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# Some characterizations of $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$-fuzzy fantastic ideals in BCH-algebras

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**ABSTRACT.** In this paper, the concepts of $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy fantastic ideals and $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$-fuzzy fantastic ideals in BCH-algebras are introduced and investigate some of their properties.

**Keywords:** BCH-algebra, fuzzy fantastic ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy ideal, $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy fantastic ideals, $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$-fuzzy fantastic ideals.

## Caracterização de ideais fantásticos de fuzzy $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ em álgebras BCH

**RESUMO.** Neste artigo, os conceitos de ideais fantásticos de fuzzy $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$ e ideais fantásticos de fuzzy $(\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta})$ em álgebras BCH são apresentados e algumas de suas propriedades são examinadas.

**Palavras-chave:** álgebra BCH, ideal fantástico de fuzzy, ideal de fuzzy $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$, ideais fantásticos de fuzzy $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$, ideais fantásticos de fuzzy.

## Introduction

In 1983, the concept of a BCH-algebra was introduced by Hu and Li (1983) and gave examples of proper BCH-algebras (HU; LI, 1985). Some classifications of BCH-algebras were calculated by Dudek and Thomys (1990) and Ahmad (1990). They also have studied a several properties of these algebras. Since then several researchers have applied this idea to different mathematical disciplines. In Saeid and Namdar (2009) applied it to BCH-algebras and he considered on n-fold ideals in BCH-algebras and computation algorithms.

The concept of a fuzzy set, which was published by Zadeh (1965), was applied by many researchers to generalize some of the basic notions of algebra. The fuzzy algebraic structures play a vital role in mathematics with wide applications in many other branches such as theoretical physics, computer sciences, control engineering, information sciences, coding theory, topological spaces, logic, set theory, real analysis, measure theory etc. In Change (1968), studied it to the topological spaces. Das (1981) and Rosenfeld (1971) applied it to the fundamental theory of fuzzy groups. In Hong et al. (2001) applied this concept to BCH-algebras and studied fuzzy dot subalgebras of BCH-algebras. Jun (1994) give characterizations of BCI/BCH-algebras. In Dudek and Rousseau (1995), give the idea of set-theoretic relations and BCH-algebras with trivial structure. In Kazanci et al. (2010) studied soft set and soft BCH-algebras. In Saeid et al. (2010) discussed fuzzy n-fold ideals in BCH-algebras.

Murali (2004) defined the concept of belongingness of a fuzzy point to a fuzzy subset under a natural equivalence on a fuzzy subset. The idea of quasi-coincidence of a fuzzy point with a fuzzy set given in Pu and Liu (1980), plays a vital role to generate some different types of fuzzy subgroups, called (α, β)-fuzzy subgroups, introduced by Bhakat and Das (1996). In particular, $(\in, \in \vee q)$-fuzzy subgroup is an important and useful generalization of the Rosenfeld's fuzzy subgroups. Bhakat (1999, 2000) studied $(\in \vee q)$-level subsets, $(\in, \in \vee q)$-fuzzy normal, quasi-normal and maximal subgroups. Jun (2009) introduced the concept of $(\in, \in \vee q)$-fuzzy subalgebras in BCK/BCI-algebras and investigated some related results. In Jun (2004, 2005) discussed (α, β)-fuzzy subalgebras (ideals) of BCK/BCI-algebras. Zhan et al. (2009) studied $(\in, \in \vee q)$-fuzzy ideals of BCI-algebras. Davvaz (2006) studied $(\in, \in \vee q)$-fuzzy subnear-rings and ideals. Davvaz and Corsini (2007) redefined fuzzy $H_v$-submodule and many-valued implications. In Ma et al. (2008, 2009) discussed some kinds of $(\in, \in \vee q)$-interval-valued fuzzy ideals of BCI-algebras. In Ma et al. (2012) studied new types of fuzzy ideals of BCI-algebras.

In this paper, the concepts of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideals and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideals in BCH-algebras are defined and investigate some of their properties.

## Preliminaries

In what follows, let X denote a BCH-algebra unless otherwise specified.

**Definition 2.1.** (HU; LI, 1983) By a BCH-algebra, we mean an algebra (X, $*$, 0) of type (2, 0) satisfying the axioms:

(BCH-I) $x * x = 0$

(BCH-II) $x * y = 0$ and $y * x = 0$ imply $x = y$

(BCH-III) $(x * y) * z = (x * z) * y$

for all x, y, z $\in$ X.

We can define a partial order "$\le$" on X by $x \le y$ if and only if $x * y = 0$.

**Proposition 2.2.** (HU; LI, 1983, 1985; SAEID; NAMDAR, 2009) In any BCH-algebra X, the following are true:

(1) $x * (x * y) \le y$

(2) $0 * (x * y) = (0 * x) * (0 * y)$

(3) $x * 0 = x$

(4) $x \le 0$ implies $x = 0$

for all x, y $\in$ X.

**Definition 2.3.** (SAEID; NAMDAR, 2009) A non-empty subset I of a BCH-algebra X is called an ideal of X if it satisfies (I1) and (I2), where

(I1) $0 \in I$,

(I2) $x * y \in I$ and $y \in I$ imply $x \in I$,

for all x, y $\in$ X.

**Definition 2.4.** (SAEID; NAMDAR, 2009) A non-empty subset I of a BCH-algebra X is called a fantastic ideal of X if it satisfies (I1) and (I3), where

(I1) $0 \in I$,

(I3) $(x * y) * z \in I$ and $z \in I$ imply $x * (y * (y * x)) \in I$,

for all x, y, z $\in$ X.

We now review some fuzzy logic concepts. A fuzzy set κ of a universe X is a function from X to the unit closed interval [0, 1], that is $\kappa : X \to [0, 1]$.

**Definition 2.5.** (DAS, 1981) For a fuzzy set κ of a BCH-algebra X and $t \in (0, 1]$, the crisp set

$\kappa_t = \{x \in X \mid \kappa(x) \ge t\}$

is called the level subset of κ.

**Definition 2.6.** (SAEID et al., 2010) A fuzzy set κ of a BCH-algebra X is called a fuzzy ideal of X if it satisfies (F1) and (F2), where

(F1) $\kappa(0) \ge \kappa(x)$,

(F2) $\kappa(x) \ge \kappa(x * y) \wedge \kappa(y)$,

for all x, y $\in$ X.

**Definition 2.7.** A fuzzy set κ of a BCH-algebra X is called a fuzzy fantastic ideal of X if it satisfies (F1) and (F3), where

(F1) $\kappa(0) \ge \kappa(x)$,

(F3) $\kappa(x * (y * (y * x))) \ge \kappa((x * y) * z) \wedge \kappa(z)$,

for all x, y, z $\in$ X.

The Theorem 2.8 is a simple consequence of the transfer principle described in Kondo and Dudek (2005).

**Theorem 2.8.** A fuzzy set κ of a BCH-algebra X is a fuzzy fantastic ideal of X if and only if each non-empty level subset $\kappa_t$ is a fantastic ideal of X.

**Definition 2.9.** (SAEID et al., 2010) A fuzzy set κ of a BCH-algebra X of the form

$$\kappa(y) = \begin{cases} t (\neq 0) & \text{if } y = x, \\ 0 & \text{if } y \neq x, \end{cases}$$

is said to be a fuzzy point with support x and value t and is denoted by $x_t$.

A fuzzy point $x_t$ is said to belong to (resp., quasi-coincident with) a fuzzy set κ, written as $x_t \in \kappa$ (resp., $x_t q \kappa$) if $\kappa(x) \ge t$ (resp., $\kappa(x) + t > 1$).

If $x_t \in \kappa$ or $x_t q \kappa$, then we write $x_t \in \vee q\kappa$. If $\kappa(x) < t$ (resp. $\kappa(x) + t \le 1$), then we say that $x_t \overline{\in} \kappa$ (resp. $x_t \overline{q} \kappa$). The symbol $\overline{\in \vee q}$ means that $\in \vee q$ does not hold.

Let $\gamma, \delta \in [0, 1]$ be such that $\gamma < \delta$. For a fuzzy point $x_r$ and a fuzzy set κ of a BCH-algebra X, we say

(1) $x_r \in_\gamma \kappa$ if $\kappa(x) \ge r > \gamma$.

(2) $x_r q_\delta \kappa$ if $\kappa(x) + r > 2\delta$.

(3) $x_r \in_\gamma \vee q_\delta \kappa$ if $x_r \in_\gamma \kappa$ or $x_r q_\delta \kappa$.

(4) $x_r \overline{\in}_\gamma \vee \overline{q}_\delta \kappa$ if $x_r \overline{\in}_\gamma \kappa$ or $x_r \overline{q}_\delta \kappa$.

## $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideals in BCH-algebras

In this section, we introduce the concept of $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideals in BCH-algebras and investigate some of their properties.

**Definition 3.1.** A fuzzy set κ of a BCH-algebra X is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of X if it satisfies (A) and (B), where

(A) $x_t \in_\gamma \kappa \Rightarrow 0_t \in_\gamma \vee q_\delta \kappa$,

(B) $(x * y)_t \in_\gamma \kappa, y_r \in_\gamma \kappa \Rightarrow x_{t \wedge r} \in_\gamma \vee q_\delta \kappa$,

for all $t, r \in (\gamma, 1]$ and for all x, y $\in$ X.

**Theorem 3.2.** Every fuzzy ideal of a BCH-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal.

**Proof.** Straightforward.

**Definition 3.3.** A fuzzy set κ of a BCH-algebra X is called an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X if it satisfies (A) and (C), where

(A) $x_t \in_\gamma \kappa \Rightarrow 0_t \in_\gamma \vee q_\delta \kappa$,

(C) $((x * y) * z)_t \in_\gamma \kappa,\ z_r \in_\gamma \kappa \Rightarrow (x * (y * (y * x)))_{t \wedge r} \in_\gamma \vee q_\delta \kappa$,

for all t, r ∈ (γ, 1] and for all x, y, z ∈ X.

**Theorem 3.4.** Every fuzzy fantastic ideal of a BCH-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal.

**Proof.** Straightforward.

**Theorem 3.5.** A fuzzy set κ of a BCH-algebra X is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X if and only if it satisfies (D) and (E), where

(D) $\kappa(0) \vee \gamma \geq \kappa(x) \wedge \delta$,

(E) $\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$,

for all x, y, z ∈ X.

**Proof.** (A) ⇒ (D)

Let x ∈ X be such that $\kappa(0) \vee \gamma < \kappa(x) \wedge \delta$. Then

$$\kappa(0) \vee \gamma < t < \kappa(x) \wedge \delta \text{ for some } \gamma < t < \delta.$$

This implies $x_t \in_\gamma \kappa$ and $0_t \overline{\in}_\gamma \kappa$. Since

$$\kappa(0) + t \leq 2\delta$$

we have $0_t \overline{q}_\delta \kappa$. It follows that $0_t \overline{\in_\gamma \vee q_\delta} \kappa$, which is a contradiction. Hence (D) holds.

(D) ⇒ (A)

Let $x_t \in_\gamma \kappa$, then $\kappa(x) \geq t$. If $0_t \in_\gamma \kappa$, then (A) holds. If $0_t \overline{\in}_\gamma \kappa$, then

$\kappa(0) < t \leq \kappa(x)$. Since

$\kappa(0) \vee \gamma \geq \kappa(x) \wedge \delta$

$\geq t \wedge \delta$

it follows that $\kappa(0) \geq \delta$. Hence

$\kappa(0) + t > \kappa(0) + \kappa(0)$

$> 2\kappa(0)$

$\geq 2\delta$

Thus (A) holds.

(C) ⇒ (E)

Suppose (E) does not hold, then there exists t such that

$$\kappa(x * (y * (y * x))) \vee \gamma < \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta.$$

Then

$$\kappa(x * (y * (y * x))) \vee \gamma < t < \kappa((x * y) * z) \wedge \kappa(z)$$

and so

$((x * y) * z)_t \in_\gamma \kappa$ or $z_t \in_\gamma \kappa$, but $(x * (y * (y * x)))_t \overline{\in_\gamma \vee q_\delta} \kappa$.

This is a contradiction.

(E) ⇒ (C)

Let $((x * y) * z)_t \in_\gamma \kappa$ and $z_r \in_\gamma \kappa$. Then

$$\kappa((x * y) * z) \geq t \text{ and } \kappa(z) \geq r$$

If $(x * (y * (y * x)))_{t \wedge r} \in_\gamma \kappa$, then (C) holds. If $(x * (y * (y * x)))_{t \wedge r} \overline{\in}_\gamma \kappa$, then

$$\kappa(x * (y * (y * x))) < t \wedge r$$

Since

$$\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$$

$$\geq t \wedge r \wedge \delta$$

it follows that $\kappa(x * (y * (y * x))) \geq \delta$ and $t \wedge r > \delta$. Thus (C) holds.

Remark 3.6. For any $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal κ of a BCH-algebra X, we can

(i) If γ = 0 and δ = 1, then κ is a fuzzy fantastic ideal of X.

(ii) If γ = 0 and δ = 0.5, then κ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X.

For any fuzzy set κ of a BCH-algebra X, we define

$$\kappa_r^\gamma = \{x \in X \mid x_r \in_\gamma \kappa\}$$

$$\kappa_r^\delta = \{x \in X \mid x_r q_\delta \kappa\}$$

and

$$[\kappa]_r^\delta = \{x \in X \mid x_r \in_\gamma \vee q_\delta \kappa\} \text{ for all } r \in [0, 1].$$

It is clear that

$$[\kappa]_r^\delta = \kappa_r^\gamma \cup \kappa_r^\delta.$$

The relationship between $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideals and the crisp fantastic ideals of a BCH-algebra X can be expressed in the form of the following theorem.

**Theorem 3.7.** Let κ be a fuzzy set of a BCH-algebra X. Then

(1) κ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X if and only if $\kappa_r^\gamma (\neq \phi)$ is a fantastic ideal of X for all $r \in (\gamma, \delta]$.

(2) If $2\delta = 1 + \gamma$, then κ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X if and only if $\kappa_r^\delta (\neq \phi)$ is a fantastic ideal of X for all $r \in (\delta, 1]$.

(3) If $2\delta = 1 + \gamma$, then κ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X if and only if $[\kappa]_r^\delta (\neq \phi)$ is a fantastic ideal of X for all $r \in (\gamma, 1]$.

**Proof.** (1) Let κ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X and $x \in \kappa_r^\gamma$ for all

$r \in (\gamma, \delta]$, we have

$\kappa(0) \vee \gamma \geq \kappa(x) \wedge \delta$

$\geq r \wedge \delta$

$= r > \gamma$

So $\kappa(0) \geq r$. Hence $0 \in \kappa_r^\gamma$. Let $(x * y) * z, z \in \kappa_r^\gamma$. Then

$\kappa((x * y) * z) \geq r > \gamma$ and $\kappa(z) \geq r > \gamma$.

It follows from Theorem 3.5 (E) that

$\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\geq r \wedge r \wedge \delta$

$\geq r \wedge \delta$

$= r > \gamma$

So $\kappa(x * (y * (y * x))) \geq r$. Thus $x * (y * (y * x)) \in \kappa_r^\gamma$. Therefore $\kappa_r^\gamma$ is a fantastic ideal of X.

Conversely, assume that $\kappa_r^\gamma$ is a fantastic ideal of X for all $r \in (\gamma, \delta]$. If there is $x \in X$ such that

$\kappa(0) \vee \gamma < r = \kappa(x) \wedge \delta$

then $x_r \in_\gamma \kappa$, but $0_r \overline{\in_\gamma \vee q_\delta} \kappa$, which is a contradiction. Suppose $x, y, z \in X$ be such that

$\kappa(x * (y * (y * x))) \vee \gamma < \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$.

Select some $t \in (\gamma, \delta]$ such that

$\kappa(x * (y * (y * x))) \vee \gamma < t = \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$.

Then

$((x * y) * z)_t \in_\gamma \kappa$, $z_t \in_\gamma \kappa$, but $(x * (y * (y * x)))_t \overline{\in_\gamma \vee q_\delta} \kappa$.

Since $\kappa_r^\gamma$ is a fantastic ideal of X, we have $x * (y * (y * x)) \in \kappa_r^\gamma$, which is a contradiction. Hence

$\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$.

Therefore κ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X.

(2) The proof is similar to (1) and we omit it.

(3) Let κ be an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideal of X and $r \in (\gamma, 1]$. Then for all

$x \in [\kappa]_r^\delta$, we have $x_r \in_\gamma \vee q_\delta \kappa$, that is

$\kappa(x) \geq r > \gamma$ or $\kappa(x) > 2\delta - r > 2\delta - 1 = \gamma$.

Since κ is an $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy ideal of X, then

$\kappa(0) \vee \gamma \geq \kappa(x) \wedge \delta$

$> \gamma \wedge \delta$

$= \gamma$

and so $\kappa(0) \geq \gamma$, that is

$\kappa(0) \geq \kappa(x) \wedge \delta$.

**Case 1:** If $r \in (\gamma, \delta]$, then $2\delta - r \geq \delta \geq r$, and so

$\kappa(0) \geq \kappa(x) \wedge \delta$

$\geq r \wedge \delta$

$= r$

or

$\kappa(0) \geq \kappa(x) \wedge \delta$

$> (2\delta - r) \wedge \delta$

$= r \wedge \delta$

$= r$

Thus, $0_r \in_\gamma \kappa$.

**Case 2:** If $r \in (\delta, 1]$, then $2\delta - r < \delta < r$ and so

$\kappa(0) \geq \kappa(x) \wedge \delta$

$= r \wedge \delta$

$= \delta$

$> 2\delta - r$

or

$\kappa(0) \geq \kappa(x) \wedge \delta$

$> (2\delta - r) \wedge \delta$

$= 2\delta - r$

Hence, $0_r q_\delta \kappa$. Thus in any case, we have $0_r \in_\gamma \vee q_\delta \kappa$.

Let $(x * y) * z, z \in [\kappa]_r^\delta$, so we have

$((x * y) * z)_r, z_r \in_\gamma \vee q_\delta \kappa$

that is

$\kappa((x * y) * z) \geq r > \gamma$

or

$\kappa((x * y) * z) > 2\delta - r$

$> 2\delta - 1$

$= \gamma$

and $\kappa(z) \geq r > \gamma$

or

$\kappa(z) > 2\delta - r$

$> 2\delta - 1$

$= \gamma$

Since κ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy fantastic ideal of X, then

$\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$> \gamma \wedge \gamma \wedge \delta$

$> \gamma \wedge \delta$

$= \gamma$

and so

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta.$

**Case 1:** If $r \in (\gamma, \delta]$, then $2\delta - r \geq \delta \geq r$ and so

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq r \wedge r \wedge \delta$

$= r \wedge \delta$

$= r$

or

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq r \wedge (2\delta - r) \wedge \delta$

$\geq r \wedge \delta \wedge \delta$

$\geq r \wedge \delta$

$= r$

or

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta$

$\geq \delta \wedge \delta \wedge \delta$

$= \delta$

$> r$

Hence, $(x * (y * (y * x)))_r \in_{\gamma} \kappa$.

**Case 2:** If $r \in (\delta, 1]$, then $2\delta - r < \delta < r$ and so

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq r \wedge r \wedge \delta$

$\geq r \wedge \delta$

$= \delta$

$> 2\delta - r$

or

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq r \wedge (2\delta - r) \wedge \delta$

$\geq \delta \wedge (2\delta - r) \wedge \delta$

$\geq \delta \wedge (2\delta - r)$

$= 2\delta - r$

or

$\kappa(x * (y * (y * x))) \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta$

$\kappa(x * (y * (y * x))) \geq (2\delta - r) \wedge (2\delta - r) \wedge \delta$

$\geq (2\delta - r) \wedge \delta$

$= 2\delta - r$

Thus, $(x * (y * (y * x)))_r q_{\delta} \kappa$. Hence in all case we have

$(x * (y * (y * x)))_r \in_{\gamma} \vee q_{\delta}\kappa,$

that is,

$x * (y * (y * x)) \in [\kappa]_r^{\delta}.$

Therefore $[\kappa]_r^{\delta}$ is a fantastic ideal of X.

Conversely, suppose that $[\kappa]_r^{\delta}$ is a fantastic ideal of a BCH-algebra X for all $r \in (\gamma, \delta]$. If there is $x \in X$ be such that

$\kappa(0) \vee \gamma < r = \kappa(x) \wedge \delta,$

then $x_r \in_{\gamma} \kappa$ but $0_r \overline{\in_{\gamma} \vee q_{\delta}} \kappa$. Since $[\kappa]_r^{\delta}$ is a fantastic ideal of X, we have $0 \in [\kappa]_r^{\delta}$, which is a contradiction. Suppose x, y, z ∈ X be such that

$\kappa(x * (y * (y * x))) \vee \gamma < \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta.$

Select some $r \in (\gamma, 1]$ such that

$\kappa(x * (y * (y * x))) \vee \gamma < r = \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta.$

Then

$((x * y) * z)_r \in_{\gamma} \kappa$, $z_r \in_{\gamma} \kappa$ but $(x * (y * (y * x)))_r \overline{\in_{\gamma} \vee q_{\delta}} \kappa$.

Since $[\kappa]_r^{\delta}$ is a fantastic ideal of X, we have $x * (y * (y * x)) \in [\kappa]_r^{\delta}$, which is a contradiction. Hence

$\kappa(x * (y * (y * x))) \vee \gamma \geq \kappa((x * y) * z) \wedge \kappa(z) \wedge \delta.$

Therefore κ is an $(\in_{\gamma}, \in_{\gamma} \vee q_{\delta})$-fuzzy fantastic ideal of X.

By setting $\gamma = 0$ and $\delta = 0.5$ in Theorem 3.7, the following corollary is obtained.

**Corollary 3.8.** Let κ be a fuzzy set of a BCH-algebra X. Then

(1) κ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X if and only if $\kappa_r (\neq \phi)$ is a fantastic ideal of X for all $r \in (0, 0.5]$.

(2) κ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X if and only if $Q(\kappa; r) (\neq \phi)$ is a fantastic ideal of X for all $r \in (0.5, 1]$, where

$Q(\kappa; r) = \{x \in X \mid x_r q \kappa\}.$

(3) κ is an $(\in, \in \vee q)$-fuzzy fantastic ideal of X if and only if $[\kappa]_r (\neq \phi)$ is a fantastic ideal of X for all $r \in (0, 1]$.

( $\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta}$ )-fuzzy fantastic ideals in BCH-algebras.

In this section, we introduce the concept of ( $\overline{\in}_{\gamma}, \overline{\in}_{\gamma} \vee \overline{q}_{\delta}$ )-fuzzy fantastic ideals in BCH-algebras and investigate some of their properties.

**Definition 4.1.** A fuzzy set κ of a BCH-algebra X is called an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideal of X if it satisfies (F) and (G), where

(F) $0_t \overline{\in}_\gamma \kappa \Rightarrow x_t \overline{\in}_\gamma \vee \overline{q}_\delta \kappa$,

(G) $(x * (y * (y * x)))_{t \wedge r} \overline{\in}_\gamma \kappa \Rightarrow ((x * y) * z)_t \overline{\in}_\gamma \vee \overline{q}_\delta \kappa$ or $z_r \overline{\in}_\gamma \vee \overline{q}_\delta \kappa$,

for all t, r ∈ (γ , 1] and for all x, y, z ∈ X.

**Theorem 4.2.** A fuzzy set κ of a BCH-algebra X is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideal of X if and only if it satisfies (H) and (I), where

(H) $\kappa(0) \vee \delta \geq \kappa(x)$,

(I) $\kappa(x * (y * (y * x))) \vee \delta \geq \kappa((x * y) * z) \wedge \kappa(z)$,

for all x, y, z ∈ X.

**Proof.** The proof is similar to the proof of Theorem 3.5.

**Remark 4.3.** For any $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideal κ of a BCH-algebra X, we can conclude that if δ = 0.5, then κ is the $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic ideal of X.

The relationship between $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideals and the crisp fantastic ideals of a BCH-algebra X can be expressed in the form of the following theorem.

**Theorem 4.4.** Let κ be a fuzzy set of a BCH-algebra X. Then

(1) κ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideal of X if and only if $\kappa_r^\gamma (\neq \phi)$ is a fantastic ideal of X for all r ∈ (δ, 1].

(2) κ is an $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideal of X if and only if $\kappa_r^\delta (\neq \phi)$ is a fantastic ideal of X for all r ∈ (γ, δ].

**Proof.** The proof is similar to the proof of Theorem 3.7.

By setting γ = 0 and δ = 0.5 in Theorem 4.4, the following corollary is obtained.

**Corollary 4.5.** Let κ be a fuzzy set of a BCH-algebra X. Then

(1) κ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic ideal of X if and only if $\kappa_r (\neq \phi)$ is a fantastic ideal of X for all r ∈ (0.5, 1].

(2) κ is an $(\overline{\in}, \overline{\in} \vee \overline{q})$-fuzzy fantastic ideal of X if and only if $Q(\kappa; r) (\neq \phi)$ is a fantastic ideal of X for all r ∈ (0, 0.5], where

$$Q(\kappa; r) = \{x \in X \mid x_r q \kappa\}.$$

## Conclusion

In the study of fuzzy algebraic system, we see that the fuzzy fantastic ideals with special properties always play an important role.

In this paper we define $(\in_\gamma, \in_\gamma \vee q_\delta)$-fuzzy fantastic ideals and $(\overline{\in}_\gamma, \overline{\in}_\gamma \vee \overline{q}_\delta)$-fuzzy fantastic ideals in BCH-algebras and give several characterizations of fuzzy fantastic ideals in BCH-algebras in terms of these notions.

We believe that the research along this direction can be continued, and in fact, some results in this paper have already constituted a foundation for further investigation concerning the further development of fuzzy BCH-algebras and their applications in other branches of algebra. In the future study of fuzzy BCH-algebras, perhaps the following topics are worth to be considered:

(1) To characterize other classes of BCH-algebras by using this notion;

(2) To apply this notion to some other algebraic structures;

(3) To consider these results to some possible applications in computer sciences and information systems in the future.

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