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The strongly generalized double difference χ sequence spaces defined by a modulus

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ABSTRACT. In this paper we introduce the strongly generalized difference sequence spaces of modulus function and $A_i = a_{i(k,l)}^{i(m,n)}$ is a non-negative four dimensional matrix of complex numbers and $(p_{i(mn)})$ is a sequence of positive real numbers. We also give natural relationship between strongly generalized difference summable sequences with respect of modulus. We examine some topological properties of the above spaces and investigate some inclusion relations between these spaces.

Keywords: De la Valle-Poussin means, difference sequence, gai sequence, analytic sequence, modulus function, double sequence.

A diferença dupla fortemente generalizada de espaços sequenciais de χ determinados por módulo

RESUMO. Os espaços sequenciais diferenciais fortemente generalizados da função modulus são apresentados. $A_i = a_{i(k,l)}^{i(m,n)}$ é uma matriz não negativa de quatro dimensões de número complexos e $(p_{i(mn)})$ é uma sequência de número reais positivos. Proporciona-se o relacionamento natural entre sequências somáveis diferenciais fortemente generalizadas referente ao modulus. Analisam-se algumas características topológicas dos espaços mencionados acima e investigam-se as relações includentes entre esses espaços.

Palavras-chave: medianas de Valle-Poussin, sequências diferenciais, sequência de Gai, sequência analítica, função de módulo, sequência dupla.

Introduction

Throughout the paper w, x and Л denote the classes of all, gai and analytic scalar valued single sequences, respectively. We write w^2 for the set of all complex sequences (x_{mn}), where $m,n \in \aleph$ the set of positive integers. Then, w^2 is a linear space under the coordinate wise addition and scalar multiplication.

Some initial works on double sequence spaces are found in Bromwich (1965). Later on these were investigated by Hardy (1917), Moricz (1991), Moricz and Rhoades (1988), Basarir and Solancan (1999), Tripathy (2003), Turkmenoglu (1999) and many others. Quite recently Zeltser (2001) in her Ph.D. thesis, had essentially studied both the theory of topological double sequence spaces and the theory of summability of double sequences. Mursaleen and Edely (2004) have recently introduced the statistical convergence and Cauchy for double sequences and given the relation between statistical convergent and strongly Ces`aro summable double sequences. Subsequently Mursaleen (2004) and Mursaleen and Edely (2004), have defined the almost strong regularity of matrices for double sequences and applied these matrices to establish a core theorem and introduced the M-core for double sequences.

They have determined four dimensional matrices transforming every bounded double sequences $x = (x_{mn})$ into one whose core is a subset of the M-core of x. Recently, Altay and Basar (2005), have defined the spaces $\mathcal{BS}$, $\mathcal{BS}(t)$, $\mathcal{CS}_p$, $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ and $\mathcal{BV}$ of double sequences consisting of all double series whose sequence of partial sums are in the spaces $\mathcal{M}_u$, $\mathcal{V}m(t)$, $\mathcal{C}_p$, $\mathcal{C}_{bp}$, $\mathcal{C}_r$ *and* $\mathcal{L}_u$ respectively, and also examined some properties of those sequence spaces and determined the α - duals of the spaces $\mathcal{BS}$, $\mathcal{CS}_{bp}$ and $\mathcal{BV}$ and the $\beta(\vartheta)$ - duals of the spaces $\mathcal{CS}_{bp}$, $\mathcal{CS}_r$ of double series. Quite recently Basar and Sever (2009), have introduced the Banach space $\mathcal{L}_q$ of double sequences corresponding to the well-known space ℓ_q of single sequences and examined some properties of the space $\mathcal{L}_q$. Quite recently Das et al. (2008), Vakeel and

Tabassum (2010, 2011a and b), Kumar (2007), Subramanian and Misra (2010), have studied the space $\chi_M^2(p,q,u)$ of double sequences and gave some inclusion relations.

Spaces of strongly summable sequences were discussed by Kuttner (1946), Maddox (1979) and others. The class of sequences which are strongly Ces`aro summable with respect to a modulus was introduced by Maddox (1986), as an extension of the definition of strongly Ces`aro summable sequences. Cannor (1989) further extended this definition to the definition of stong A – summability, with respect to a modulus where $A = (a_{n,k})$ is a no-negative regular matrix and established some connection between strong A – summabilty with respect to a modulus and A – statistical convergence. The notion of double sequence was presented by Pringsheim (1900). The four dimensional matrix transformations

$$(Ax)_{k,l} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{kl}^{mn} x_{mn}$$

was also studied extensively by Hamilton (1936, 1938a and b, 1939). In his work and throughout this paper, the four dimensional matrices and double sequences have real-valued entries unless specified otherwise. In this paper we extend a few results known in the literature for ordinary (single) sequence spaces to apply sequence spaces. A sequence $x = (x_{i(mn)})$ is said to be strongly (V_2, λ_2) summable to zero, if $t_{rs}(|x|) \to 0$ as $r, s \to \infty$. Let $A = \left(a_{i(k,l)}^{i(mn}\right)$ be an infinite four dimensional matrix of complex numbers. We write

$$Ax = \left(A_i(x)\right)_{i=1}^{\infty}, \text{ if}$$
$$A_i(x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(a_{i(k,l)}^{i(mn)}\right) x_{mn}$$

converges for each $i \in \aleph$.

Let $p = (p_{mn})$ be a sequence of positive real numbers with $0 < p_{mn} < \sup p_{mn} = G$ and let $D = \max(1, 2^{G-1})$. Then, for $a_{mn}, b_{mn} \in \aleph$, the set of complex numbers, and for all $m, n \in \aleph$ we have

$$\left|a_{mn} + b_{mn}\right|^{\frac{1}{m+n}} \leq D\left\{\left|a_{mn}\right|^{\frac{1}{m+n}} + \left|b_{mn}\right|^{\frac{1}{m+n}}\right\} \qquad (1)$$

The double series $\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} x_{mn}$ is said to be convergent if and only if the double sequence (S_{mn}) is convergent, where $s_{mn} = \sum_{i,j=1}^{m,n} x_{ij}\,(m, n \in \aleph)$.

A sequence $x = (x_{mn})$ is said to be double analytic if $\sup_{mn} |x_{mn}|^{\frac{1}{m+n}} < \infty$. The vector space of all double analytic sequences is denoted by Λ^2.

A sequence $x = (x_{mn})$ is said to be a double gai sequence if $\left((m+n)!|x_{mn}|\right)^{\frac{1}{m+n}} \to 0$, as $m, n \to \infty$. The set of all double gai sequences is denoted by x^2. We denote ϕ as the set of all finite sequences.

The $(m,n)^{th}$ section, usually denoted by $x^{[m,n]}$, of the sequence $x = (x_{mn})$ is defined by $x^{[m,n]} = \sum_{i,j=0}^{m,} x_{ij}\Im_{ij}$ for all $m, n \in \aleph$; where $\Im_{ij}$ denotes the double sequence whose only non-zero term is $\frac{1}{(i+j)!}$ in the $(i,j)^{th}$ place for each $i, j \in \aleph$.

The difference sequence space (for single sequences), usually denoted by $Z(\Delta)$, is defined as (KIZMAZ, 1981)

$$Z(\Delta) = \left\{x = (x_k) \in w : (\Delta x_k) \in Z\right\}$$

for $Z = c, c_0$ *and* l_∞, $\Delta(x_k) = x_k + x_{k+1}$, for all $k \in$, where w, c, c_0 and l_∞ denote the class of all, convergent, null, and bounded scalar valued single sequences respectively. The above space is a Banach space normed by $\|x\| = |x_1| + \sup_{k\geq 1}|\Delta x_k|$.

In this paper we define the difference double sequence space as follows:

$$Z(\Delta) = \left\{x = (x_{mn}) \in w^2 : (\Delta x_{mn} \in Z)\right\}$$

where: $Z = \Lambda^2, x^2$ and
$\Delta x_{mn} = \left(x_{mn} - x_{m,n+1}\right) - \left(x_{m+1,n} - x_{m+1,n+1}\right)$ for all $m, n \in \aleph$.
We also have, for all $m, n \in \aleph$.

$$\Delta^m x_{mn} = \Delta(\Delta^{m-1} x_{mn}) = \Delta^{m-1} x_{mn} - \Delta^{m-1} x_{m,n+1} -$$
$$- \Delta^{m-1} x_{m+1,n} + \Delta^{m-1} x_{m+1,n+1}$$

A function $f:[0,\infty) \to [0, \infty)$ is said to be a modulus function (NAKANO, 1953) if and only if it satisfies

(i) $f(x) = 0$, if and only if, $x = 0$,
(ii) $f(x + y) \leq f(x) + f(y)$, for all $x \geq 0, y \geq 0$,
(iii) f is increasing,
(iv) f Is continuous from the right at 0.

Since $|f(x) + f(y)| \leq f(|x - y|)$, it follows from (iv) that is continuous on $[0, \infty)$.

A double sequence $\lambda_2 = \{(\beta_r, u_s)\}$ is said to be a double λ_2 sequence if there exist two non-decreasing sequences of positive numbers tending to infinity such that $\beta_{r+1} \leq \beta_r + 1$, $\beta_1 = 1$ and $u_{S+1} \leq u_s + 1$, $u_1 = 1$. The generalized double de Vallee-Poussin mean is defined as

$$t_{rs} = t_{rs}(x_{mn}) = \frac{1}{\lambda_{rs}} \sum_{(m,n)\in I_{rs}} x_{mn}$$

where:

$\lambda_{rs} = \beta_r u_s$ and $I_{rs} = \{(mn): r - \beta_r + 1 \leq m \leq r, s - u_s + 1 \leq n \leq s\}$.

A double sequence $x = (x_{mn})$ is said to be (V_2, λ_2)- summable to a number L if $P-\lim_{rs} t_{rs} = L$. If $\lambda_{rs} = rs$, then (V_2, λ_2)- summability is reduced to $(C, 1, 1)$ - summability.

Main results

Let $A = \left(a_{i(k,l)}^{i(m,n)}\right)$ is an infinite four dimensional matrix of complex numbers and $p = \left(p_{i(m,n)}\right)$ be a double analytic sequence of positive real numbers such that $0 < h = \inf_i p_{i(mn)} \leq sup_i\, p_{i(mn)} = H < \infty$, and f be a modulus. We define

$$V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right] =$$

$$\left\{x = (x_{mn}) \in w^2 : \lim_{r,s\to\infty} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\right.$$

$$\left.\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} = 0\right\}$$

$$V_{2_{\Lambda^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right] =$$

$$\left\{x = (x_{mn}) \in w^2 : \sup_{r,s} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\right.$$

$$\left.\left[f\left(\left|A_i\left(\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}} < \infty\right\}$$

where: $A_i(\Delta^m x) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{i(k,l)}^{i(m,n)} \Delta^m x_{mn}$. In what follows in this paper we establish some of the topological properties of the above spaces and investigate inclusion relations between them. We prove:

Theorem-1

Let f be a modulus function. Then $V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ is a linear space over the complex field C.

Proof: Let $x, y \in V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ and $\alpha, \mu \in \aleph$. Then there exist integers D_α and D_u such that $|\alpha|^{\frac{1}{m+n}} \leq D_\alpha$ and $|\mu|^{\frac{1}{m+n}} \leq D_\mu$. By using (1) and the properties of modulus f, we have:

$$\lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{i(k,l)}^{i(mn)}\left((m+n)!\Delta^m(\alpha x_{mn} + \mu y_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$\leq DD_\alpha^H \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \alpha^{\frac{1}{m+n}} a_{i(k,l)}^{i(mn)}\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$+ DD_\mu^H \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \mu^{\frac{1}{m+n}} a_{i(k,l)}^{i(mn)}\left((m+n)!\Delta^m y_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}},$$

As $r, s \to \infty$.

This proves that $V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ is linear.

Theorem-2

Let be a modulus function. Then the inclusion

$$V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right] \subset V_{2_{\lambda^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]$$

holds.

Proof: Let $x \in V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]$ such that $x \to \left(V_{2_{\chi^2}}^{\lambda_2}\left[A, \Delta^m, p, f\right]\right)$. Then we have

$$\sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$= \sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0 + 0\right|\right)\right]^{p_{i(mn)}}$$

$$\leq D \sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}} - 0\right|\right)\right]^{p_{i(mn)}}$$

$$+ D \sup_{rs} \lambda_{rs}^{-1} \sum_{mn\in I_{rs}}\left[f\left(|0|\right)\right]^{p_{i(mn)}}$$

$$\leq D\sup_{rs}\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}-0\right|\right)\right]^{p_{i(mn)}}$$

$$+\ T\max\left\{f\left(|0|\right)^h, f\left(|0|\right)^H\right\} < \infty$$

Hence $x\in V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p,f\right]$. Therefore the inclusion

$$V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p,f\right]\subset V_{2_{\lambda^2}}^{\lambda_2}\left[A,\Delta^m,p,f\right]$$

holds . This completes the proof.

Theorem-3

Let $p=\left(p_{i(mn)}\right)\in\Lambda^2$. Then $V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p,f\right]$ is a paranormed space with paranorm

$$g(x)=\sup_{rs}\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}},$$

where: $M=\max\left(1,\sup_i p_i\right)$.

Proof: Clearly $g(-x)=g(x)$. It is trivial that $\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}=0$ for $x_{mn}=0$. Hence we get $g(0)=0$. Further since $\frac{p_i}{M}\leq 1$ and $M\geq 1$, using Minkowski's inequality and definition of modulus f, for each (r,s), we have

$$\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m(x_{mn}+y_{mn})\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}$$

$$\leq\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}$$

$$+\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m y_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}$$

This follows that $g(x)$ is sub-additive. Next, for any complex number α and the definition of modulus function, we have

$$g(\alpha x)=\sup_{rs}\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m\alpha x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}\right)^{\frac{1}{M}}$$

$$\leq K^{\frac{H}{M}}g(x),$$

where $K=1+\left[|\alpha|^{\frac{1}{m+n}}\right]$ and $\left[|t|\right]$ denotes the integral part of t.

Since f is modulus, we have $x\to 0$ implies $g(x)\to 0$. Similarly $x\to 0$ and $\alpha\to 0$ implies $g(\alpha x)\to 0$. Thus we have for g x fixed and $\alpha\to 0$, $g(\alpha x)\to 0$. This completes the proof.

Theorem-4

Le f be a modulus function. Then $V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p\right]\subset V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p,f\right]$.

Proof: Let $x\in V_{2_{\chi^2}}^{\lambda_2}\left[A,\Delta^m,p\right]$. Then for every $\varepsilon>0$, there exists δ, $0<\delta<1$, such that for every $t\in[0,\infty)$ with $0\leq t\leq\delta$, $f(t)<\varepsilon$. Now we have

$$\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}-0\right|\right)\right]^{p_{i(mn)}}$$

$$=\lambda_{rs}^{-1}\sum_{mn\in I_{rs}\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\leq\delta}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$+\ \lambda_{rs}^{-1}\sum_{mn\in I_{rs}\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|>\delta}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$\leq\max\left\{f(\varepsilon)^h, f(\varepsilon)^H\right\}$$

$$+\max\left\{1,\left(2f(1)\delta^{-1}\right)^H\right\}\lambda_{rs}^{-1}\sum_{mn\in I_{rs}\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|>\delta}$$

$$\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

Therefore $x \in V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$. This completes the proof.

Theorem-5

Let $0 < p_{i(mn)} < q_{i(mn)}$ and $\left\{\frac{q_{i(mn)}}{p_{i(mn)}}\right\}$ be bounded. Then $V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,q,f\right] \subset V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$

Proof: Let

$$x \in V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,q,f\right] \quad (2)$$

Then

$$\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}} \quad (3)$$

$$\to 0,\ as\ \ r,s\to\infty$$

Let

$$t_i = \lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}$$ and

$$\gamma_{i(mn)} = \frac{p_{i(mn)}}{q_{i(mn)}}$$

Since $p_{i(mn)} \le q_{i(mn)}$, we have $0 \le \gamma_{i(mn)} \le 1$. Take $0 < \gamma < \gamma_{i(mn)}$ Define

$$u_i = t_i\,(t_i \ge 1)\,;\ u_i = 0\,(t_i < 1);$$
$$v_i = 0\,(t_i \ge 1)\,;\ v_i = t_i\,(t_i < 1);$$

$$t_i = u_i + v_i\ ;\ t_i^{\gamma_{i(mn)}} = u_i^{\gamma_{i(mn)}} + v_i^{\gamma_{i(mn)}}.$$

Then it follows that

$$u_i^{\gamma_{i(mn)}} \le u_i \le t_i \text{ and } v_i^{\gamma_{i(mn)}} \le v_i^{\gamma} \quad (4)$$

Hence by (4)

$$t_i^{\gamma_{i(mn)}} = u_i^{\gamma_{i(mn)}} + v_i^{\gamma_{i(mn)}} \le t_i + v_i^{\gamma}$$

That is

$$\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}\right)^{\gamma_{i(mn)}}$$

$$\le \lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}$$

$$\left(\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}\right)^{\frac{p_{i(mn)}}{q_{i(mn)}}}$$

$$\le \lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}$$

$$\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$\le \lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}$$

But as, by (3)

$$\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{q_{i(mn)}}$$

$$\to 0\ ,\ as\ r,s\to\infty\,,$$

$$\lambda_{rs}^{-1}\sum_{mn\in I_{rs}}\left[f\left(\left|A_i\left((m+n)!\Delta^m x_{mn}\right)^{\frac{1}{m+n}}\right|\right)\right]^{p_{i(mn)}}$$

$$\to 0\ ,\ as\ r,s\to\infty\,.$$

Hence $x \in V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$. This establishes the theorem.

Theorem-6

(*i*) Let $0 < \inf p_i \le 1$. Then $V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right] \subset V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,f\right]$,

(*ii*) Let $1 \le p_i \sup p_i < \infty$. Then $V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,f\right] \subset V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$,

(*iii*) Let $0 < p_i \le q_i < \infty$. Then

$$V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]\subset V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,q,f\right].$$

Proof: The proof is a routine verification.

Conclusion

We give natural relationship between strongly generalized difference $V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$-summable sequences with respect to *f*. We also examine some topological properties of $V_{2_{\chi^2}}{}^{\lambda_2}\left[A,\Delta^m,p,f\right]$ spaces and investigate some inclusion relations between these spaces.

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