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On a class of α_γ-open sets in a topological space

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ABSTRACT. In this paper, we introduce the concept of α_γ-open sets as a generalization of γ-open sets in a topological space (X, τ). Using this set, we introduce $\alpha_\gamma T_0$, α_γ-$T_{1/2}$, $\alpha_\gamma T_1$, $\alpha_\gamma T_2$, $\alpha_\gamma D_0$, $\alpha_\gamma D_1$ and $\alpha_\gamma D_2$ spaces and study some of its properties. Finally we introduce $\alpha_{(\gamma,\gamma')}$-continuous mappings and give some properties of such mappings.

Keywords: γ-open set, α_γ-open set, α_γ-g.closed set.

Uma classe de conjuntos α_γ-aberto em um espaço topológico

RESUMO. Neste artigo, apresentamos o conceito de conjuntos α_γ-abertos como uma generalização de conjuntos γ-aberto em um espaço topológico (X, τ). Usando este conjunto, introduzimos espaços $\alpha_\gamma T_0$, α_γ-$T_{1/2}$, $\alpha_\gamma T_1$, $\alpha_\gamma T_2$, $\alpha_\gamma D_0$, $\alpha_\gamma D_1$ e $\alpha_\gamma D_2$ e estudamos algumas de suas propriedades. Finalmente introduzimos mapeamentos contínuos de $\alpha_{(\gamma,\gamma')}$ e damos algumas propriedades de tais mapeamentos.

Palavras-chave: γ-conjunto aberto, α_γ-conjunto aberto, α_γ-g conjunto fechado.

Introduction

Njastad (1965) introduced α-open sets. Kasahara (1979) defined the concept of an operation on topological spaces and introduce the concept of α-closed graphs of an operation. Ogata (1991) called the operation α (respectively α-closed set) as γ-operation (respectively γ-closed set) and introduced the notion of τ_γ which is the collection of all γ-open sets in a topological space. Also he introduced the concept of γ-T_i ($i = 0, 1/2, 1, 2$) and characterized γ-T_i using the notion of γ-closed and γ-open sets. In this paper, we introduce the concept of α_γ-open sets by using an operation γ on $\alpha O(X, \tau)$ and we introduce the concept of α_γ-generalized closed sets and α_γ-$T_{1/2}$ spaces and characterize α_γ-$T_{1/2}$ spaces using the notion of α_γ-closed or α_γ-open sets. Also, we show that some basic properties of $\alpha_\gamma T_i$, $\alpha_\gamma D_i$ for $i = 0, 1, 2$ spaces and we introduce $\alpha_{(\gamma,\gamma')}$-continuous mappings and study some of its properties. Let (X, τ) be a topological space and A be a subset of X. The closure of A and the interior of A are denoted by Cl(A) and Int(A), respectively. A subset A of a topological space (X, τ) is said to be α-open (NJASTAD, 1965) if $A \subseteq Int(Cl(Int(A)))$. The complement of an α-open set is said to be α-closed. The intersection of all α-closed sets containing A is called the α-closure of A and is denoted by $\alpha Cl(A)$. The family of all α-open (resp. α-closed) sets in a topological space (X, τ) is denoted by $\alpha O(X, \tau)$ (resp. $\alpha C(X, \tau)$). An operation γ on a topology τ is a mapping from τ in to power set P(X) of X such that $V \subseteq \gamma(V)$ for each $V \in \tau$, where $\gamma(V)$ denotes the value of γ at V. A subset A of X with an operation γ on τ is called γ-open if for each $x \in A$, there exists an open set U such that $x \in U$ and $\gamma(U) \subseteq A$. Clearly $\tau_\gamma \subseteq \tau$. Complements of γ-open sets are called γ-closed. The γ-closure of a subset A of X with an operation γ on τ is denoted by τ_γ-Cl(A) and is defined to be the intersection of all γ-closed sets containing A. A topological X with an operation γ on τ is said to be γ-regular if for each $x \in X$ and for each open neighborhood V of x, there exists an open neighborhood U of x such that $\gamma(U)$ contained in V. It is also to be noted that $\tau = \tau_\gamma$ if and only if X is a γ-regular space (OGATA, 1991).

α_γ-open sets

Definition 2.1. Let $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ be a mapping satisfying thefollowing property, $V \subseteq \gamma(V)$ for each $V \in \alpha O(X, \tau)$. We call the mapping γ an operation on $\alpha O(X, \tau)$.

Definition 2.2. Let (X, τ) be a topological space and $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ an operation on $\alpha O(X, \tau)$.

A nonempty set A of X is called an α_γ-open set of (X, τ) if for each point $x \in A$, there exists an α-open set U containing x such that $\gamma(U) \subseteq A$. The complement of an α_γ-open set is called α_γ-closed in (X, τ). We suppose that the empty set is α_γ-open for any operation $\gamma : \alpha O(X, \tau) \rightarrow P(X)$. We denote the set of all α_γ-open (resp. α_γ-closed) sets of (X, τ) by $\alpha O(X, \tau)_\gamma$ (resp. $\alpha C(X, \tau)_\gamma$).

Remark 2.3. A subset A is an α_{id}-open set of (X, τ) if and only if A is α-open in (X, τ). The operation $id : \alpha O(X, \tau) \rightarrow P(X)$ is defined by $id(V) = V$ for any set $V \in \alpha O(X, \tau)$, this operation is called the identity operation on αO(X, τ). Therefore, we have that $\alpha O(X, \tau)_{id} = \alpha O(X, \tau)$.

Remark 2.4. The concept of α_γ-open and open are independent.

Example 2.5. Consider X = {a, b, c} with the topology $\tau = \{\varphi, \{a\}, \{a, b\}, X\}$ and $\alpha O(X, \tau) = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on αO(X, τ) by $\gamma(A) = A$ if $A = \{a, c\}$ or $A = \varphi$ and $\gamma(A) = X$ otherwise. Then α_γ-open sets are φ, {a, c} and X.

Remark 2.6. It is clear from the definition that every α_γ-open subset of a space X is α-open, but the converse need not be true in general as shown in the following example.

Example 2.7. Consider X = {a, b, c} with the topology $\tau = \{\varphi, \{a\}, X\}$ and $\alpha O(X, \tau) = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on αO(X, τ) by $\gamma(A) = A$ if $b \in A$ and $\gamma(A) = X$ if $b \notin A$. Then $\alpha O(X, \tau)_\gamma = \{\varphi, \{a, b\}, X\}$ and $\{a\} \in \alpha O(X, \tau)$, but $\{a\} \notin \alpha O(X, \tau)_\gamma$.

Theorem 2.8. If A is a γ-open set in (X, τ), then A is an α_γ-open set.

Proof. Follows from that every open set is α-open.

The converse of the above theorem need not be true in general as it is shown below.

Example 2.9. Consider X = {a, b, c} with the topology $\tau = \{\varphi, \{a\}, X\}$. Define an operation γ on αO(X, τ) by $\gamma(A) = A$. Then {a, b} is an α_γ-open set but not a γ-open set.

The proof of the following result is easy and hence it is omitted.

Proposition 2.10. If (X, τ) is γ-regular space, then every open set is α_γ-open.

Theorem 2.11. Let $\{A_\alpha\}_{\alpha \in J}$ be a collection of α_γ-open sets in a topological space (X, τ), then $\cup_{\alpha \in J} A_\alpha$ is α_γ-open.

Proof. Let $x \in \cup_{\alpha \in J} A_\alpha$, then $x \in A_\alpha$ for some $\alpha \in J$. Since A_α is an α_γ-open set, implies that there exists an α-open set U containing x such that $\gamma(U) \subseteq A_\alpha \subseteq \cup_{\alpha \in J} A_\alpha$. Therefore $\cup_{\alpha \in J} A_\alpha$ is an α_γ-open set of (X, τ).

If A and B are two α_γ-open sets in (X, τ), then the following example shows that $A \cap B$ need not be α_γ-open.

Example 2.12. Consider X = {a, b, c} with the discrete topology on X. Define an operation γ on αO(X, τ) by $\gamma(A) = \{a, b\}$ if $A = \{a\}$ or {b} and $\gamma(A) = A$ otherwise. Then $A = \{a, b\}$ and $B = \{a, c\}$ are α_γ-open sets but $A \cap B = \{a\}$ is not an α_γ-open set.

From the above example we notice that the family of all α_γ-open subsets of a space X is a supratopology and need not be a topology in general.

Proposition 2.13. The set A is α_γ-open in the space (X, τ) if and only if for each $x \in A$, there exists an α_γ-open set B such that $x \in B \subseteq A$.

Proof. Suppose that A is α_γ-open set in the space (X, τ). Then for each $x \in A$, put $B = A$ is an α_γ-open set such that $x \in B \subseteq A$.

Conversely, suppose that for each $x \in A$, there exists an α_γ-open set B such that $x \in B \subseteq A$, thus $A = \cup B_x$ where $B_x \in \alpha O(X, \tau)_\gamma$ for each x. Therefore, A is an α_γ-open set.

Definition 2.14. An operation γ on αO(X, τ) is said to be α-regular if for every α-open sets U and V of each $x \in X$, there exists an α-open set W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V)$.

Definition 2.15. An operation γ on αO(X, τ) is said to be α-open if for every α-open set U of each $x \in X$, there exists an α_γ-open set V such that $x \in V$ and $V \subseteq \gamma(U)$.

In the following two examples, we show that α-regular operation is incomparable with the α-open operation.

Example 2.16. Consider X = {a, b, c} with the topology $\tau = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on αO(X, τ) by $\gamma(A) = \{a, b\}$ if $A = \{a\}$ and $\gamma(A) = X$ if $A \neq \{a\}$. Then γ is α-regular but not α-open.

Example 2.17. Consider X = {a, b, c} with the topology $\tau = \{\varphi, \{a\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on αO(X, τ) by $\gamma(A) = A$ if $A = \{a, b\}$ or {a, c} and $\gamma(A) = X$ otherwise. Then γ is not α-regular but γ is α-open.

In the following proposition the intersection of two α_γ-open sets is also an α_γ-open set.

Proposition 2.18. Let γ be an α-regular operation on $\alpha O(X, \tau)$. If A and B are α_γ-open sets in X, then $A \cap B$ is also an α_γ-open set.

Proof. Let $x \in A \cap B$, then $x \in A$ and $x \in B$. Since A and B are α_γ-open sets, there exist α-open sets U and V such that $x \in U$ and $\gamma(U) \subseteq A$, $x \in V$ and $\gamma(V) \subseteq B$. Since γ is an α-regular operation, then there exists an α-open set W of x such that $\gamma(W) \subseteq \gamma(U) \cap \gamma(V) \subseteq A \cap B$. This implies that $A \cap B$ is α_γ-open set.

Remark 2.19. By the above propositon, if γ is an α-regular operation on $\alpha O(X, \tau)$. Then $\alpha O(X, \tau)_\gamma$ form a topology on X.

Definition 2.20. A point $x \in X$ is in αCl_γ-closure of a set $A \subseteq X$, if $\gamma(U) \cap A \neq \varphi$ for each α-open set U containing x. The αCl_γ-closure of A is denoted by $\alpha Cl_\gamma(A)$.

Definition 2.21. Let A be a subset of (X, τ), and $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ be an operation on $\alpha O(X, \tau)$. Then the α_γ-closure of A is denoted by $\alpha_\gamma Cl(A)$ and defined as follows, $\alpha_\gamma Cl(A) = \cap\{F : F \text{ is } \alpha_\gamma\text{-closed and } A \subseteq F\}$.

The proof of the following theorem is obvious and hence omitted.

Theorem 2.22. Let (X, τ) be a topological space and γ be an operation on $\alpha O(X, \tau)$. For any subsets A, B of X, we have the following properties:

(1) $A \subseteq \alpha_\gamma Cl(A)$.

(2) $\alpha_\gamma Cl(A)$ is α_γ-closed set in X.

(3) A is α_γ-closed set if and only if $A = \alpha_\gamma Cl(A)$.

(4) $\alpha_\gamma Cl(\varphi) = \varphi$ and $\alpha_\gamma Cl(X) = X$.

(5) If $A \subseteq B$, then $\alpha_\gamma Cl(A) \subseteq \alpha_\gamma Cl(B)$.

(6) $\alpha_\gamma Cl(A \cup B) \supseteq \alpha_\gamma Cl(A) \cup \alpha_\gamma Cl(B)$.

(7) $\alpha_\gamma Cl(A \cap B) \subseteq \alpha_\gamma Cl(A) \cap \alpha_\gamma Cl(B)$.

Theorem 2.23. For a point $x \in X$, $x \in \alpha_\gamma Cl(A)$ if and only if for every α_γ-open set V of X containing x such that $A \cap V \neq \varphi$.

Proof. Let $x \in \alpha_\gamma Cl(A)$ and suppose that $V \cap A = \varphi$ for some α_γ-open set V which contains x. Then $(X\backslash V)$ is α_γ-closed and $A \subseteq (X\backslash V)$, thus $\alpha_\gamma Cl(A) \subseteq (X\backslash V)$. But this implies that $x \in (X\backslash V)$, a contradiction. Therefore $V \cap A \neq \varphi$.

Conversely, Let $A \subseteq X$ and $x \in X$ such that for each α_γ-open set U which contains x, $U \cap A \neq \varphi$. If $x \notin \alpha_\gamma Cl(A)$, there is an α_γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $(X\backslash F)$ is an α_γ-open set with $x \in (X\backslash F)$, and thus $(X\backslash F) \cap A \neq \varphi$, which is a contradiction.

The proof of the following theorems are obvious and hence omitted.

Theorem 2.24. Let A be any subset of a topological space (X, τ) and γ be an operation on $\alpha O(X, \tau)$. Then the following relation holds.

$A \subseteq \alpha Cl(A) \subseteq \alpha Cl_\gamma(A) \subseteq \alpha_\gamma Cl(A) \subseteq \tau_\gamma\text{-}Cl(A)$.

Theorem 2.25. Let A be a subset of a topological space (X, τ) and γ be an operation on $\alpha O(X, \tau)$. Then, the following conditions are equivalent:

(1) A is α_γ-open.

(2) $\alpha Cl_\gamma(X\backslash A) = X\backslash A$.

(3) $\alpha_\gamma Cl(X\backslash A) = X\backslash A$.

(4) $X\backslash A$ is α_γ-closed.

Theorem 2.26. Let $\gamma : \alpha O(X, \tau) \rightarrow P(X)$ be an operation on $\alpha O(X, \tau)$ and A be a subset of X, then:

(1) A subset $\alpha Cl_\gamma(A)$ is an α-closed set in (X, τ).

(2) If γ is α-open, then $\alpha Cl_\gamma(A) = \alpha_\gamma Cl(A)$, and $\alpha Cl_\gamma(\alpha Cl_\gamma(A)) = \alpha Cl_\gamma(A)$, and $\alpha Cl_\gamma(A)$ is α_γ-closed.

Proof. To prove that $\alpha Cl_\gamma(A)$ is α-closed. Let $x \in \alpha Cl(\alpha Cl_\gamma(A))$. Then $U \cap \alpha Cl_\gamma(A) \neq \varphi$ for every α-open set U of x. Let $y \in U \cap \alpha Cl_\gamma(A)$, $y \in U$ and $y \in \alpha Cl_\gamma(A)$. Since U is α-open set containing y, implies $\gamma(U) \cap A \neq \varphi$. Therefore $x \in \alpha Cl_\gamma(A)$. Hence $\alpha Cl(\alpha Cl_\gamma(A)) \subseteq \alpha Cl_\gamma(A)$. This implies $\alpha Cl_\gamma(A)$ is an α-closed set.

(2) By Theorem 2.24, we have $\alpha Cl_\gamma(A) \subseteq \alpha_\gamma Cl(A)$. Now to prove that $\alpha_\gamma Cl(A) \subseteq \alpha Cl_\gamma(A)$. Let $x \notin \alpha Cl_\gamma(A)$, then there exists an α-open set U such that $\gamma(U) \cap A = \varphi$. Since γ is α-open, there exists an α_γ-open set V such that $x \in V \subseteq \gamma(U)$. Therefore $V \cap A = \varphi$. This implies $x \notin \alpha_\gamma Cl(A)$. Hence $\alpha_\gamma Cl(A) \subseteq \alpha Cl_\gamma(A)$. Therefore $\alpha Cl_\gamma(A) = \alpha_\gamma Cl(A)$. Now, $\alpha Cl_\gamma(\alpha Cl_\gamma(A)) = \alpha_\gamma Cl(\alpha_\gamma Cl(A)) = \alpha_\gamma Cl(A) = \alpha Cl_\gamma(A)$.

Definition 2.27. A subset A of the space (X, τ) is said to be α_γ-generalized closed (Briefly. α_γ-g.closed) if $\alpha_\gamma Cl(A) \subseteq U$ whenever $A \subseteq U$ and U is an α_γ-open set in (X, τ). The complement of an α_γ-g.closed set is called an α_γ-g.open set.

It is clear that every α_γ-closed subset of X is also an α_γ-g.closed set. The following example shows that an α_γ-g.closed set need not be α_γ-closed.

Example 2.28. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, \{a, c\}, X\}$. Define an operation γ on $\alpha O(X, \tau)$ by $\gamma(A) = A$ if $A = \{b\}$ or $\{a, c\}$ or φ and $\gamma(A) = X$ otherwise. Now, if we let $A = \{a\}$, since the only α_γ-open supersets of A are $\{a, c\}$ and X, then A is α_γ-g.closed. But it is easy to see that A is not α_γ-closed.

Theorem 2.29. A subset A of (X, τ) is α_γ-g.closed if and only if $\alpha_\gamma Cl(\{x\}) \cap A \neq \varphi$, holds for every $x \in \alpha_\gamma Cl(A)$.

Proof. Let U be an α_γ-open set such that $A \subseteq U$ and let $x \in \alpha_\gamma Cl(A)$. By assumption, there exists a $z \in \alpha_\gamma Cl(\{x\})$ and $z \in A \subseteq U$. It follows from Theorem 2.23, that $U \cap \{x\} \neq \varphi$, hence $x \in U$, this implies $\alpha_\gamma Cl(A) \subseteq U$. Therefore A is α_γ-g.closed.

Conversely, suppose that $x \in \alpha_\gamma Cl(A)$ such that $\alpha_\gamma Cl(\{x\}) \cap A = \varphi$. Since, $\alpha_\gamma Cl(\{x\})$ is α_γ-closed, therefore $X \backslash \alpha_\gamma Cl(\{x\})$ is an α_γ-open set in X. Since $A \subseteq X\backslash(\alpha_\gamma Cl(\{x\}))$ and A is α_γ-g.closed implies that $\alpha_\gamma Cl(A) \subseteq X\backslash\alpha_\gamma Cl(\{x\})$ holds, and hence $x \notin \alpha_\gamma Cl(A)$. This is a contradiction. Therefore $\alpha_\gamma Cl(\{x\}) \cap A \neq \varphi$.

Theorem 2.30. A set A of a space X is α_γ-g.closed if and only if $\alpha_\gamma Cl(A)\backslash A$ does not contain any non-empty α_γ-closed set.

Proof. Necessity. Suppose that A is α_γ-g.closed set in X. We prove the result by contradiction. Let F be an α_γ-closed set such that $F \subseteq \alpha_\gamma Cl(A)\backslash A$ and $F \neq \varphi$. Then $F \subseteq X\backslash A$ which implies $A \subseteq X\backslash F$. Since A is α_γ-g.closed and $X\backslash F$ is α_γ-open, therefore $\alpha_\gamma Cl(A) \subseteq X\backslash F$, that is $F \subseteq X\backslash\alpha_\gamma Cl(A)$. Hence $F \subseteq \alpha_\gamma Cl(A) \cap (X\backslash\alpha_\gamma Cl(A)) = \varphi$. This shows that, $F = \varphi$ which is a contradiction. Hence $\alpha_\gamma Cl(A)\backslash A$ does not contains any non-empty α_γ-closed set in

Sufficiency. Let $A \subseteq U$, where U is α_γ-open in (X, τ). If $\alpha_\gamma Cl(A)$ is not contained in U, then $\alpha_\gamma Cl(A) \cap X\backslash U \neq \varphi$. Now, since $\alpha_\gamma Cl(A) \cap X\backslash U \subseteq \alpha_\gamma Cl(A)\backslash A$ and $\alpha_\gamma Cl(A) \cap X\backslash U$ is a non-empty α_γ-closed set, then we obtain a contradication and therefore A is α_γ-g.closed.

Corollary 2.31. If a subset A of X is α_γ-g.closed set in X, then $\alpha_\gamma Cl(A)\backslash A$ dose not contain any non-empty γ-closed set in X.

Proof. Proof follows from the Theorem 2.8.

The converse of the above corollary is not true in general as it is shown in the following example.

Example 2.32. Consider $X = \{a, b, c\}$ with the topology $\tau = \{\varphi, \{c\}, X\}$. Define an operation γ on $\alpha O(X, \tau)$ by $\gamma(A) = A$. If we let $A = \{a, c\}$ then A is not α_γ-g.closed, since $A \subseteq \{a, c\} \in \alpha O(X, \tau)_\gamma$ and $Cl(A) = X \not\subset \{a, c\}$, where $\alpha_\gamma Cl(A)\backslash A = \{b\}$ dose not contain any non-empty γ-closed set in X.

Theorem 2.33. If A is an α_γ-g.closed set of a space X, then the following are equivalent:

(1) A is α_γ-closed.

(2) $\alpha_\gamma Cl(A)\backslash A$ is α_γ-closed.

Proof. (1) $\Rightarrow$ (2). If A is an α_γ-g.closed set which is also α_γ-closed, then by Theorem 2.30, $\alpha_\gamma Cl(A)\backslash A = \varphi$ which is α_γ-closed.

(2) $\Rightarrow$ (1). Let $\alpha_\gamma Cl(A)\backslash A$ be α_γ-closed set and A be α_γ-g.closed. Then by Theorem 2.30, $\alpha_\gamma Cl(A)\backslash A$ does not contain any non-empty α_γ-closed subset. Since $\alpha_\gamma Cl(A)\backslash A$ is α_γ-closed and $\alpha_\gamma Cl(A)\backslash A = \varphi$, this shows that A is α_γ-closed.

Theorem 2.34. For a space (X, τ), the following are equivalent:

(1) Every subset of X is α_γ-g.closed.

(2) $\alpha O(X, \tau)_\gamma = \alpha C(X, \tau)_\gamma$.

Proof. (1) $\Rightarrow$ (2). Let $U \in \alpha O(X, \tau)_\gamma$. Then by hypothesis, U is α_γ-g.closed which implies that $\alpha_\gamma Cl(U) \subseteq U$, so, $\alpha_\gamma Cl(U) = U$, therefore $U \in \alpha C(X, \tau)_\gamma$. Also let $V \in \alpha C(X, \tau)_\gamma$. Then $X\backslash V \in \alpha O(X, \tau)_\gamma$, hence by hypothesis $X\backslash V$ is α_γ-g.closed and then $X\backslash V \in \alpha C(X, \tau)_\gamma$, thus $V \in \alpha O(X, \tau)_\gamma$ according above we have $\alpha O(X, \tau)_\gamma = \alpha C(X, \tau)_\gamma$.

(2) $\Rightarrow$ (1). If A is a subset of a space X such that $A \subseteq U$ where $U \in \alpha O(X, \tau)_\gamma$, then $U \in \alpha C(X, \tau)_\gamma$ and therefore $\alpha_\gamma Cl(U) \subseteq U$ which shows that A is α_γ-g.closed.

Proposition 2.35. If A is γ-open and α_γ-g.closed then A is α_γ-closed.

Proof. Suppose that A is γ-open and α_γ-g.closed. As every γ-open is α_γ-open and $A \subseteq A$, we have $\alpha_\gamma Cl(A) \subseteq A$, also $A \subseteq \alpha_\gamma Cl(A)$, therefore $\alpha_\gamma Cl(A) = A$. That is A is α_γ-closed.

Theorem 2.36. If a subset A of X is α_γ-g.closed and $A \subseteq B \subseteq \alpha_\gamma Cl(A)$, then B is an α_γ-g.closed set in X.

Proof. Let A be α_γ-g.closed set such that $A \subseteq B \subseteq \alpha_\gamma Cl(A)$. Let U be an α_γ-open set of X such that $B \subseteq U$. Since A is α_γ-g.closed, we have $\alpha_\gamma Cl(A) \subseteq U$. Now $\alpha_\gamma Cl(A) \subseteq \alpha_\gamma Cl(B) \subseteq \alpha_\gamma Cl[\alpha_\gamma Cl(A)] = \alpha_\gamma Cl(A) \subseteq U$. That is $\alpha_\gamma Cl(B) \subseteq U$, where U is α_γ-open. Therefore B is an α_γ-g.closed set in X.

The converse of the above theorem need not be true as seen from the following example.

Example 2.37. Consider X = {a, b, c} with the topology τ = {φ, {a}, {c}, {a, c}, {b, c}, X}. Define an operation γ on $\alpha O(X, \tau)$ by $\gamma(A) = A$. Let A = {b} and B = {b, c}. Then A and B are α_γ-g.closed sets in (X, τ). But $A \subseteq B \not\subset \alpha_\gamma Cl(A)$.

Proposition 2.38. Let γ be an operation on $\alpha O(X, \tau)$. Then for each $x \in X$, {x} is α_γ-closed or X\{x} is α_γ-g.closed in (X, τ).

Proof. Suppose that {x} is not α_γ-closed, then X\{x} is not α_γ-open. Let U be any α_γ-open set such that $X\backslash\{x\} \subseteq U$, implies U = X. Therefore $\alpha_\gamma Cl(X\backslash\{x\}) \subseteq U$. Hence X\{x} is α_γ-g.closed.

α_γ-Separation axioms

Definition 3.1. A space (X, τ) is said to be α_γ-$T_{½}$ if every α_γ-g.closed set is α_γ-closed.

Theorem 3.2. The following statements are equivalent for a topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$:

(1) (X, τ) is α_γ-$T_{½}$.

(2) Each singleton {x} of X is either α_γ-closed or α_γ-open.

Proof. (1) ⇒ (2). Suppose {x} is not α_γ-closed. Then by Proposition 2.38, X\{x} is α_γ-g.closed. Now since (X, τ) is α_γ-$T_{½}$, X\{x} is α_γ-closed, that is {x} is α_γ-open.

(2) ⇒ (1). Let A be any α_γ-g.closed set in (X, τ) and $x \in \alpha_\gamma Cl(A)$. By (2) we have {x} is α_γ-closed or α_γ-open. If {x} is α_γ-closed then $x \notin A$ will imply $x \in \alpha_\gamma Cl(A)\backslash A$, which is not possible by Theorem 2.30. Hence $x \in A$. Therefore, $\alpha_\gamma Cl(A) = A$, that is A is α_γ-closed. So, (X, τ) is $\alpha\gamma$-$T_{½}$. On the other hand, if {x} is α_γ-open then as $x \in \alpha_\gamma Cl(A)$, $\{x\} \cap A \neq \varphi$. Hence $x \in A$. So A is α_γ-closed.

Definition 3.3. A subset A of a topological space (X, τ) is called an α_γDset if there are two U, V $\in \alpha O(X, \tau)_\gamma$ such that $U \neq X$ and $A = U\backslash V$. It is true that every α_γ-open set U different from X is an α_γD-set if A = U and V = φ. So, we can observe the following.

Remark 3.4. Every proper α_γ-open set is an α_γD-set.

Definition 3.5. A topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$ is said to be

(1) $\alpha_\gamma D_0$ if for any pair of distinct points x and y of X there exists an α_γD-set of X containing x but not y or an α_γD-set of X containing y but not x.

(2) $\alpha_\gamma D_1$ if for any pair of distinct points x and y of X there exists an α_γD-set of X containing x but not y and an α_γD-set of X containing y but not x.

(3) $\alpha_\gamma D_2$ if for any pair of distinct points x and y of X there exist disjoint α_γD-sets G and E of X containing x and y, respectively.

Definition 3.6. A topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$ is said to be:

(1) $\alpha_\gamma T_0$ if for any pair of distinct points x and y of X there exists an α_γ-open set U in X containing x but not y or an α_γ-open set V in X containing y but not x.

(2) $\alpha_\gamma T_1$ if for any pair of distinct points x and y of X there exists an α_γ-open set U in X containing x but not y and an α_γ-open set V in X containing y but not x.

(3) $\alpha_\gamma T_2$ if for any pair of distinct points x and y of X there exist disjoint α_γ-open sets U and V in X containing x and y, respectively.

Remark 3.7. For a topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$, the following properties hold:

(1) If (X, τ) is $\alpha_\gamma T_i$, then it is $\alpha_\gamma T_{i-1}$, for i = 1, 2.

(2) If (X, τ) is $\alpha_\gamma T_i$, then it is $\alpha_\gamma D_i$, for i = 0, 1, 2.

(3) If (X, τ) is $\alpha_\gamma D_i$, then it is $\alpha_\gamma D_{i-1}$, for i = 1, 2.

Theorem 3.8. A topological space (X, τ) is $\alpha_\gamma D_1$ if and only if it is $\alpha_\gamma D_2$.

Proof. Sufficiency. Follows from Remark 3.7.

Necessity. Let x, y $\in$ X, $x \neq y$. Then there exist α_γD-sets G_1, G_2 in X such that $x \in G_1$, $y \notin G_1$ and $y \in G_2$, $x \notin G_2$. Let $G_1 = U_1\backslash U_2$ and $G_2 = U_3\backslash U_4$, where U_1, U_2, U_3 and U_4 are α_γ-open sets in X. From $x \notin G_2$, it follows that either $x \notin U_3$ or $x \in U_3$ and $x \in U_4$. We discuss the two cases separately.

(i) $x \notin U_3$. By $y \notin G_1$ we have two subcases:

(a) $y \notin U_1$. From $x \in U_1\backslash U_2$, it follows that $x \in U_1\backslash (U_2 \cup U_3)$, and by $y \in U_3\backslash U_4$ we have $y \in U_3\backslash(U_1 \cup U_4)$. Therefore $(U_1\backslash(U_2 \cup U_3)) \cap (U_3\backslash(U_1 \cup U_4)) = \varphi$.

(b) $y \in U_1$ and $y \in U_2$. We have $x \in U_1\backslash U_2$, and $y \in U_2$. Therefore $(U_1\backslash U_2) \cap U_2 = \varphi$.

(ii) $x \in U_3$ and $x \in U_4$. We have $y \in U_3\backslash U_4$ and $x \in U_4$. Hence $(U_3\backslash U_4) \cap U_4 = \varphi$. Therefore, X is $\alpha_\gamma D_2$.

Theorem 3.9. A topological space (X, τ) with an operation γ on $\alpha O(X, \tau)$ is $\alpha_\gamma T_0$ if and only if for each pair of distinct points x, y of X, $\alpha_\gamma Cl(\{x\}) \neq \alpha_\gamma Cl(\{y\})$.

Proof. Clear.

Theorem 3.10. A topological space (X, τ) with an operation γ on αO(X, τ) is $\alpha_\gamma T_1$ if and only if the singletons are α_γ-closed sets.

Proof. Let (X, τ) be $\alpha_\gamma T_1$ and x any point of X. Suppose y ∈ X\{x}, then

x ≠ y and so there exists an α_γ-open set U such that y ∈ U but x ∉ U. Consequently y ∈ U ⊆ X\{x} that is X\{x} = ∪{U : y ∈ X\{x}} which is α_γ-open.

Conversely, suppose {p} is α_γ-closed for every p ∈ X. Let x, y ∈ X with x ≠ y. Now x ≠ y implies y ∈ X\{x}. Hence X\{x} is an α_γ-open set contains y but not x. Similarly X\{y} is an α_γ-open set contains x but not y. Accordingly X is an $\alpha_\gamma T_1$ space.

Proposition 3.11. The following statements are equivalent for a topological space (X, τ) with an operation γ on αO(X, τ):

(1) X is $\alpha_\gamma T_2$.

(2) Let x ∈ X. For each y ≠ x, there exists an α_γ-open set U containing x such that y ∉ α_γCl(U).

(3) For each x ∈ X, ∩{ α_γCl(U) : U ∈ $\alpha O(X, \tau)_\gamma$ and x ∈ U} = {x}.

Proof. (1) ⇒ (2). Since X is $\alpha_\gamma T_2$, there exist disjoint α_γ-open sets U and V containing x and y respectively. So, U ⊆ X\V. Therefore, α_γCl(U) ⊆ X\V. So y ∉ α_γCl(U).

(2) ⇒ (3). If possible for some y ≠ x, we have y ∈ α_γCl(U) for every α_γ-open set U containing x, which then contradicts (2).

(3) ⇒ (1). Let x, y ∈ X and x ≠ y. Then there exists an α_γ-open set U containing x such that y ∉ α_γCl(U). Let V = X\ α_γCl(U), then y ∈ V and x ∈ U and also U ∩ V = φ.

$\alpha_{(\gamma,\gamma')}$-Continuous maps

Throughout this section, let (X, τ) and (Y, σ) be two topological spaces and let γ : αO(X, τ) → P (X) and γ' : αO(Y, σ) → P(Y) be the operations on αO(X, τ) and αO(Y, σ), respectively.

Definition 4.1. A mapping f : (X, τ) → (Y, σ) is said to be $\alpha_{(\gamma,\gamma')}$-continuous if for each x of X and each $\alpha_{\gamma'}$-open set V containing f(x), there exists an α_γ-open set U such that x ∈ U and f(U) ⊆ V.

Theorem 4.2. Let f : (X, τ) → (Y, σ) be an $\alpha_{(\gamma,\gamma')}$-continuous mapping. Then:

(1) f(α_γCl(A)) ⊆ $\alpha_{\gamma'}$Cl(f(A)) holds for every subset A of (X, τ).

(2) For every $\alpha_{\gamma'}$-closed set B of (Y, σ), f^{-1}(B) is α_γ-closed in (X, τ).

Proof. (1) Let y ∈ f(α_γCl(A)) and V be the $\alpha_{\gamma'}$-open set containing y, then there exists a point x ∈ X and an α_γ-open set U such that f(x) = y, x ∈ U and f(U) ⊆ V. Since x ∈ α_γCl(A), we have U ∩ A ≠ φ, and hence φ ≠ f(U ∩ A) ⊆ f(U) ∩ f(A) ⊆ V ∩ f(A). This implies y ∈ $\alpha_{\gamma'}$Cl(f(A)).

(2) It is sufficient to prove that (1) implies (2). Let B be the $\alpha_{\gamma'}$-closed set in (Y, σ). That is $\alpha_{\gamma'}$Cl(B) = B. By using (1) we have f(α_γCl(f^{-1}(B))) ⊆ $\alpha_{\gamma'}$Cl(f(f^{-1}(B))) ⊆ $\alpha_{\gamma'}$Cl(B) = B holds. Therefore α_γCl(f^{-1}(B)) ⊆ f^{-1}(B), and hence f^{-1}(B) = α_γCl(f^{-1}(B)). Hence f^{-1}(B) is α_γ-closed set in (X, τ).

Definition 4.3. A mapping f : (X, τ) → (Y, σ) is said to be $\alpha_{(\gamma,\gamma')}$-closed if for any α_γ-closed set A of (X, τ), f(A) is $\alpha_{\gamma'}$-closed (Y, σ).

Definition 4.4. If f is $\alpha_{(id,\ \gamma')}$-closed, then f(F) is $\alpha_{\gamma'}$-closed for any α-closed set F of (X, τ).

Remark 4.5. If f is bijective mapping and f^{-1}: (Y, σ) → (X, τ) is $\alpha_{(\gamma',id)}$-continuous, then f is $\alpha_{(id,\gamma')}$-closed.

Proof. Proof follows from the Definitions 4.3 and 4.4.

Theorem 4.6. Suppose f : (X, τ) → (Y, σ) is $\alpha_{(\gamma,\ \gamma')}$-continuous and f is $\alpha_{(\gamma,\ \gamma')}$-closed, then

(1) For every α_γ-g.closed set A of (X, τ) the image f(A) is $\alpha_{\gamma'}$-g.closed.

(2) For every $\alpha_{\gamma'}$-g.closed set B of (Y, σ) the inverse set f^{-1}(B) is α_γ-g.closed.

Proof. (1) Let V be any $\alpha_{\gamma'}$-open set in (Y, σ) such that f(A) ⊆ V, then by Therem 4.2 (2), f^{-1}(V) is α_γ-open. Since A is α_γ-g.closed and A ⊆ f^{-1}(V), we have α_γCl(A) ⊆ f^{-1}(V), and hence f(α_γCl(A)) ⊆ V. By assumption f(α_γCl(A)) is an $\alpha_{\gamma'}$-closed set, therefore $\alpha_{\gamma'}$Cl(f(A)) ⊆ $\alpha_{\gamma'}$Cl(f(α_γCl(A))) = f(α_γCl(A)) ⊆ V. This implies f(A) is $\alpha_{\gamma'}$-g.closed.

(2) Let U be any α_γ-open set such that f^{-1}(B) ⊆ U. Let F = α_γCl(f^{-1}(B)) ∩ (X\U), then F is α_γ-closed in (X, τ). This implies f(F) is $\alpha_{\gamma'}$-closed set in (Y, σ). Since f(F) = f(α_γCl(f^{-1}(B)) ∩ (X\U))) ⊆ $\alpha_{\gamma'}$Cl(B) ∩ f(X\U) ⊆ $\alpha_{\gamma'}$Cl(B) ∩ (Y\B). This implies f(F) = φ and hence F = φ. Therefore α_γCl(f^{-1}(B)) ⊆ U. This implies f^{-1}(B) is α_γ-g.closed.

Theorem 4.7. Suppose f : (X, τ) → (Y, σ) is $\alpha_{(\gamma,\gamma')}$-continuous and $\alpha_{(\gamma,\gamma')}$-closed, then:

(1) If f is injective and (Y, σ) is $\alpha_{\gamma'}$-$T_{½}$, then (X, τ) is α_γ-$T_{½}$.

(2) If f is surjective and (X, τ) is α_γ-$T_{½}$, then (Y, σ) is $\alpha_{\gamma'}$-$T_{½}$.

Proof. (1) Let A be an α_γ-g.closed set of (X, τ). Now to prove that A is α_γ-closed. By Theorem 4.6 (1), f(A) is $\alpha_{\gamma'}$-g.closed. Since (Y, σ) is $\alpha_{\gamma'}$-$T_{½}$, this implies that f(A) is $\alpha_{\gamma'}$-closed. Since f is $\alpha_{(\gamma,\gamma')}$-continuous, then by Theorem 4.2, we have A = f^{-1}(f(A)) is α_γ-closed. Hence (X, τ) is α_γ-$T_{½}$.

(2) Let B be an $\alpha_{\gamma'}$-g.closed set in (Y, σ). Then f^{-1}(B) is α_γ-closed, since (X, τ) is α_γ-$T_{½}$ space. It follows from the assumption that B is $\alpha_{\gamma'}$-closed.

Definition 4.8. A mapping f : (X, τ) → (Y, σ) is said to be $\alpha_{(\gamma,\gamma')}$-homeomorphic, if f is bijective, $\alpha_{(\gamma,\gamma')}$-continuous and f^{-1} is $\alpha_{(\gamma',\gamma)}$-continuous.

Remark 4.9. If f : (X, τ) → (Y, σ) is bijective and f^{-1}: (Y, σ) → (X, τ) is $\alpha_{(\gamma',\gamma)}$-continuous, then f is $\alpha_{(\gamma,\gamma')}$-closed.

Theorem 4.10. Let f : (X, τ) → (Y, σ) be $\alpha_{(\gamma,\gamma')}$-homeomorphic. The space (X, τ) is α_γ-$T_{½}$ if and only if (Y, σ) is $\alpha_{\gamma'}$-$T_{½}$.

Proof. Necessity. Let B be an $\alpha_{\gamma'}$-g.closed set of (Y, σ). By Theorem 4.6, f^{-1}(B) is α_γ-g.closed and hence α_γ-closed. Since f is $\alpha_{(\gamma,\gamma')}$-closed, we have B = f($f^{-1}$(B)) is $\alpha_{\gamma'}$-closed.

Sufficiency. Let A be an α_γ-g.closed set of (X, τ). By Theorem 4.6, f(A) is $\alpha_{\gamma'}$-g.closed and hence $\alpha_{\gamma'}$-closed. Since f is $\alpha_{(\gamma,\gamma')}$-continuous, then by Theorem 4.2, we have A = f^{-1}(f(A)) is $\alpha_{\gamma'}$-closed.

Theorem 4.11. If f : (X, τ) → (Y, σ) is an $\alpha_{(\gamma,\gamma')}$-continuous surjective mapping and E is an $\alpha_{\gamma'}$D-set in Y, then the inverse image of E is an α_γD-set in X.

Proof. Let E be an $\alpha_{\gamma'}$D-set in Y. Then there are $\alpha_{\gamma'}$-open sets U_1 and U_2 in Y such that E = $U_1 \backslash U_2$ and $U_1 \neq$ Y. By the $\alpha_{(\gamma,\gamma')}$-continuous of f, $f^{-1}(U_1)$ and $f^{-1}(U_2)$ are α_γ-open in X. Since $U_1 \neq$ Y and f is surjective, we have $f^{-1}(U_1) \neq$ X. Hence, f^{-1}(E) = $f^{-1}(U_1)\backslash f^{-1}(U_2)$ is an α_γD-set.

Theorem 4.12. If (Y, σ) is $\alpha_{\gamma'}D_1$ and f : (X, τ) → (Y, σ) is $\alpha_{(\gamma,\gamma')}$-continuous bijective, then (X, τ) is $\alpha_\gamma D_1$.

Proof. Suppose that Y is an $\alpha_{\gamma'}D_1$ space. Let x and y be any pair of distinct points in X. Since f is injective and Y is $\alpha_{\gamma'}D_1$, there exist $\alpha_{\gamma'}$D-sets G_x and G_y of Y containing f(x) and f(y) respectively, such that f (x) ∉ G_y and f(y) ∉ G_x. By Theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are α_γD-sets in X containing x and y, respectively, such that x ∉ $f^{-1}(G_y)$ and y ∉ $f^{-1}(G_x)$. This implies that X is an $\alpha_\gamma D_1$ space.

Theorem 4.13. A topological space (X, τ) is $\alpha_\gamma D_1$ if for each pair of distinct points x, y ∈ X, there exists an $\alpha_{(\gamma,\gamma')}$-continuous surjective mapping f : (X, τ) → (Y, σ), where Y is an $\alpha_{\gamma'}D_1$space such that f(x) and f(y) are distinct.

Proof. Let x and y be any pair of distinct points in X. By hypothesis, there exists an $\alpha_{(\gamma,\gamma')}$-continuous, surjective mapping f of a space X onto an $\alpha_{\gamma'}D_1$space Y such that f(x) ≠ f(y). By Theorem 3.8, there exist disjoint $\alpha_{\gamma'}$D-sets G_x and G_y in Y such that f(x) ∈ G_x and f(y) ∈ G_y. Since f is $\alpha_{(\gamma,\gamma')}$-continuous and surjective, by Theorem 4.11, $f^{-1}(G_x)$ and $f^{-1}(G_y)$ are disjoint α_γD-sets in X containing x and y, respectively. Hence by Theorem 3.8, X is $\alpha_\gamma D_1$ space.

Conclusion

In this paper, we introduce the concept of an operation γ on a family of α-open sets in a topological space (X, τ). Using this operation γ, we introduce the concept of α_γ-open sets as a generalization of γ-open sets in a topological space (X, τ). Using this set, we introduce $\alpha_\gamma T_0$, α_γ-$T_{½}$, $\alpha_\gamma T_1$, $\alpha_\gamma T_2$, $\alpha_\gamma D_0$, $\alpha_\gamma D_1$ and $\alpha_\gamma D_2$ spaces and study some of their properties. Finally, we introduce $\alpha_{(\gamma,\gamma')}$-continuous mappings and give some properties of such mappings.

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