an α^γ-symmetric topological space (X, τ) the following are equivalent:

(1) (X, τ) is α^γ-T_0.

(2) (X, τ) is α^γ-D_1.

(3) (X, τ) is α^γ-T_1.

Proof. (1) ⇒ (3). Corollary 5.20.

(3) ⇒ (2) ⇒ (1). Remark 5.10 (2) and Corollary 5.13.

Remark 5.22. If (X, τ) is an α^γ-symmetric space with an operation γ on τ, then the following statements are equivalent:

(1) (X, τ) is an α^γ-T_0 space.

(2) (X, τ) is an α^γ-$T_{½}$ space.

(3) (X, τ) is an α^γ-T_1 space.

Definition 5.23. Let A be a subset of a topological space (X, τ) and γ an operation on τ. The α^γ-kernel of A, denoted by $\alpha^\gamma ker(A)$ is defined to be the set $\alpha^\gamma ker(A) = \cap\{U \in \alpha^\gamma O(X): A \subseteq U\}$.

Theorem 5.24. Let (X, τ) be a topological space with an operation γ on τ and $x \in X$. Then $y \in \alpha^\gamma ker(\{x\})$ if and only if $x \in \alpha^\gamma cl(\{y\})$.

Proof. Suppose that $y \notin \alpha^\gamma ker(\{x\})$. Then there exists an α^γ-open set V containing x such that $y \notin V$. Therefore, we have $x \notin \alpha^\gamma cl(\{y\})$. The proof of the converse case can be done similarly.

Theorem 5.25. Let (X, τ) be a topological space with an operation γ on τ and A be a subset of X. Then, $\alpha^\gamma ker(A) = \{x \in X: \alpha^\gamma cl(\{x\}) \cap A \neq \varphi\}$.

Proof. Let $x \in \alpha^\gamma ker(A)$ and suppose $\alpha^\gamma cl(\{x\}) \cap A \neq \varphi$. Hence $x \notin X\backslash\alpha^\gamma cl(\{x\})$ which is an α^γ-open set containing A. This is impossible, since $x \in \alpha^\gamma ker(A)$. Consequently, $\alpha^\gamma cl(\{x\}) \cap A \neq \varphi$. Next, let $x \in X$ such that $\alpha^\gamma cl(\{x\}) \cap A \neq \varphi$ and suppose that $x \notin \alpha^\gamma ker(A)$. Then, there exists an α^γ-open set V containing A and $x \notin V$. Let $y \in \alpha^\gamma cl(\{x\}) \cap A$. Hence, V is an α^γ-neighborhood of y which does not contain x. By this contradiction $x \in \alpha^\gamma ker(A)$ and the claim.

Theorem 5.26. If a singleton {x} is an $\alpha^\gamma D$-set of (X, τ), then $\alpha^\gamma ker(\{x\}) \neq X$.

Proof. Since {x} is an $\alpha^\gamma D$-set of (X, τ), then there exist two subsets $U_1 \in \alpha^\gamma O(X)$ and $U_2 \in \alpha^\gamma O(X)$ such that $\{x\} = U_1\backslash U_2$, $\{x\} \subseteq U_1$ and $U_1 \neq X$. Thus, we have that $\alpha^\gamma ker(\{x\}) \subseteq U_1 \neq X$ and so $\alpha^\gamma ker(\{x\}) \neq X$.

Theorem 5.27. If f: (X, τ) → (Y, σ) is an $\alpha^{(\gamma,\beta)}$-irresolute surjective function and A is an $\alpha^\beta D$-set in Y, then the inverse image of A is an $\alpha^\gamma D$-set in X.

Proof. Let A be an $\alpha^\beta D$-set in Y. Then there are α^β-open sets O_1 and O_2 in Y such that $A = O_1\backslash O_2$ and $O_1 \neq Y$. By the $\alpha^{(\gamma,\beta)}$-irresolute of f, $f^{-1}(O_1)$ and $f^{-1}(O_2)$ are α^γ-open in X. Since $O_1 \neq Y$ and f is surjective, we have $f^{-1}(O_1) \neq X$. Hence, $f^{-1}(A) = \} f^{-1}(O_1)\backslash f^{-1}(O_2)$ is an $\alpha^\gamma D$-set.

Theorem 5.28. If (Y, σ) is α^β-D_1 and f: (X, τ) → (Y, σ) is $\alpha^{(\gamma,\beta)}$-irresolute bijective, then (X, τ) is α^γ-D_1.

Proof. Suppose that Y is an α^β-D_1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is α^β-D_1, there exist $\alpha^\beta D$-set O_x and O_y of Y containing f(x) and f(y) respectively, such that $f(x) \notin O_y$ and $f(y) \notin O_x$. By Theorem 5.27, $f^{-1}(O_x)$ and $f^{-1}(O_y)$ are $\alpha^\gamma D$-set in X containing x and y, respectively, such that $x \notin f^{-1}(O_y)$ and $y \notin f^{-1}(O_x)$. This implies that X is an α^γ-D_1space.

Theorem 5.29. A topological space (X, τ) is α^γ-D_1 if for each pair of distinct points x, $y \in X$, there exists an $\alpha^{(\gamma,\beta)}$-irresolute surjective function f: (X, τ) → (Y, σ), where Y is an α^β-D_1 space such that f(x) and f(y) are distinct.

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists an $\alpha^{(\gamma,\beta)}$-irresolute, surjective function f of a space X onto an α^β-D_1 space Y such that $f(x) \neq f(y)$. It follows from

Theorem 5.11 that α^β-D_1 = α^β-D_2. Hence, there exist disjoint α^βD-set O_x and O_y in Y such that f(x) ∈ O_x and f(y) ∈O_y. Since f is $\alpha^{(\gamma,\beta)}$-irresolute and surjective, by Theorem 5.27, $f^{-1}(O_x)$ and $f^{-1}(O_y)$ are disjoint α^γD-sets in X containing x and y, respectively. So, the space (X, τ) is α^γ-D_1.

Functions With α^γ-Closed Graphs

In this section, functions with α^γ-closed graphs are introduced and studied, and some properties and characterizations of α^γ-closed graphs are explained.

Definition 6.1. Let f: (X, τ) → (Y, σ) be any function, the graph of the function f is denoted by G(f) and is said to be α^γ-closed if for each (x, y) ∉ G(f), there exist U ∈ α^γO(X, x) and an open set V of Y containing y such that (U x V) ∩ G(f) = φ.

A useful characterisation of functions with α^γ-closed graph is given below.

Lemma 6.2. The function f: (X, τ) → (Y, σ) has an α^γ-closed graph if and only if for each x ∈ X and y ∈ Y such that y ≠ f(x), there exist an α^γ-open set U and an open set V containing x and y respectively, such that f(U) ∩V = φ.

Proof. It follows readily from the above definition.

Theorem 6.3. Suppose that a function f: (X, τ) → (Y, σ) has an α^γ-closed graph, then the following are true:

(1) If f is surjective, then Y is T_1.

(2) If f is injective, then X is α^γ-T_1.

(3) If a function f is α^γ-continuous and injective, then X is α^γ-T_2.

(4) For each x ∈ X, {f(x)} = ∩{f(cl(f(U)): U ∈ α^γO(X, x)}.

Proof. (1) Let y_1 and y_2 be two distinct points of Y. Since f is surjective, there exists x in X such that f(x) = y_2, then (x, y_1) ∉ G(f). By Lemma 6.2, there exist α^γ-open set U and open set V containing x and y_1 respectively, such that f(U) ∩ V = φ. We obtain an open set V containing y_1 which does not contain y_2. Similarly we can obtain an open set containing y_2 but not y_1. Hence, Y is T_1.

(2) Let x_1 and x_2 be two distinct points of X. The injectivity of f implies f(x_1) ≠ f(x_2) whence one obtains that (x_1, f(x_2)) ∈ (X x Y)\G(f). The α^γ-closedness of G(f), by Lemma 6.2, ensures the existence of U ∈ α^γO(X, x_1), V ∈ O(Y, f(x_2)) such that f(U) ∩V = φ. Therefore, f(x_2) ∉ f(U) and a fortiori x_2 ∉ U. Again (x_2, f(x_1)) ∈ (X x Y)\G(f) and α^γ-closedness of G(f), as before gives A ∈ α^γO(X, x_2), B ∈ O(Y, f(x_1)) with f(A) ∩ B = φ, which guarantees that f(x_1) ∉ f(A) and so x_1 ∉ A. Therefore, we obtain sets U and A ∈ α^γO(X) such that x_1 ∈ U but x_2 ∉U while x_2 ∈ A but x_1 ∉A. Thus X is α^γ-T_1.

(3) Let x_1 and x_2 be any distinct points of X. Then f(x_1) ≠ f(x_2), so (x_1, f(x_2)) ∈ (X x Y)\G(f). Since the graph G(f) is α^γ-closed, there exist an α^γ-open set U containing x_1 and open set V containing f(x_2) such that f(U) ∩V = φ. Since f is α^γ-continuous, f^{-1}(V) is an α^γ-open set containing x_2 such that U ∩ f^{-1}(V) = φ. Hence X is α^γ-T_2.

(4) Suppose that y ≠ f(x) and y ∈{cl(f(U)): U ∈ α^γO(X, x)}. Then y ∈ cl(f(U)) for each U ∈ α^γO(X, x). This implies that for each open set V containing y, V ∩ f(U) ≠ φ. Since (x, y) ∉ G(f) and G(f) is an α^γ-closed graph, this is a contradiction.

Theorem 6.4. If f: (X, τ) → (Y, σ) is α^γ-continuous and Y is T_2 space, then G(f) is α^γ-closed graph.

Proof. Suppose that (x, y) ∉ G(f), then f(x) ≠ y. By the fact that Y is T_2, there exist open sets W and V such that f(x) ∈ W, y ∈ V and V ∩ W = φ. Since f is α^γ-continuous, there exists U ∈ α^γO(X, x) such that f(U) ⊆ W. Hence, we have f(U) ∩ V = φ. This means that G(f) is α^γ-closed.

Conclusion

In this paper, we introduce the notion of α^γ-open sets, α^γ-continuity and $\alpha^{(\gamma,\beta)}$-irresoluteness in topological spaces. By utilizing these notions we introduce some weak separation axioms. Also we show that some basic properties α^γ-T_i (i = 0, ½, 1, 2), α^γ-D_i (i = 0, 1, 2) spaces and we offer a new notion of the graph of a function called an α^γ-closed graph and investigate some of their fundamental properties.

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